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APPLIED MECHANICS

A Treatise for the use of Students who have time to work
Experimental, Numerical, and Graphical Exercises
illustrating the subject

BY

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WITH 371 ILLUSTRATIONS

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LA BELLE SAUVAGE

PREFACE.

THIS book describes what has for many years been the course of instruction in Applied Mechanics at the Finsbury Technical College. All mechanical and electrical engineering students in their first year had two lectures a week; the substance of these lectures is here printed in the larger type. Mechanical engineers had three lectures a week in their second year; the substance of these lectures is here printed in small type. As I found our arrangement of hours per week to work fairly well, I give it here :—

	Mechanical Engineers.		Electrical Engineers.	
	1st year.	2nd year.	1st year.	2nd year.
Mathematics	4	4	4	4
Graphics and Machine Drawing	7	13	4	3
Mechanics: Lectures	2	3	2	0
Mechanical Laboratory ...	3	2	2	0
Mechanics: Numerical Exercises	2	1	2	0
Mechanism	1	0	0	0
Wood and Iron Workshops ...	4	7	3	6

Chemists and building trade students also attended in the mechanical department, and the mechanical engineering students had courses of study in the physics and chemistry departments of the College. The Mechanics course included work on the steam and gas engine not given in this book. When, after *much experience* in teaching at an English public [illegible] at the Imperial College of Engineering, Japan, and many years. I ventured sixteen years ago to publish my method of teaching mechanics. I was met with some ridicule. Even without encouragement I was prepared to pursue the course which I had begun and found to be good, but I met with a great deal of encouragement from thoughtful men of about my own age. I had myself to scheme out and make drawings for every piece of apparatus for the laboratory. I knew of no collection of numerical and graphical exercises which were suitable for students, but gradually a collection was made from those given out in the lectures which seemed to me to be less objectionable, less academic, less misleading, than those hitherto available. I had difficulty in getting clever assistants trained in academic ways to sympathise with me. It was found in time that students took very eagerly to the quantitative experimental work, and that the whole system, faithfully followed, produced men whose knowledge was always ready for use in practical problems, and who knew the limits of usefulness of their knowledge. I am glad to say that more than twenty complete sets of the apparatus have been made and sent to various institutions by my workshop assistant, Mr. Shepherd.

It would have made this book too large if I had included in it, as I should have liked to do, copies of the instructions which each student receives when he begins on a new piece of apparatus.

Professor (now Sir Robert) Ball, at the Royal College of Science, Dublin, started quantitative experimental mechanical work. He used the well-known frame of the late Professor Willis, which was taken to pieces and built up in new forms for fresh experiments. What I have done has been to carry out Professor Ball's idea, using a distinct piece of apparatus for each fresh kind of experiment. A student measures things for himself; illustrates mechanical principles; finds the limits to which the notions of the books as to friction and properties of materials are correct; learns the use of squared paper, and the accuracy of graphical methods of calculation; and, above all, really learns to think for himself. Professor Ewing, at Cambridge, has developed the ideas of Professor Ball to a far greater extent than what I have had opportunity for, and I know of no place in which a better engineering education can be obtained at the present time than at Cambridge. I am glad to think that a system begun under the Science and Art Department by Sir Robert Ball is now likely to be adopted generally in science classes.

I am under great obligations to my assistant, Mr. G. A. Baxandall, who has been to great trouble in adding to the exercises, verifying answers, and correcting proofs. Professor Willis, D.Sc., has been kind enough to read through the proofs, and consequently I feel that there can be no important mistake anywhere.

I should like to think that, before a student begins the part in small type, he has worked through Thomson and Tait's small book on "Natural Philosophy," and that he has read the early part of my book on "The Calculus for Engineers."

JOHN PERRY.

16th July, 1897.
Royal College of Science,
London, S.W.

graphics; another, that it is all shading and colouring and the production of pretty pictures without centre lines or dimensions. Probably the greatest mistake is that of wasting time in a school in giving the information that one cannot help picking up in one's ordinary practical work after leaving school.

We believe that the principles which an engineer really recollects and keeps ready for mental use are very few. By means of lectures, models, drawing-office and laboratory, and numerical exercise work, we show a man how these simple principles enter, in curiously different-looking shapes, into his engineering practice. We give him the use of all the necessary methods of study, and we send him out into practical life prepared to study things for himself. We ought to recognise the fact that his real study of his profession is not at school or college. We ought to teach him how to learn for himself. Any child can state Newton's second law of motion, and the other half-dozen all-important principles of mechanics, so as to get full marks in an examination paper; the engineer knows that the phenomena he deals with are exceedingly complex, and that only a long experience will enable him to utilise the so easily stated principles. Schools and colleges are the places in which men ought to learn the uses of all mental tools; they are sure to specialise afterwards, but in the meantime we ought to give them plenty of tools to choose from. The average student cannot take in more than the elementary principles; the best students need not take in more.

2. The most important lesson for a beginner, however he may have studied mathematics and mechanics, and however able he may be as a mathematician, is this—that he must not go on merely assuming that he knows how to do things; he must know things by **actual trial** and not mere hearsay. He must actually calculate certain numerical results; he must actually illustrate principles with laboratory apparatus; and, if there is a school workshop, he must get to know the properties of materials by chipping and filing and paring and planing and turning. It is just the same as in one's after-school work. There is no great mechanical engineer who has not himself worked like a workman with other workmen, and got to understand men and things by actual contact with them. The man who shirks the following exercises and laboratory work will lose a great deal more than he is aware of.

Teachers will notice that things requiring even a little preparation more than other things will gradually become neglected. Therefore, let it be part of the *daily* work for every student to use logarithms, drawing instruments in graphical exercise work, and squared paper. In the drawing-office, blue prints ought to be *made by some* student or other every day; the **planimeter** ought

> to be used every day, and some student ought to be resetting his drawing-pens and other instruments every day. It ought to be a rule that all apparatus must always be ready for use, and that it is always in use. Teachers can arrange their own work in such a way that they cannot help seeing every day how the practical work of students is being done. When we find our system to be going with clockwork regularity and we feel no worry, we ought to believe that some change is necessary. If we find that the students are not absorbed in their work, we must understand that we teachers are in fault.

3. Students cannot spend too much time in multiplying, factoring, and simplifying **algebraical and trigonometrical expressions.** These are our tools, and we must get familiar with them. We may easily spend too much time in studying roots of equations, permutations and combinations, etc., and in the solution of triangles; and therefore, if it is possible, we try to learn all our mathematics, mechanics, physics, and chemistry from teachers who are engineers. What acquaintance with these subjects we have, ought to be a real knowledge, not the glib pretence which suffices for examinations; it must not be something apart from our life and work. To effect this object we must work many numerical and graphical exercises, and try to conquer our contempt for simple laboratory experiments, and illustrate the forty-seventh proposition of the First Book of Euclid by actually drawing some right-angled triangles and measuring their sides—to illustrate rules about triangles by actual measurement. In this way we learn much more than the ordinary geometrician knows; among other things, we obtain a valuable knowledge of the errors we are likely to make in graphical calculation.

4. Every student is supposed to be able to calculate the values of any algebraical or trigonometrical expression when numerical values are given. He ought to be able, when given at random such an expression as

$$w = a^{2/3}\, b^{-1/2} \div \sqrt{m^2 + n^2 a \log_e b \,.\, \cos.\, \theta},$$

to be able to calculate w when he knows the values of a, b, m, n, and θ. He ought to be able to use logarithms in multiplication and division and extraction of roots; to know all the usual mathematical symbols, and how it is sometimes convenient to use $\sqrt{a}$ and sometimes $a^{\frac{1}{2}}$. It is pedantic to say that a man must not use a formula unless he is able to prove its truth. It is *usually of great* help in learning to prove a formula to have previously used the formula and know the value and meaning of what we are to prove. A living Northern professor of great eminence has declared that a boy ought not to be allowed to use logarithms until he is able to calculate them; he has not said that a boy ought not to use a watch or wear a coat until he is able to make them.

EXERCISES.

1. Find 4·326 × 0·003457 to four significant figures, leaving out all unnecessary figures in the work. Find 0·01584 ÷ 2·104 to four significant figures. Also do these using logarithms. Find $\log_e 7$. Calculate

$5^{2·43}$, $3^{-0·246}$, $·042^{0·476}$, $\sqrt[5]{246·3}$, $31·01^{\frac{2}{3}} \times 0·02641^{\frac{1}{7}}$.

Ans., 0·01495, 0·007529, 1·94591, 49·95, 0·7632, 0·2211, 3·008, 5·872.

2. If $m = (a^3 + 2a^2b + s - ·345)^{\frac{2}{3}} \div (a^2 b^2)^5$

find m if $a = 0·504$, $b = 0·309$, $s = 1·567$.

Ans., 1·220.

3. What errors are there in assuming

$$(1 + a)^n = 1 + na$$

to be true in

$$(1·001)^3 = 1·003, \ (1·01)^{\frac{1}{3}} = 1·0033.$$

$$(0·99)^2 = (1 - ·01)^2 = 1 - ·02 = ·98.$$

$$\frac{1}{·99} = \frac{1}{1 - ·01} = (1 - ·01)^{-1} = 1 + ·01 = 1·01.$$

$$\sqrt[3]{\frac{1}{1·01}} = (1 + ·01)^{-\frac{1}{3}} = (1 - ·0033) = ·9967.$$

$$\sqrt{99} = \sqrt{100\,(1 - ·01)} = 10\,(1 - ·01)^{\frac{1}{2}} = 10\,(1 - ·005).$$

$$= 9·95\,?$$

The above answers are very nearly correct: the student is expected to find the correct answers.

4. How much error is there in the assumptions

$$\frac{1 + \alpha}{1 + \beta} = 1 + \alpha - \beta, \quad (1 + \alpha)(1 + \beta) = 1 + \alpha + \beta,$$

when $\alpha = ·01$, $\beta = ·01$, $\alpha = -·003$, $\beta = -·005$?

Ans., No error; ·01 per cent., ·004 per cent., ·0015 per cent.

5. If d is the diameter of the bore or the "calibre" of a gun, it is usually assumed that the weight of the gun is proportional to d^3, and that the thickness of armour which its projectile will pierce is proportional to d. If an 8-inch gun weighs 14 tons and can pierce 11 inches of armour, what thickness will be pierced by a 10-inch gun, and what is the weight of the gun? *Ans.*, 13·75 inches, 27·34 tons.

5. The linear expansion of bodies by heat is practically proportional to the rise of temperature. The values of a, the coefficient for linear expansion (the fractional increase in length for a rise in temperature of 1° Centigrade), are the

following numbers divided by 10^5:—Aluminium, 2·34; copper, 1·79; gold, 1·45; iron, 1·2; lead, 2·95; platinum, 0·9; silver, 1·94; tin, 2·27; zinc, 2·9; brass (71 copper to 29 zinc), 1·87; bronze (86 copper to 10 tin to 4 zinc), 1·8; German silver, 1·8; steel, 1·11; brick, 0·5; glass, 0·9; granite, 0·9; sandstone, 1·2; slate, 1·04; boxwood (across the fibre), 6·1; boxwood (along the fibre), 0·3; oak (across), 5·4; oak (along), 0·5; pine (across), 3·4; pine (along), 0·5.

The co-efficient, k, of **cubical expansion** is three times the co-efficient of linear expansion, because $(1 + a)^3 = 1 + 3a$ is practically correct for these small values of a. The average values of k between 0° and 100° C. are the following numbers divided by 10^3:—Alcohol, 1·26; mercury, 0·18; olive oil, 0·8; petroleum, 1·04; pure water, 0·43; sea-water, 0·5.

The student is supposed to have worked many exercises like the following ones:—

1. Steel rails of 0° C. have an aggregate length of 1 mile. What is the length at 33° C.? *Ans.*, 1 mile 24·2 inches.

2. A ring of wrought iron has an inside diameter of 5 feet when at a temperature of 970° C. What is the diameter at 0° C.? *Ans.*, 4·9 feet.

3. A cylindric plug of copper just fits into a hole 4 inches diameter in a piece of cast iron. After heating the mass to 1,240° C., by how much is the diameter of the hole too small for the plug? *Ans.*, ·0293 inch.

4. A bar of iron is 70 centimetres long at 0° C. What is its length in boiling water (100° C.)? What is its length at 50° C.?
Ans., 70·079 centimetres, 70·039 centimetres.

5. Two rods—one of copper, the other of iron—measure 98 centimetres each at 0° C. What is the difference in their lengths at 57° C.?
Ans., ·027 centimetre.

6. Bars of wrought iron, each 3·4 metres long, are laid down at a temperature of 10° C. What space is left between every two if they are intended to close up completely at 40° C.? *Ans.*, 1·26 millimetres.

7. A wrought-iron connecting-rod is 12 feet long at 10° C. What is the increase of length at 80° C.? *Ans.*, 0·121 inch.

8. A wrought-iron Cornish boiler is 33 feet long; the shell is at 0° C., the flue at 100° C. What would the difference of the lengths be if the flue were not prevented from expansion? *Ans.*, 0·475 inch.

9. A steel pump rod is 1,000 feet long. What is its change of length for a change of 10° C. *Ans.*, 1·44 inch.

10. In a thermometer ·01 cubic inch of mercury at 10° C. is raised to 15° C., and rises 1 inch in the tube. What is the cross-section of the tube?
Ans., 9×10^{-6} square inch.

11. The volume of a mass of iron being 5 cubic feet at 10° C., find its volume at 80° C. *Ans.*, 5·0126 cubic feet.

6. A student's knowledge of mathematics ought to be such that he can work out for *himself all* the rules given in such an excellent book on mensuration as that of Professor A. Lodge. The thorough study of such a book is one of several ways which may be recommended of getting familiar with mathematical principles. But nobody's life is long enough to use all these ways, and, besides, unnecessary study leads to dulness. Hence, if a student has taken some other way, he need not be alarmed at his ignorance of the more complex rules in mensuration; he may feel absolutely certain that he can work out such rules for himself, given time and necessity. He will study the more complex rules, such as prismoidal formulæ, if he needs to use them practically, not otherwise. The following rules are in constant use and must be familiar to the student, whether or not he knows the reasons for them. If he is familiar with the rules and does not anxiously search for the reasons for them, he lacks the necessary spirit of the practical engineer.

RULES IN MENSURATION.

An area is found in square inches if all the dimensions are given in inches. It is found in square feet if all the dimensions are given in feet.

Area of a *parallelogram.*—Multiply the length of one side by the perpendicular distance from the opposite side.

The *centre* of *gravity* of a parallelogram is at the point of intersection of its diagonals.

Draw a *right-angled triangle*; measure very accurately the lengths of the sides. You will find that, no matter what scale of measurement you use, the square of the length of the hypothenuse is equal to the sum of the squares of the lengths of the other two sides.

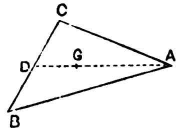


Fig. 1.

Area of a *triangle.*—Any side multiplied by its perpendicular distance from the opposite corner and divided by two.

The *centre* of *gravity*, or, rather, the centre of area, of a triangle is found by joining (Fig. 1) any corner, A, with the middle point, D, of the opposite side, B C, and making D G one-third of D A. G is the centre of gravity.

Area of an *irregular figure.*—Divide into triangles, and add the areas of the triangles together.

Circumference of a circle.—Multiply the diameter by 3·1416.

Arc of a circle.—From eight times the chord of half the arc subtract the chord of the whole arc; one-third of the remainder will give the length of the arc, nearly.

Area of a *trapezium.*—Half the sum of the parallel sides multiplied by the perpendicular distance between them.

Area of a *circle.*—Square the radius, and multiply by 3·1416; or square the diameter, and multiply by 0·7854.

Area of a *sector* of a circle.—Multiply half the length of the arc by the radius of the circle.

Area of a *segment* of a circle.—Find the area of the sector having the same arc, and the area of the triangle formed by the chord of the segment and the two radii of the sector. Take the sum or difference of these areas as the segment is greater or less than a semicircle.

Otherwise, for an approximate answer:—Divide the cube of the height of the segment by twice the chord, and add the quotient to two-thirds of the product of the chord and height of the segment. When the segment is greater than a semicircle, subtract the area of the remaining segment from the area of the circle.

The *areas of curves* may be found by **Simpson's rule.**—Divide the area into any even number of parts by an odd number of equidistant parallel lines or ordinates, the first and last touching the bounding curve. Take the sum of the extreme ordinates (in many cases each of the extreme ordinates is of no length), four times the sum of the even ordinates, and twice the sum of the odd ordinates (omitting the first and last); multiply the total sum by one-third of the distance between any two successive ordinates.

The ordinary rule for an **indicator diagram** is:—Draw lines at right angles to the atmospheric line, touching the extreme ends of the diagram. Divide the distance between them into ten equal parts (a parallel ruler with ten pieces is sometimes supplied), and at the middle of each part draw a line at right angles to the atmospheric line. Measure the breadth of the diagram on each of these ten lines, and take one-tenth of their sum. This gives the **average breadth,** and represents the average pressure to scale. The better plan is to find the area by a **planimeter.** The earnest student will practise the use of the planimeter, finding its error by tests on rectangles and circles. The average breadth of a diagram is used in many ways.

Exercise for Advanced Students.—Prove that Simpson's rule gives the area correctly if the arcs of curve between the odd ordinates follow, each of them, any such law as

$$y = a + bx + cx^2 + dx^3.$$

Surface of a sphere.—Multiply the diameter by the circumference.

Surface of a *cylinder.*—Multiply the circumference by the length, and add the areas of the two ends.

Surface of *a right circular cone.*—Multiply half the circumference of the base by the slant side, and add the area of the base.

Lateral surface of the *frustum of a right cone.*—Multiply the slant side by the circumference of the section equidistant from its parallel faces.

Area of an *ellipse.*—Multiply the product of the major and minor axes by ·7854.

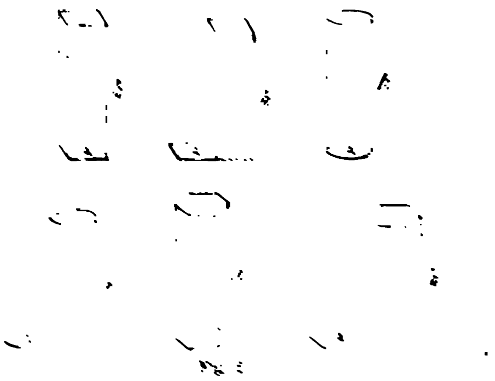
The areas of two similar figures are as the squares of their like dimensions. The volumes are as the cubes of their like dimensions.

The *cubic content* of a body is calculated in cubic inches if all the dimensions are given in inches; in cubic feet if all the dimensions are given in feet.

Cubic content of a *plate.*—Multiply area of plate by its thickness.

Cubic content of a *sphere.*—Cube the diameter, and multiply by ·5236.

Cubic content of the *segment of a sphere.*—Subtract twice



the height of the segment from three times the diameter of the sphere; multiply the remainder by the square of the height, and this product by ·5236.

The cubic content and surface of a sphere are each two-thirds of that of the cylindric vessel which just encloses it.

Cubic content of *any prismatic body* (Fig. 2).—Multiply the area of the base by the perpendicular height. This will give the same product as, Multiply the area of cross section by length along the axis of the prism. (The axis of a prismatic body goes from centre of gravity of base to centre of gravity of top.) The centre of gravity of a prismatic body is half-way along the axis.

Cubic content of *any pyramidal or conical body* (Fig. 3). —Multiply the area of the base by one-third of the perpendicular height.

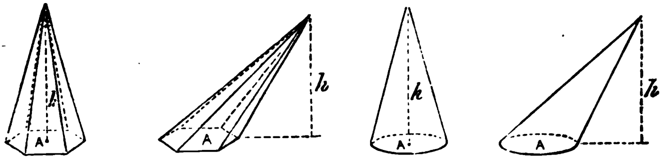


Fig. 3.

Centre of gravity is one-quarter of the way along the axis from the base. (The axis of any such body joins the centre of area of base with the vertex.)

The cubic content of the *rim of a wheel* is found by multiplying the area of a cross section by the circumference of the circle which passes through the centres of gravity of the cross sections.

The weight of a cubic inch of each of the following materials is given in lbs.:—Cast iron, ·26; wrought iron, ·28; steel, ·28; brass, ·3; copper, ·32; bronze, ·3; lead, ·4; tin, ·27; zinc, ·26. Hence to find the weight of a body of cast iron or any other of these substances, find the volume in cubic inches and multiply by one of the above numbers.

Very often it is only the approximate weight that is wanted, so that a **moulder** may know how much metal to melt, or for other purposes. Now, suppose we want the approximate weight of a cast-iron beam. Find roughly the average section and get its area in square inches, multiply by the length in inches, add to this the cubic content of any little gusset plates or other excrescences, multiply by ·26 and we have the weight in pounds.

The specific gravity of a substance means its weight as compared with the weight of the same bulk of water. Now, it is known that a cubic foot of water weighs very nearly 1,000 ounces, or rather 62·3 lbs. The specific gravity of a brick varies from 2 to 2·167, and therefore the weight of a cubic foot of brick varies from 2 × 1,000 or 2,000 to 2·167 × 1,000, or 2,167 ounces.

We see, then, that from a table of specific gravities we can get the weight of a cubic foot of a substance, and therefore if we know the cubic content of a body formed of this substance we can calculate its weight.

Various plans for saving labour in calculation suggest themselves to people working at any particular trade. For instance, if a pattern has no prints for cores, the weight of the pattern bears nearly the same proportion to the weight of the casting as the weight of a cubic inch of the wood bears to the weight of a cubic inch of cast iron. This is not always a convenient rule, because the pattern is a little larger than the casting, and the density of wood alters as it dries.

The *area of an irregular figure* may be obtained approximately by cutting it out of a uniform sheet of cardboard and weighing it. Now cut out a rectangle or square whose area it is easy to calculate. Weigh this also. The areas are in the same proportion as the weights.

The *area of cross section of a fine wire in square inches* can be determined with some accuracy by weighing a considerable length of the wire, dividing by the weight of the material per cubic inch, and dividing by the length of the wire in inches.

EXERCISES.

1. Find the number of revolutions per mile made by a rolling wheel $4\frac{1}{2}$ feet diameter. *Ans.*, 373.

2. Find the area and circumference of a circle of 4 inches radius. Find the circumference and diameter of a circle whose area is 20 square inches. *Ans.*, 50·27 square inches, 25·13 inches; 15·85 inches, 5·046 inches.

3. Find the area of a parallelogram whose adjacent sides are 50 and 30 feet and the angle between them 65°. *Ans.*, 1359·45 square feet.

4. Find the area of a sector of a circle, radius 4 inches, angle 50°. *Ans.*, 6·982 square inches.

5. Find the area of the segment of a circle, chord 20 inches, height 3 inches. If this were a parabolic segment, its area would be $\frac{2}{3}$rds of chord multiplied by height. *Ans.*, 40·6 square inches.

6. Ordinates of a curve, 1·5 inches apart, are 2·30, 2·35, 2·46, 2·57, 2·42, 2·21, 2·10. Find the area between the first and last by Simpson's rule. Test your answer by drawing the curve and using a planimeter. *Ans.*, 21·34.

7. Find the area of the surface of a sphere, its radius being 8 inches. *Ans.*, 804·2 square inches.

8. A boiler has 300 tubes 8 feet long, 3 inches diameter. What is the total cross-sectional area? What is the area of tube-heating surface? *Ans.*, 2,121 square inches, 1,885 square feet.

9. Right circular cone, base 3 inches radius, height 8 inches; find its curved surface and volume. *Ans.*, 80·52 square inches, 75·41 cubic inches.

10. Segment of sphere, height 4 inches, diameter of base 15 inches: find volume. *Ans.*, 309·9 cubic inches.

11. Find by Simpson's rule the number of cubic feet in a log of timber 36 feet long, the cross-sections at intervals of 6 feet being 8·20, 5·68, 4·04, 2·92, 2·16, 1·54, 1·02 square feet. *Ans.*, 124·36.

12. A railway cutting is $\frac{1}{2}$ mile long; the thirteen cross-sectional areas, measured at equal intervals of 220 feet, are respectively 280, 462, 594, 685, 757, 742, 500, 346, 320, 418, 512, 626, 560 square feet. Find the volume by Simpson's rule. *Ans.*, 157,439 cubic yards.

13. Two models of terrestrial globes; the areas of Africa are in the ratio 3 to 2. What is the ratio of the diameters of the globes? What is the ratio of their volumes? *Ans.*, 1·225, 1·837.

14. Find the surface and volume of a sphere of 2·361 inches radius. If it is of cast iron, what is its weight? Find the weight of a segment of this sphere 1 inch in height. *Ans.*, 70·1 square inches, 55·12 cubic inches; 14·44 lbs., 1·67 lb.

15. Find the volume and weight of the rim of a cast-iron wheel; section circular, outside and inside radii 20 feet and 18 feet 6 inches. *Ans.*, 213·7 cubic feet, 42·87 tons.

16. Find the circumference and area of a circle whose radius is 5 inches. This is the base of a cylinder of 11 inches height. What is its volume? Find its curved surface. If made of cast iron, what is its weight? What is the area of a section making 70° with the axis? (A × cos. 20° is the area of the circular base which is its projection.)

Ans., 31·42 inches, 78·54 square inches, 863·9 cubic inches, 2·4 square feet, 226·4 lbs., 83·58 square inches.

17. A cone on an elliptic base, whose principal diameters are 12 inches and 8 inches, is 20 inches in vertical height. What is its volume? What is its weight if made of cast iron? *Ans.*, 502·7 cubic inches, 131·7 lbs.

18. A plate of wrought iron $\frac{1}{16}$ inch thick weighs 0·6 lb. What is its area? *Ans.*, 34·3 square inches.

19. A disc of zinc 10 inches diameter, with a hole of 2 inches diameter in it, weighs 0·2 lb. Find its thickness. *Ans.*, 0·0106 inch.

20. 4 lbs. of copper is drawn into wire 0·14 inch diameter. Find its length. *Ans.*, 67·66 feet.

21. A piece of round copper wire 100 feet long weighs 5 lbs. What is its diameter? *Ans.*, ·128 inch.

22. A spherical shell 10 inches outside and 8 inches inside diameter, of cast iron: what is its weight? *Ans.*, 66·45 lbs.

23. A rolled girder of wrought iron 12 inches outside depth, flanges 6 inches by $\frac{1}{2}$ inch, web $\frac{1}{2}$ inch thick: what is its weight if 18 feet long? *Ans.*, 695·6 lbs.

24. Cylindric boiler 30 feet long, 7 feet diameter. Two cylindric flues each 2 feet 6 inches diameter: what is the area of the plates? Plate everywhere $\frac{1}{2}$ inch thick: what is its weight? When in this boiler water covers the flues and its level is a quarter of the diameter of the shell from the top, what are the volumes of water and steam? *Ans.*, 1,189 square feet, 10·7 tons, 2·8 to 1.

25. A circular plate of lead 2 inches thick, 8 inches diameter, is converted into spherical shot of the same density, each of ·075 inch radius. How many shot does it make? *Ans.*, 56,889.

26. Calculate the weight of one foot of each of the double flanged girders of the table at page 393, iron and steel.

27. Walking from the centre towards the end of one span of a lattice girder railway bridge, I count 10 wrought iron bars of rectangular section, each 14 feet long; the cross section of the first is 5 inches × $\frac{5}{8}$ inch. If the widths increase by $\frac{1}{2}$ inch and the thicknesses by $\frac{1}{16}$ inch, find the total weight of bars. *Ans.*, 1·08 ton.

7. The idea of **velocity** involves two things—the **direction** and the **speed**. When the direction does not alter, we speak of the speed as if it were the whole idea. Find the time in seconds taken by a body to traverse a certain distance measured in feet. This distance divided by the time is called the **average velocity**. Thus, if a railway train moves through 200 feet in 4 seconds, its average velocity during this time is 200 ÷ 4, or 50 feet per second. If we find, with careful measuring instruments, that it moves through 20 feet in ·4 second, or through 2 feet in ·04 second, the velocity is 20 ÷ ·4, or 2 ÷ ·04, or 50 feet per second. It is important to remember that the velocity may be always changing during an interval of time, however short. To get the **velocity at any instant,** we must make very exact measurements of the time taken to pass over a very short distance, and even this will only give us the average velocity during this short time. But if we make a number of measurements, using shorter and shorter periods of time, the average velocity becomes more and more nearly the velocity which we want. Thus, at 10 o'clock, **a man in a railway train** making a careful measurement finds that the train passes over 200 feet in the next 4 seconds. He finds the average speed for 4 seconds after 10 o'clock to be 200 ÷ 4, or 50 feet per second. Another man finds that it passes over 100·4 feet in the two seconds after 10 o'clock, and finds during these two seconds an average velocity of 100·4 ÷ 2, or 50·2 feet per second. Another man finds 50·25 feet passed over in

one second after 10 o'clock, which shows a velocity of 50·25 feet per second. Another man finds 25·132 feet passed over in half a second after 10 o'clock, and finds 25·132 ÷ 0·5, or 50·264 feet per second. Another man finds 12·567 feet in a quarter second after 10 o'clock, and his observation gives 50·268 feet per second, and so on. It is evident that the values given by these various observations are approaching the **real value** of the velocity at 10 o'clock.

Tabulating these results, we have :—

Intervals of Time in Seconds after 10 o'clock.	Average Velocity in Feet per Second.
4	50·00
2	50·20
1	50·25
½	50·264
¼	50·268

Plot the two sets of numbers on squared paper, and draw a curve through the points so found. Produce the curve, and we have the means of finding the average velocity for an infinitely small interval of time after 10 o'clock. This is the required velocity.

8. Acceleration.—This is the time rate of change of the velocity of a body. Thus it is known that the velocity of a body falling freely in London—

At the end of one second is 32·2 feet per second.
" " two seconds is 64·4 " "
" " three " 96·6 " "
" " four " 128·8 " "

and we see that there is an *increase* to the velocity of 32·2 every second. The acceleration in this case is always of the same amount—hence we call it *uniform* acceleration, and say it is 32·2 feet per second per second.

EXERCISES.

1. One English mile per hour; also one knot; convert each of these into feet per minute and feet per second. *Ans.*, 88, 1·467; 101·3, 1·689.

2. A torpedo-catcher travels at 32 knots; convert this into English miles per hour. *Ans.*, 36·85.

3. Prove that 60 miles per hour means ·0268 kilometres * per second.

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* *See tables on page 654.*

4. An eccentric disc is 10 inches diameter; the shaft makes 300 revolutions per minute. What is the rubbing velocity on the straps in feet per second? *Ans.*, 13·09.

5. In running a race of 1 mile long, A beats B by 100 yards, and B beats C by 90 yards. By how many does A beat C? *Ans.*, 185 yards, nearly.

6. Ten miles per hour; state this in feet per second and in centimetres per second. *Ans.*, $14\frac{2}{3}$; 447.

7. An acceleration of 32·2 feet per second per second; state this in miles per hour per second; state it in centimetres per second per second. *Ans.*, 21·96; 981·4.

8. 200 gallons of water per minute; how many pounds per second? How many cubic feet per second? *Ans.*, 33·3; ·535.

9. A round pipe 6 inches diameter has 30 gallons per second flowing through it. What is the velocity? If the diameter becomes 10 inches, what is the velocity?

Calculate in the two cases the kinetic energy of one pound of water, this being the square of the velocity divided by 64·4.

Ans., 24·5 feet per second; 8·8 feet per second; 9·3 foot-pounds; 1·2 foot-pound.

9. *Example.*—Two fine wires are 10 feet apart; a **bullet** breaks them both. The breaking of each wire causes an electric spark to make a mark underneath a fixed platinum pointer on a revolving drum. If the drum is 4 feet in diameter, and revolves at 300 revolutions per minute, and when the drum is at rest the spark-marks are found to be 1·32 feet asunder on the curved surface, assuming that the intervals of time between the breaking of the wires and making the marks were the same, find the time between the breaking of the wires, and find the velocity of the bullet. The surface velocity is $\frac{300 \times 4\pi}{60}$, or 62·83 feet per second; 1·32 divided by this gives ·02101 seconds; dividing this into 10 feet gives 476 feet per second as the velocity of the bullet.

Exercise.—In some **gun experiments** screens 150 feet apart were cut by a bullet at the following times (in seconds), counting from the time of cutting the first screen:—0, 0·0666, 0·1343, 0·2031, 0·2729, 0·3439, 0·4159. Find the average velocity between every two successive screens.

Ans., 2,252, 2,216, 2,180, 2,149, 2,113, and 2,083 feet per second.

10. Speed or velocity is a **rate**—the rate of increase of space with regard to time. If a body has passed through the space s at the time t, and if it goes over the additional space δs in the additional time of δt, then $\frac{\delta s}{\delta t}$ is the average velocity. Observe that δs is one symbol;

it does not mean a quantity called δ multiplied by a quantity called s. When we imagine δt to get smaller and smaller without limit, the average velocity becomes what we call the real velocity, and we indicate it by the symbol $\frac{ds}{dt}$. When, instead of space and time, we have other quantities, we generally use y for the dependent and x for the independent variable, and the symbol $\frac{dy}{dx}$ means "the **rate** of increase of y with regard to x." Thus, if y is the ordinate of a curve and x is the abscissa at the point P (Fig. 4), and $QR = \delta y$, $PR = \delta x$, then $\delta y/\delta x$ is the tangent of the angle QVX. As δx and δy become smaller and smaller, so that $\delta y/\delta x$ is to be written dy/dx, then dy/dx is evidently tangent of the angle which the tangent at P makes with the axis of x. We usually call this the **slope** of the curve at the point P.

If y is any quantity that depends upon another, x, and if we plot the values of x and y to scale on a sheet of **squared paper**, the curve shows by its slope everywhere the rate of increase of y with regard to x. If we know the algebraic law connecting y and x, we can find this rate by certain easy rules. Thus, if

$$y = \mathrm{A}x^n + \mathrm{B} \qquad (1),$$

$$u = \frac{dy}{dx} = n\,\mathrm{A}\,x^{n-1} \qquad (2),$$

whatever kind of number n may be.

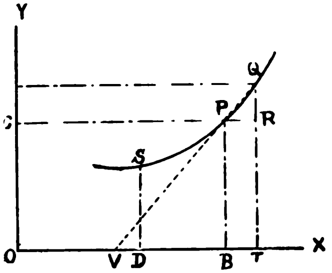

Fig. 4.

11. In teaching beginners it is well to start on the assumption that students already possess the notions of the differential and integral calculus, and it is a teacher's duty to put before them the symbols used in the calculus at once. It is surely much better to do this than to **evade the calculus** in the fifty usual methods which we sometimes see adopted. Unfortunately many readers of this book are likely to be preparing for examinations in which only academic methods of elementary study are recognised, so we must keep our calculus symbols for the smaller print paragraphs. We may say, however, that we think even beginners in this subject will be able to understand the author's book on the calculus; and if so, they will find the study of applied mechanics very greatly simplified. The language of the calculus is the natural, easy, simple language of the engineer; but it is in *writing*, whereas most engineers only *speak* of rates and integrals.

If y is known as a function of x, we can find $\frac{dy}{dx}$, which we may call u. Conversely, *if u is known* as a function of x, we can find y. The two symbols are $\frac{dy}{dx}$, called a rate, or "**the differential co-efficient** of y with regard to x": and $\int u \,.\, dx$, called "**the integral** of u with regard to x."

12. The following formulæ will suffice for nearly all engineering calculations:—

y	$\frac{dy}{dx}$	u	$\int u \,.\, dx$
C	0	C	C x
A x^n	n A x^{n-1}	B x^m	$\frac{B}{m+1} x^{m+1}$
		A x^{-1}	A log. x
A e^{ax}	a A e^{ax}	A e^{ax}	$\frac{1}{a}$ A e^{ax}
A log. $(x + a)$	$\frac{A}{x + a}$	$\frac{A}{x + a}$	A log. $(x + a)$
A sin. $(ax + b)$	a A cos. $(ax + b)$	A sin. $(ax + b)$	$-\frac{A}{a}$ cos. $(ax + b)$
A cos. $(ax + b)$	$-a$ A sin. $(ax + b)$	A cos. $(ax + b)$	$\frac{A}{a}$ sin. $(ax + b)$

and all the important part of the author's book on the calculus is devoted to illustrating the use of them.

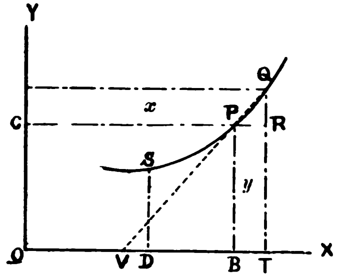

Fig. 5.

13. For example, if x is the abscissa O B, and y is the ordinate P B of the curve shown in Fig. 5; if the area between the curve and O X and P B, and any other ordinate nearer O, say D S, be called A; and if the area to Q T be called A + δ A, O T being called $x + \delta x$, the area δA = P Q T B, and as δx is made smaller and smaller this area gets more and more nearly $(y + \frac{1}{2}\delta y)\,\delta x$, as Q T is $y + \delta y$. Thus

$$\frac{\delta A}{\delta x} = y + \frac{1}{2}\,\delta y,$$

and as δx gets smaller and smaller this in the limit gets to be

$$\frac{dA}{dx} = y.$$

Hence, if we know y in terms of x, we can find A by integration. If we know A, we can find y by differentiation. Thus, if we have the parabolic curve,

$$y = ax^2 \quad \ldots . \ (3)$$

$$A = \frac{1}{3}\,ax^3 + C \ldots . \ (4),$$

a constant is added, because the rate of change of a constant is 0, and we wish our answer to be as general as possible. The value of C is fixed as soon as we settle from which ordinate A is to be calculated. Thus, if A is 0 when x is 0, C is 0. If $F(x)$ is the general integral of u,

$$\int_{x_1}^{x_2} u\,.\,dx$$

tells us to use x_2 for x in the general integral, then to use x_1 for x and subtract, or

$$\int_{x_1}^{x_2} u\,.\,dx = F(x_2) - F(x_1) \ \ldots . \ (5).$$

If u is the ordinate of a curve, (5) means the area between the curve, the axis of x, and the two ordinates at x_1 and x_2.

Any summation which may be indicated by an area can be effected in this way by integration. It is very easy to recollect the rule that the rate of change of x^n is nx^{n-1}, and that the integral of x^m is $\frac{1}{m+1}\,x^{m+1}$. This rule suffices for most engineering problems. There is one case in which the rule for integrating x^m is useless to us—namely, when m is -1; in this particular case

$$\int x^{-1}\,.\,dx \quad \text{is} \quad \log_e x.$$

14. Integrations.—In a great number of problems we have the ordinates like AP, called the y, of every point of a curve, given us, the abscissa OA of the point being called x.

If we are given a list of corresponding values of y and x, we can draw the curve.

Now, if AQ represents to scale the area of APRO, and if we find *many points like* Q

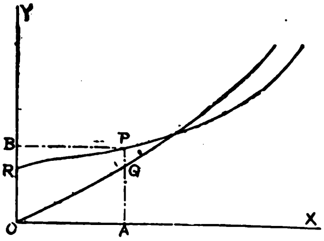

Fig. 6.

and join them, we get a new curve, which is said to represent the integral or area of the first. The areas may be found by means of a planimeter. I always use the following method myself. It is inaccurate to this extent, that the curve B P is taken to be a many-sided polygon instead of a curve. But it leads to a very quick solution of complicated-looking problems. Thus the following values of y are given for the corresponding values of x.

x	y	$\int ydx$
0		
0·5	1·869	
1		1·869
1·5	1·756	
2		3·625
2·5	1·657	
3		5·282
3·5	1·571	
4		6·853
4·5	1·497	
5		8·350
5·5	1·432	
6		9·782
6·5	1·375	
7		11·157
7·5	1·326	
8		12·483
8·5	1·247	
9		13·730

It will be seen that we assume the ordinate at any point like $x = 3·5$ to be the average height of the curve from $x = 3$ to $x = 4$. We always take the values of y at equal intervals in the values of x. If the above values of x had been 0, 0·05, 0·10, 0·15, etc., beginning with the first, we should have had to divide all the numbers in the last column by 10.

Exercise.—For values of $x = 0·5, 1·5, 2·5$, etc., calculate various values of y from $y = 2 - 3x - 0·5x^2$, and find the integrals. The true value of the integrals is $A = 2x - 1·5x^2 - \frac{1}{6}x^3$. Calculate this for $x = 0, 1, 2, 3$, etc., and note the errors in our numerical method.

15.—The symbol $\frac{d^2y}{dx^2}$ means the rate of change of $\frac{dy}{dx}$ with regard to x; $\frac{d^3y}{dx^3}$ means the rate of change of $\frac{d^2y}{dx^2}$ with regard to x.

It will be found that much of the difficulty which some students find in using the calculus is due to their not being able to differentiate t^n with regard to t, although they know how to differentiate x^n with regard to x—the mere use of another letter than the one to which they have been accustomed causing the difficulty. It is well, therefore, from the beginning to get used to many other letters, such as t, v, s, p, w, etc.

If s is space and t time, $\frac{ds}{dt}$ is velocity v, and $\frac{d^2s}{dt^2} = \frac{dv}{dt}$ is acceleration. Newton's symbols were: s for space, $\dot{s}$ for velocity, $\ddot{s}$ for acceleration.

Example.—If acceleration $\frac{d^2s}{dt^2} = a$, a constant; integrate with regard to t, and we have $\frac{ds}{dt} = \text{velocity} = at + b \ldots.$ (6), where b is the constant which we always add to make the answer general. We note the meaning of b to be the velocity when $t = 0$; and perhaps we had better use v_0 instead of b, so that $\frac{ds}{dt} = at + v_0$ Integrate again, and we have $s = \frac{1}{2}at^2 + v_0 t + c \ldots.$ (7). Evidently c means the value of s when $t = 0$. (6) and (7) are well-known laws of uniformly accelerated motion.

16. *Example.*—**A chain** of length x, hanging vertically downwards, is being lowered from a capstan. Its weight per foot of its length is w. When the weight wx is lowered through the distance δx, the work done is $wx\ \delta x$, and the whole work done by it from the time it is of no length, till it is of length l, is—

$$\int_0^l wx\, dx, \text{ or } \left[\tfrac{1}{2} wx^2\right]_0^l, \text{ or, } \tfrac{1}{2} wl^2,$$

or its total weight multiplied by half its vertical length.

Suppose the chain to get thicker as more of it is let down: say that $w = a + bx$, then the above summation becomes—

$$\int_0^l (a + bx)\, x\, dx = \int_0^l (ax + bx^2)\, dx = \left[\tfrac{1}{2} ax^2 + \tfrac{1}{3} bx^3\right]_0^l =$$

$\frac{1}{2}al^2 + \frac{1}{3}bl^3$. These two answers represent also the work done in lifting the chain.

17. *Example.*—**Fluid expanding** through the volume δv at the pressure p does work $p.\ \delta v$, and

$$\int_{v_1}^{v_2} p \, . \, \delta v$$

is the work done in expanding from volume v_1 to volume v_2. Thus, suppose that expanding fluid follows the law

$$pv^k = c \qquad \text{or} \qquad p = cv^{-k},$$

[illegible] with [illegible] the answer is

[illegible]

[illegible]

[illegible]

28. The Compound Interest Law.—[illegible]

[illegible]

29. The Harmonic Law.—[illegible]

[illegible]

Thus if $y = A \sin. (ax + b)$,

$$\frac{dy}{dx} = Aa \cos. (ax + b),$$

$$\frac{d^2y}{dx^2} = -Aa^2 \sin. (ax + b) = -a^2y.$$

With this knowledge a student can put in a few words all the theory of Chapter XXV. He will see that **simple harmonic motion** is stated by $x = a \sin. (qt + e)$ . . . . (1), where a is the **amplitude**, q is $2\pi f$ or $\frac{2\pi}{T}$, where f is the **frequency** or T is the **periodic time**, and e is the *epoch*, a quantity introduced because x is not 0 when $t = 0$; that is, we give generality to it by assuming that we may begin to count time from any position of the body. Observe that the motion may be angular, x and a being angles. $\frac{dx}{dt}$ will then be angular velocity, and $\frac{d^2x}{dt^2}$ will be angular acceleration. Differentiating twice we get

$$\text{velocity} = \frac{dx}{dt} = aq \cos. (qt + e) \;.\;.\;.\;. (2).$$

$$\text{acceleration} = \frac{d^2x}{dt^2} = -aq^2 \sin. (qt + e) \;.\;.\;.\;. (3).$$

Notice the sign of $\frac{d^2x}{dt^2}$, and see how it agrees with the statements of Chapter XXV. If we only think of the numerical value of $\frac{d^2x}{dt^2}$

we notice at once that $x \div \frac{d^2x}{dt^2} = \frac{1}{q^2}$ .... (4), and the square root of this is $\tau/2\pi$, which is the rule found in another way in Chapter XXV.

20. Students will do well to **graph** on squared paper some curves like the following :—

1. If $y = 0{\cdot}1\,x^2 + 3$, take $x = 0$, $x = 1$, $x = 2$, etc., and in each case calculate y. Plot the values of x and y as co-ordinates of points on squared paper, and draw the curve passing through these points.

2. Graph $y = a + bx$.
   1st, when $a = 0$, $b = 1$.
   2nd, when $a = 1$, $b = 1$.
   3rd, when $a = 1$, $b = 1{\cdot}5$.
   4th, when $a = -1$, $b = 1{\cdot}5$.
   5th, when $a = 1$, $b = -1$.

3. Graph $y = {\cdot}1\,e^x$.

4. Graph $y = 10 \sin.\left(\frac{7}{44}x + \frac{\pi}{3}\right)$.

5. Graph $y = 120\,x^{-1}$.

6. Graph $y = 120\,x^{-1{\cdot}3}$.

7. Graph $y = 120\,x^{-0{\cdot}7}$.

8. Graph $y = 10\,x^{\frac{1}{2}}$.

9. Graph $y = \pm\sqrt{25 - x^2}$,
   and also $y = \pm 1{\cdot}3\sqrt{25 - x^2}$.

21. The following observed numbers are known to follow a law like $y = a + bx$; but there are errors of observation. Find by the use of squared paper the most probable values of a and b.

x	2	3	$4\frac{1}{2}$	6	7	9	12	13
y	5·6	6·85	9·27	11·65	12·75	16·32	20·25	22·33

Ans., $y = 2{\cdot}5 + 1{\cdot}5\,x$.

22. The following observed numbers are known to follow a law like $y = ax\,/\,(1 + sx)$:—

Find by plotting the values of y/x and y on squared paper that these follow a law $y/x + sy = a$, and so find a and s.

x	·5	1	2	·3	1·4	2·5
y	·78	·97	1·22	·55	1·1	1·24

Ans., $y = 3x\,/\,(1 + 2x)$.

23. Vertical Line.—A line showing the direction in which that force which we call the resultant force of gravity acts. It is a line at right angles to the surface of still water or mercury.

24. Level Surface.—A surface like that of a still lake, everywhere at the same level, and everywhere at right angles to the force of gravity or other resultant force which is acting upon matter. It is not a plane surface.

25. Curvature.—For any curve we can find at any place what circle will best coincide with the curve just there. The radius of this circle is called the radius of curvature at the place. But since we say, for instance, that a railway line curves much when we mean that the radius is small, the name curvature is always given to the reciprocal of the radius. Thus, if the radius is 8 feet, we say that the curvature is $\frac{1}{8}$. If at another place the curvature is [illegible], the change of curvature in going from the one place to the other is the difference between these two fractions.

Curvature may also be defined as the angle turned through by a tangent to the curve per unit of its length. A student ought to see for himself that the two definitions are the same. If in Fig. 5 the distance along the curve between P and Q is called δs, and if $\delta\theta$ is the angle which the tangent at Q makes with the tangent at P, then the average curvature between P and Q is $\delta\theta/\delta s$. As P and Q become closer and closer, without limit, the curvature is $\frac{d\theta}{ds}$.

If a curve is defined by its x and y co-ordinates, the curvature is $\frac{d^2y}{dx^2} \div \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$.

Example.—Find the curvature, where $x=0$, of the parabola $y=ax^2$. Here $\frac{dy}{dx}=2ax$ and $\frac{d^2y}{dx^2}=2a$, so that the curvature anywhere is $2a \div \left\{1+4a^2x^2\right\}^{\frac{3}{2}}$, and at the vertex where $x=0$ the curvature is $2a$.

When, as in ordinary beams, $\frac{dy}{dx}$ is small, it is evident that the curvature may be taken to be $\frac{d^2y}{dx^2}$. This is what we take to be true in the discussion of **beams and struts.**

Example.—In making 100 steps round a curve, my compass, showing the direction of motion, changes from N. to N.E. What is the average curvature? Answer—from N. to N.E. is 45 degrees, or 0·7854 radians, and this divided by 100 steps, or ·007854 radians per step, is the average curvature. The reciprocal of this, or 127·3 steps, is the radius of curvature, if the curvature is constant—that is, if the curve is an arc of a circle.

Many **unpractical rules** will be found in books, requiring us to draw a tangent to a given curve at a given point, or to find its curvature there by trial. These are only **academic** suggestions. If half a dozen students get tracings of the same curve, and two points to measure the angle between the tangents there, they will obtain six very different answers.

EXERCISES.

1. Through what angle must a rail 10 feet long be bent to fit a curve of half a mile radius? *Ans.*, 0·22 degrees.

2. Arc s is 10 feet long, radius r half a mile; find the versed sine x—that is, the greatest deflection from straightness. Prove that, practically, $s^2 = 8\,rx$; so that in this case $x = \cdot 0047$ feet.

3. Find the radius of curvature of the catenary $y = \frac{a}{2}\left\{ e^{\frac{x}{a}} + e^{\frac{-x}{a}} \right\}$ at the vertex.

Ans., a.

4. Find the radius of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (1) At the end of the major axis; (2) at the end of the minor axis.

Ans., (1) $\frac{b^2}{a}$; (2) $\frac{a^2}{b}$.

26. Angle.—An angle can be drawn: First, if we know its magnitude in **degrees**; a right angle has 90 degrees. Second, if we know its magnitude in **radians**; a right angle contains 1·5708 radians. Two right angles contain 3·1416 radians. One radian is equal to 57·2958 degrees. One radian has an arc, B C, equal in length to the radius A B or A C. It sometimes gets the clumsy name "a unit of circular measure." Third, we can draw an angle if we know either its *sine cosine*, or *tangent*, etc. Draw any angle, B A C (Fig. 7). Take any point, P in A B, and draw P Q at right angles to A C. Then measure P Q, A P and A Q in inches and decimals of an inch. P Q ÷ A P is called the **sine** of the angle. A Q ÷ A P is called the **cosine** of the angle. P Q ÷ A Q is called the **tangent** of the angle. Calculate each of these for any angle we may draw, and measure with a protractor the number of degrees in

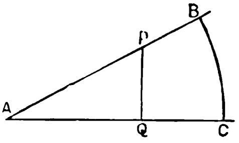

Fig. 7.

the angle. We shall find from a book of mathematical tables whether our three answers are exactly the sine, cosine, and tangent of the angle. This exercise will impress on our memory the meaning of the three terms. It will also impress upon us the fact that if we know the angle in degrees, we can find, by means of a book of tables, its sine, or cosine, or tangent; and if we know any one of the sides A P, O P, or O A, of the right-angled triangle A O P and the angle A, we can find the other sides.

Divide the number of degrees in an angle by 57·2958, and we find the number of radians. Suppose we know the number of radians in the angle B A C and we know the radius A B or A C, then the arc B C is

A B × number of radians in the angle.

Given, then, a radius to find the arc, or given an arc to find the radius, are very easy problems.

A student becomes accustomed in seeing an angle drawn on paper, to judge from a mere glance how many degrees the angle contains. It would be an advantage to acquire the habit of judging how many radians there are in the angle. What we mean is, that he ought to be as ready to *think* in *radians* as in *degrees*, and to do this he requires to be familiar with the size of a radian.

EXERCISES.

1. Draw an angle of 35°. Find by measurement the sine, cosine, *tangent*, *cotangent*, secant, and cosecant of the angle, and compare with the numbers in a book of tables. Calculate the number of radians. Try if $\sin.^2 35° + \cos.^2 35° = 1$: if $\sin. 35° \div \cos. 35° = \tan. 35°$; if $\tan.^2 35° + 1 = \sec.^2 35°$; and if $\cot.^2 35° + 1 = \operatorname{cosec.}^2 35°$.

2. If $\alpha = 55°$, $\beta = 20°$, illustrate the following important formulæ by numerical calculation:—

$$\text{Sin.}\ (\alpha + \beta) = \sin.\ \alpha \cos.\ \beta + \cos.\ \alpha \sin.\ \beta.$$
$$\text{Sin.}\ (\alpha - \beta) = \sin.\ \alpha \cos.\ \beta - \cos.\ \alpha \sin.\ \beta.$$
$$\text{Cos.}\ (\alpha + \beta) = \cos.\ \alpha \cos.\ \beta - \sin.\ \alpha \sin.\ \beta.$$
$$\text{Cos.}\ (\alpha - \beta) = \cos.\ \alpha \cos.\ \beta + \sin.\ \alpha \sin.\ \beta.$$
$$\text{Sin.}\ \alpha \cos.\ \beta = \frac{1}{2}\left\{\sin.\ (\alpha + \beta) + \sin.\ (\alpha - \beta)\right\}.$$
$$\text{Cos.}\ \alpha \cos.\ \beta = \frac{1}{2}\left\{\cos.\ (\alpha + \beta) + \cos.\ (\alpha - \beta)\right\}.$$
$$\text{Sin.}\ \alpha \sin.\ \beta = \frac{1}{2}\left\{\cos.\ (\alpha - \beta) - \cos.\ (\alpha + \beta)\right\}.$$
$$\textit{Cos.}\ 2\alpha = 2 \cos.^2\alpha - 1 = 1 - 2 \sin.^2\alpha.$$

3. What are sin. 150°, cos. 130°, tan. 170°, cos. 240°, sin. 220°, tan. 218°, sin. 290°, cos. 310°, tan. 320°? Express all these angles in radians.

Ans., ·5, − ·6428, − ·1763, − ·5, − ·6428, ·7813, − ·9397, ·6428, − ·8391; 2·6180, 2·2689, 2·9671, 4·1888, 3·8397, 3·8048, 5·0614, 5·4105, 5·5851.

4. The sine of an angle is 0·25; find its cosine, tangent, cotangent, secant, and cosecant. Find the angle by actual drawing. How many radians? *Ans.*, ·9683, ·2582, 3·875, 1·033, 4; 14°·5; 2528.

5. What are the sine, tangent, and radians of $1\frac{1}{2}$ degrees? *Ans.*, Each ·0262.

6. If in Fig. 7 A is 47°, and AP 5·23 feet, find AQ and PQ. *Ans.*, AQ = 3·567, PQ = 3·824.

27. Angular Velocity.—If a wheel makes 90 turns per minute, this means that it makes 1·5 turns per second. But in making one turn any radial line moves through the angle of 360 degrees, which is 6·2832 radians; so that 1·5 turns per second means 6·2832 × 1·5, or 9·4248 radians per second. This is the common **scientific way** in which the *angular velocity* of a wheel is measured—so many radians per second. If a wheel makes 30 turns per minute, its angular velocity is 3·1416 radians per second; n turns per minute mean $2\pi n$ radians per minute, or $2\pi n \div 60$ radians per second. One turn is the angular space traversed in one revolution.

Exercise.—Show that the linear speed in feet per second of a point in a wheel is equal to the angular velocity of the wheel multiplied by the distance in feet of the point from the axis.

28. Angular Acceleration. — The increase of angular velocity per second. If a wheel starts from rest, and has an angular velocity of 1 radian per second at the end of the first second, its average angular acceleration during this time is 1 radian per second per second.

EXERCISES.

1. A shaft revolves at 800 revolutions per minute. What is its angular velocity in radians per second? *Ans.*, 83·79.

2. A point is 3,000 miles from the earth's axis, and revolves once in 23 hours 56 minutes 4 seconds. What is its velocity in miles per hour? *Ans.*, 787·5.

3. The average radius of the rim of a fly-wheel is 10 feet. When the wheel makes 150 revolutions per minute what is the average velocity of the rim? *Ans.*, 157·1 feet per second.

4. An acceleration of 1 turn per minute every second; how much is this in radians per second per second? *Ans.*, ·1047.

5. A wheel is revolving at the rate of 90 turns a minute. What is its angular velocity?

A point on the wheel is 6 feet from the axis; what is its linear speed? If its distance from the centre be increased by 50 per cent., what does its speed become? If at the same time the speed of the wheel increases 50 per cent., what is now the linear speed of the point?

Ans., **9·425 radians per second; 56·55 feet per second; 84·82 feet per second; 127·2 feet per second.**

6. There is a lever, A O, 30 inches long, works about an axis at O. The lever is made to turn by applying a force at a point B in A O, 15 inches from O, so that B receives a velocity of 2 feet per second. What is the angular velocity of the lever?

If the same velocity had been given to the point A instead of B, what would the angular velocity have been?

Ans., **1·6 radians per second; 0·8 radian per second.**

29. Length of Belt.—Let D and d be the diameters of pulleys, c the distance between their centres, L the length of belt, let $D + d$ be called S. Prove that for a **crossed** belt

$$L = \left(\frac{\pi}{2} + \theta\right) S + 2c \cos.\, \theta \;\ldots\; (1),$$

where $\sin.\, \theta = S \div 2c$.

Example.—Find the length of a crossed belt for pulleys of 20 and 15 inches diameter, the distance of their centres apart being 120 inches. First find θ, if $\sin.\, \theta = 35 \div 240 = {\cdot}1458$. So that $\theta = 8^\circ\, 23'$, or ·1463 radians. Also $\cos.\, \theta = {\cdot}9893$. Hence we have

$$L = (1{\cdot}5708 + {\cdot}1463)\, 35 + 240 \times {\cdot}9893 = 297{\cdot}5 \text{ inches.}$$

Notice from the formula (1) that as L depends only on c and S, if c and S are the same, the same length of belt will do. Thus the same crossed belt does for any two corresponding steps in **stepped cones** if $D + d$ is the same. It is exceedingly easy to calculate the sizes of these steps when we know the speeds. Thus the above-mentioned pulleys were 20 and 15 inches, and their speeds were as 3 to 4. If there were two steps, and we want another pair to give a speed ratio, say, of 1 to 2 using the same belt, we know that if their diameters are D and d

$$D + d = 35, \qquad D = 2d.$$

$$\text{Hence,}\quad 3\, d = 35 \text{ and } d = 11\tfrac{2}{3},\ D = 23\tfrac{1}{3}.$$

Exercise.—The centres of two pulleys, $3\frac{1}{2}$ and $1\frac{1}{2}$ feet in diameter respectively, are 10 feet apart. Find the length of crossed belt required. *Ans.*, 28·48 feet.

If the belt is an **open** belt, the length L is easy to obtain. It is

$$L = \frac{\pi}{2}(D + d) + \theta\,(D - d) + 2c \cos.\, \theta,$$

where $\sin.\, \theta = (D - d) \div 2c$. It is easy to show that the following answer, which is much easier to deal with later, is practically correct:

$$L = \frac{\pi}{2}(D + d) + 2c + \frac{1}{4}\frac{(D - d)^2}{c} \;\ldots\; (2).$$

The length of belt depends upon $D + d$, c, and $D - d$.

Exercise.—Pulleys of 20 and 15 inches diameter, whose axes are 120 inches apart, are connected by an open belt. Find its length.

$$\textit{Ans.}, \ \text{L} = \frac{\pi}{2}(35) + 240\left(1 + \frac{1}{8}\frac{25}{(120)^2}\right) = 295{\cdot}03''.$$

The student ought to find the correct answer, and see how small is the error in the use of our approximate formula.

Suppose we have a pair of pulleys D_1 and d_1, and we want another pair D_2 and d_2 on the same shafts to work with the same length of belt, the ratio of D_2 to d_2 being known. Putting the two expressions for L equal, we have

$$\text{D}_2 + d_2 + \frac{1}{2\pi c}(\text{D}_2 - d_2)^2 = \text{D}_1 + d_1 + \frac{1}{2\pi c}(\text{D}_1 - d_1)^2 \ \ldots\ldots (3).$$

The right-hand side is known, and we know D_2 in terms of d_2, so that it is easy to calculate.

Example.—In the above example of $\text{D}_1 = 20$, $d_1 = 15$, $c = 120$, let us calculate D_2 and d_2, if $\text{D}_2 = 2\,d_2$. Our equation (3) becomes

$$35 + \frac{1}{2\pi \times 120}(5)^2 = 3\,d_2 + \frac{1}{2\pi \times 120}(d_2)^2.$$

The solution of this quadratic gives us $d_2 = 33$; and, therefore, $\text{D}_2 = 66$. In practice we usually calculate D_2 and d_2 as if the belt were crossed. These are nearly the right answers. Then, taking the $\text{D}_2 - d_2$ so found, we find from (3) a corrected value of $\text{D}_2 + d_2$, and we use this, knowing the ratio of D_2 to d_2, to find them more correctly. We may, if we please, approximate more nearly by repeating this process, but this is seldom necessary.

Example.—On the driving shaft, going at 100 revolutions per minute, the diameter of the first step is 20 inches. From this step the driven shaft, which is 120 inches away, is to go at 200 revolutions. From other steps the speed of the driven shaft is to be 150, 100, 75, and 50 revolutions. Find the sizes of the steps if the belt is crossed.

Here $\text{D}_1 = 20$, $d_1 = 10$, $\text{D}_2 + d_2 = 30$, $\text{D}_2 = \frac{3}{2}\,d_2$. Hence $\frac{3}{2}\,d_2 + d_2 = 30$, $2\frac{1}{2}\,d_2 = 30$, $d_2 = 12$ inches, $\text{D}_2 = 18$ inches.

In the same way we find $d_3 = 15$, $\text{D}_3 = 15$; $d_4 = 17\frac{1}{7}$, $\text{D}_4 = 12\frac{6}{7}$; $d_5 = 20$, $\text{D}_5 = 10$.

30. If a line, A B, makes an angle θ with the horizontal, the **projection of its length** on the horizontal is A B cos. θ. Its projection on a vertical line is A B sin. θ.

Exercise.—Draw two lines O X, O Y at right angles to each other. Now draw lines O P, O Q, O R, O S, of lengths 3, 5, $2\frac{1}{2}$, and 4 inches, and making angles of 35°, 72°, 130°, and 220° with O X. Find the projection of each line on O X and on O Y, and the sum of the projections on O X and on O Y.

Ans., 2·457; 1·545; −1·607; −3·064; 1·721; 4·755; 1·915; −2·571; −0·669; 5·820.

If a plane area of A square inches is inclined at an angle θ with the horizontal, its area, as **projected on the horizontal**, is A cos. θ square inches.

Try to prove that this must be so by dividing the area into strips by horizontal lines.

EXERCISES.

1. A plane area of 35 square inches is inclined at 20° to the horizontal; find its horizontal and vertical projections.
Ans., 32·89 square feet; 11·97 square feet.

2. The cross-section (a cross-section always means a section by a plane at right angles to the axis or line of centres of sections) of a cylinder is a circle of 0·7 inch radius. Find the areas of sections which make angles of 25° and 45° with the cross-section. Note that the cross-section is a projection of any other section.
Ans., 1·699, 2·177 square inches.

3. The above cylinder is a tie bar of wrought iron. The total tensile load is 12,000 lb.; how much is this per square inch of the cross-section? How much is it per square inch of either of the other sections?
Ans., 7794 lbs., 7063 lbs., 5512 lbs.

4 The cross-section of a pipe is a circle of 15 inches diameter; what is the area in square feet? If 13 gallons flow per second, what is the velocity v_0? What is the area of a section at 28° to the cross-section? What is the velocity v, *normal* to this section, if normal velocity × area = cubic feet per second? Show that v is the resolved part of v_0 normal to the section. *Ans.*, 1·228, 1·7 feet per second; 1·39, 1·5 feet per second.

5. Part of a roof, shown in plan as 4,000 square feet, is inclined at 24° to the horizontal; what is its area? *Ans.*, 4378·7 square feet.

6. A tie bar or short strut of 2 square inches cross-section; what is the area of a section making 45° with the cross-section? If the total tensile or compressive load is 20,000 lbs, how much is this per square inch on each of the sections? Resolve the total load normal to and tangential to the oblique section, and find how much it is per square inch each way. *Ans.*, 2·828 square inches; 10,000 lbs., 7,070 lbs., 5,000 lbs.

CHAPTER II.

VECTORS. RELATIVE MOTION.

31. ANY quantity which is **directive** is called a **vector** quantity—for example, a velocity or a force. It can be **represented by a line.** Its *amount* can be represented to some scale by the *length* of the line. The clinure of the line and an arrow-head represent the **clinure** and **sense** of the vector. Vector quantities are distinguished from **scalar** quantities, such as a sum of money, the mass of a body, energy, etc.

The **resolved part** of a vector in any new direction is represented by the projection of its representative length in the new direction. Thus in Fig. 8, if O P represents to scale a velocity or a force, its resolved part in the direction O X is O A, the amount of which is O P cos. A O P, and its resolved part in the direction O Y is O B, the amount of which is O P cos. B O P. Thus, if a ship is going at 9 knots north-eastward, the northerly component of its velocity is 9 cos. 45°, or 6·363 knots, and its easterly component is 6·363 knots.

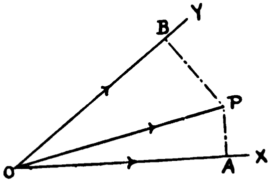

Fig. 8.

If a body has an acceleration of 20 feet per second per second in the direction 25° east of north, the northerly component of this is 20 cos. 25°, or 18·13 feet per second per second; and the easterly component is 20 sin. 25°, or 8·452 feet per second per second.

If a force of 30 lbs. is in a northerly direction, its component in a north-easterly direction is 30 cos. 45°, or 21·21 lbs.

32. The **resultant** of **two or more forces** is a force which might be substituted for them without changing the effect. If two strings pull a small body with forces of 5 lbs. and 7 lbs. (Fig. 9), and if the angle between them is 30°, draw O P equal in length to 5 inches, and make the angle Q O P equal to 30°. Make the length of O Q 7 inches. Complete the parallelogram Q O P R, and draw the diagonal O R. Measure O R in inches; we find it to be 11·6 inches, so that the resultant of the two forces

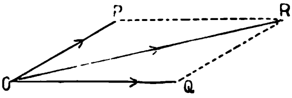

Fig. 9.

is 11·6 lbs. One string acting in the direction OR with a pull of 11·6 lbs. will produce the same effect at O as the two strings did. In using this construction, take care that the arrow-heads are **confluent**—that is, that they all point away from O, or they all point towards O. Suppose when the two strings were acting we had found by experiment that a third string OE (Fig. 10) would just prevent the two strings from causing motion at O, then experiment would also show that the force in OE, which may be called the **equilibrant** of OP and OQ, is exactly equal and opposite to the resultant of OP and OQ.

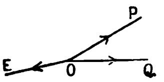

Fig. 10.

We see that when OQ and OP are given, and the angle between them, we may use the above principle, called the **parallelogram of forces**; or, what comes to the same thing, the **triangle of forces**. Draw OQ one of the forces, draw QR the other, and let their arrow-heads be **circuital**; then the non-circuital force OR is the resultant; or a circuital force RO would be the equilibrant. It is in this way that we find the resultant or vector sum of any two vectors.

In **vector language**, $OQ + QR = OR$, or $OQ = OR - QR$. This principle is very easy to express; to be able to apply it implies a considerable experience. We mention it now merely to introduce a few exercises.

It is very easy to solve problems graphically. The student must work some numerical exercises, and test his answers graphically.

Example.—Two forces, OP and OQ, of 5 lbs. and 7 lbs. respectively, act a point, at an angle of 56°; find their resultant by calculation.

In Fig. 11 OP and OQ represent, to scale, the two forces, the angle POQ being 56°. Find the resolved part of OP (Art. 31) in the direction OX, say; it is $OA = OP \cos. 56° = 5 \times \cdot5592 = 2\cdot796$. Now obtain the resolved part of OP in the direction OY, taken at right angles to OX; it is $OB = OP . \cos. POB = 5 \sin. 56° = 5 \times \cdot829 = 4\cdot145$. Instead of the given forces we now have OA and OQ acting along OX, and OB along OY. Draw OH (Fig. 11) equal to $OA + OQ$, that is, 9·796, and $OV = 4\cdot145$. The resultant of these is OR, and we have

$$OR^2 = OH^2 + OV^2 = (9\cdot796)^2 + (4\cdot145)^2 = 24\cdot996;$$

$$\therefore OR = \text{[illegible]} \text{ very nearly.}$$

To obtain the direction of O R, we have

$$\tan. \alpha = \frac{\text{R H}}{\text{O H}} = \frac{4 \cdot 145}{9 \cdot 796} = \cdot 4231\ ;$$

$$\therefore \alpha = 23^\circ \text{ nearly.}$$

The student should test this result by finding O R **graphically** by the method explained above. He should also observe that if the angle P O Q had been, say, 130°, O A would have been directly opposed to O Q, in which case O H would have been obtained by subtracting O A from O Q. It is of the **utmost importance** that he should **work many exercises** similar to this one, so as to become familiar with the method.

It will be shown later (Art. 94) that the same method is employed for calculating the resultant of any number of forces. In every case we resolve the forces along any line O X, and let O H represent their resultant; then resolve along O Y taken at right angles to O X, and let O V represent the resultant of these resolved parts. O R and α are then easily calculated.

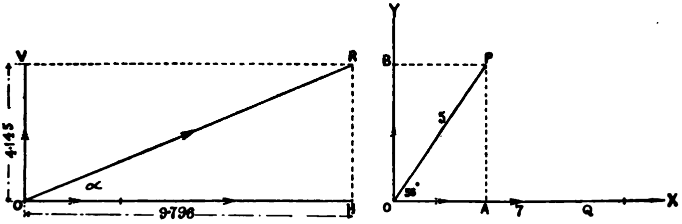

Fig. 11.

EXERCISES.

1. Force of 20 lbs. at an angle of 72° with the horizontal; what are its horizontal and vertical components? *Ans.*, 6·18 lbs., 19·02 lbs.

2. A man walks towards the north-north-east at 4 miles per hour; at what rate is he getting towards the east? And at what rate towards the north? *Ans.*, 1·53 miles an hour; 3·69 miles an hour.

3. Forces O P of 10 lbs. and O Q of 7 lbs. at an angle Q O P equal to 35°; find the resultant. Test your answer by working the problem graphically. *Ans.*, 16·24 lbs. inclined 14° 18′ with the force 10 lbs.

4. On a horizontal surface there is a normal pressure of 4 tons per square inch and a tangential force of 3 tons per square inch from N.E. to S.W. What is the total force per square inch?

Ans., An oblique pressure of 5 tons per square inch, making an angle $\tan.^{-1} 1\frac{1}{3}$ with the N.E. to S.W. *direction* in a vertical plane.

5. An anvil is carried by three ropes, which make angles 20°, 30°, 25° with the vertical: the tensions in the ropes are known to be 1,000 lbs., 700 lbs., and 1,200 lbs. What is the weight of the anvil? If the horizontal components of these three forces are drawn as balancing one another, find the azimuthal angles which the vertical planes through the ropes make with each other—that is, find the angles in the *plan* of the ropes.
Ans., 2,633 lbs.; 85° 47′, 137° 43′, 136° 30′.

33. If the magnitude of O P (Fig. 11) is called P, of O Q is Q, of O R is R, and if the angle Q O P is θ, it can be proved, that

$$R = \sqrt{P^2 + Q^2 + 2\,P\,Q\cos.\,\theta}.$$

It will be seen from Fig. 11 that

$$\tan.\,\alpha = \frac{RH}{OH} = \frac{OB}{OA + OQ} = \frac{P\sin.\,\theta}{P.\cos.\,\theta + Q}.$$

If θ is a right angle, P and Q are the resolved parts of R in their two directions—

$$R = \sqrt{P^2 + Q^2}, \text{ and } \tan.\,ROQ = P/Q.$$

EXERCISES.

1. $P = 3$, $Q = 4$, $\theta = 90°$. Then $R = 5$ and α is an angle whose tangent is 0·75. [The neat way of making this statement is to write $\alpha = \tan.^{-1}\,0{\cdot}75$.]

2. $P = 3{\cdot}045$, $Q = 7{\cdot}462$, $\theta = 37°$: find R, α. *Ans.*, $R = 10{\cdot}06$. $\alpha = 10°30'$.

3. $P = 3{\cdot}045$, $Q = 7{\cdot}462$, $\theta = 143°$; find R, α. *Ans.*, $R = 5{\cdot}353$. $\alpha = 20°$.

4. $P = 12{\cdot}06$, $Q = 1{\cdot}002$, $\theta = 184°$; find R, α. *Ans.*, $R = 13{\cdot}05$. $\alpha = 187°\ 42'$.

34. A river flows at 1 mile per hour; a swimmer has a velocity of 2 miles per hour **relatively to the water.** What is his velocity relatively to the bank? This will depend upon the direction in which he swims.

Make A B represent 1 mile per hour down the river (as some students cannot get out of their heads the wrong notion that these lines represent distance, imagine the drawing to be infinitely small but greatly magnified merely that it may be examined); make B C represent the velocity of swimming to scale, the direction and *sense* being correct. Take care that the arrow-heads are circuital. Then A C is the **sum of the two velocities** possessed by the swimmer, and is therefore his velocity *relatively to the bank.* Let the student draw B C in all sorts of directions and reflect upon his answers. If he has ever swam in a broad river or has watched a swimming dog trying to reach his master, he will understand his answers more readily.

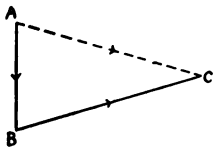

Fig. 12.

EXERCISES.

1. Given the above velocities and that the stream flows due south, if the absolute motion of the swimmer is to be south-east, in what direction ought he to swim? *Ans.*, 16° 20′ S. of E.

2. A steamer moves westward at 10 feet per second: a boy throws a ball across the deck northwards at 4 feet per second. What is the velocity of the ball relatively to the water? *Ans.*, 10·77 feet per second, 21° 48′ N. of W.

3. A steamer has a velocity of 14 knots due west; the wind blows with a velocity of 7 knots from the north. What will be the apparent velocity of the wind to a person on board the steamer? *Ans.*, 15·7 knots from W. 26° N.

4. Velocity of a ship westwards 10 feet per second; velocity of a ball on deck 5 feet per second north-east relatively to the ship. What is the total velocity of the ball? *Ans.*, 7·37 feet per second, 28° 40′ N. of W.

5. If the total velocity of the ball is 12 feet per second northwards, what is its velocity relatively to the ship? *Ans.*, 15·62 feet per second, 50° 12′ N. of E.

6. A railway train is going at 30 feet per second; how must a man throw a stone from the window so that it shall leave the train laterally at 1 foot per second, but have no velocity in the direction of the train's motion?

Ans., 30·03 feet per second; at an angle of 178° 6′ with direction of motion of train.

35. A **bicyclist** is ordered to travel so that he shall be more to the north at the rate of 3 miles every hour, and he must keep to roads. Notice that if he is on a north and south road his task is very easy. If his road is directed N.W., he must travel at 4·242 miles per hour. If his road is W.N.W., he must travel at 7·839 miles per hour. If his road is due west, his task is an impossible one. If his road makes an angle θ with due north, he must travel at the rate of $3 \div \cos. \theta$ miles per hour.

36. Water flowing from the inner to the outer part of a motionless wheel of a **centrifugal pump** is guided by vanes to follow a curved path. Suppose its radial velocity v_r known; show that its real velocity anywhere is $v_r \div \cos. \theta$, if θ is the angle which the vane there makes with the radial direction.

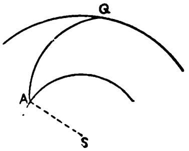

Fig. 13.

A student will find this an excellent **graphical exercise.** Draw circles of 1 and 2 feet radii to represent the inner and outer cylindric surfaces of the wheel of a pump. Draw any shape of vane connecting these, but you had better take a shape from an actual wheel (see Art. 427). Let the angle at A with the tangent to the circle there (Fig. 13) be 18½°. Now imagine a particle of water travelling out radially at 0·1 foot per second. Imagine it to take 10 seconds to get to Q. Mark its successive positions along the vane. You had better also trace its positions backward for a few seconds towards A from S. Now *imagine the wheel to revolve* about its centre so that A moves at 0·3 foot per second. Map out the real positions of the points which you found on the vane, and therefore the *real* path of a particle of water. Try to follow it after it has left Q, assuming plenty of space outside; but this is a problem which you perhaps may return to later.

37. Let P (Fig. 14) be a point on the rim of the wheel of a centrifugal pump. Suppose that we know the velocity of P, and represent it by the distance P B; also that we know the velocity of the water along the vane of the wheel at P, and represent it by the distance A P. Now, a particle of water at P has both these velocities; A P relatively to the wheel, together with P B because the wheel is in motion. Hence the total velocity of the particle of water is represented in direction, amount, and sense by the *vector sum* A P + P B, and we can either use the parallelogram method or the triangle method to find it.

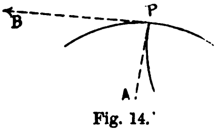

Fig. 14.

Exercise.—The rim of the wheel of a centrifugal pump goes at 30 feet per second; water flows radially at 5 feet per second; the vanes are inclined backward at an angle of 35° to the rim. What is the absolute velocity of the water? What is the component of this parallel to the rim? *Ans.*, 23·4 feet per second; 22·8 feet per second.

When we know the velocity of water before it enters a wheel and the velocity of the wheel at the place, and we wish the water to enter **without shock**, this simply means that the total velocity of the water the instant after it enters the wheel shall be exactly the same as before it entered. Thus in Fig. 15, if the velocity, C P, of water is known before it enters the turbine wheel, and P B represents the velocity of the wheel, find the velocity which, added to P B, will have C P for a resultant. It is P A if P Q represents the velocity C P to scale. Make, then, P A the direction of the vane at P. The water will flow in the direction P A *relatively to the wheel*, and it has also the velocity P B because it moves with the wheel; its total velocity is the same as before it entered, and it has entered without shock. We don't much care how the vane curves afterwards so long as it curves gradually; it is its direction at P that is important.

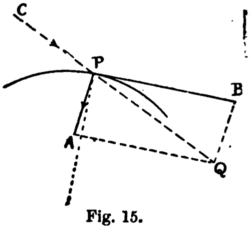

Fig. 15.

Exercise.—The inner circumference of a centrifugal pump wheel goes at 15 feet per second; water approaches it radially at 5 feet per second. What is the angle of the vanes if the water is to enter without shock? *Ans.*, 18° 26′.

Note that the angle chosen by us for the vane at A (Fig. 13) enabled the water to enter the wheel **without shock.** We may at *once say that it* is only the angles at A and at Q that are of real importance in the design of a pump. The actual shape of the vane is unimportant so long as the curve is fairly direct. But the exercise ought to be worked carefully throughout.

We made the tangent of the angle at A equal to the radial velocity ·1, divided by the velocity of the wheel at A (see Art. 33). We always endeavour to have this sort of relation true at the inner side of the wheel of a centrifugal pump or turbine.

38. Usually we consider the earth and frame of a machine to be fixed. The student will find it very instructive to think of some link or wheel as fixed (which he usually thinks of as moving) and now note what the motions are. One simple example of this is given in Art. 467, and we there show that to understand the relative motions in a **four-link mechanism** is to understand the motions in a very great number of other mechanisms.

Train A is passing train B. Looked at by an observer on the ground, how very different is their appearance from what it is to an observer in either train!

The mathematics of the subject is quite easy. It is simply this: If a, b, c, etc., are lines drawn representing what may be called the *absolute* displacements of points A, B, C, etc., or the rotations of bars about axes, then their displacements *relatively* to a frame F, whose own absolute displacement or rotation is f, are $a - f$, $b - f$, $c - f$, etc., the — sign meaning vector subtraction.

But mathematics does not satisfy us; we want the **instinct** of comprehending easily these relative motions. That we do not possess it is evidenced by the fact that all the mechanisms of Art. 467 do really seem to us different, and that we need to give the name **epicyclic train** to a train of wheels when the framework which connects their centres is allowed to move instead of remaining at rest, as it does in our usual way of studying things. Notice that this, which ought to be an easy subject, follows in smaller printing.

39. Thus, for example, in Fig. 16 we have a train of three wheels (the student ought to take two or four or more). When the frame F is at rest (that is, we take speeds relatively to the frame) let A, B, C have the angular velocities a, ba, ca (evidently in the figure b is negative). Now let F have an absolute angular velocity, f, and the absolute velocities of the wheels are $a + f$, $ba + f$, $ca + f$.

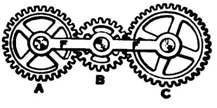

Fig. 16.

1. Suppose A is at rest; $a+f=0$, $a=-f$. B's velocity is $(-b+1)f$; C's velocity is $(-c+1)f$. Thus, let C have the same number of teeth as A; so that $c=1$, C's velocity is 0. If C has 100 teeth, and alongside it on what is practically the same spindle, let there be wheels of 99 and 101 teeth also gearing with B. Evidently the absolute speeds of the three will be (since c has the three values $\frac{100}{99}$, $\frac{100}{100}$, and $\frac{100}{101}$) as $-\frac{1}{99}$ to 0 to $\frac{1}{101}$. When the arm goes round and the motions of the three wheels are observed, we call this **Ferguson's paradox**. If we could, like flies, move with the arm, there would be nothing paradoxical about it.

Notice that A, B, and C may be bevel wheels, as shown in Fig. 17.

2. *Simpler Case.*—Wheels A and B connected by arm F; arm rotates. The absolute rotational velocity of B is 0. What is A's velocity?

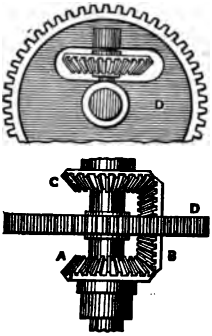

Fig. 17.

Here $ba+f=0$. Hence $a=-f/b$; so that A's velocity is $-f/b+f$ or $f\left(1-\frac{1}{b}\right)$. Thus, let $b=-1$, as it is in **Watts' sun and planet motion**; A's velocity is $2f$. In this case, however, because of the angularity of the connecting-rod, B has some angular velocity, fluctuating between, say, $+\beta$ and $-\beta$. Hence (since $b=-1$), $-a+f=\pm\beta$, and A's velocity is $2f\mp\beta$.

3. In Fig. 17 we have A, B, and C, three bevel wheels. B may be carried round the axes of the others on a spur wheel D. Suppose we are looking from the point D at every wheel, and the numbers of teeth on A and C are as 1 to c, then relatively to D the speeds of A and C are a and $-ac$. If D rotates at speed f, the absolute velocities of A and C are $a+f$ and $-ac+f$.

Example.—Let $c=1$; then, if A's speed $a+f$ is called α, *so that* $a=\alpha-f$, C's speed is $-a+f$ or $-\alpha+2f$. Thus, suppose A goes at 20 revolutions per minute or $a = 20$, then C's speed is $-20 + 2f$; so that if f is gradually changed in the following way, we get the following speeds for C shown in the table :—

Speed of F.	Speed of C.
4	− 12
6	− 8
8	− 4
10	0
12	4
14	8
16	12
18	16
20	20

Again, we may imagine the speed of F to keep constant and that of A to vary, and we get the same result.

By means of a **cam** we may gradually change from negative to positive velocities, but here we have a very much better means by ordinary gearing.

In Fig. 17, if the shaft of A is rotated by coned pulleys, by which it may be given varying speeds, and if A is kept rotating at a fixed speed, C gets speeds which may be negative or positive or zero.

Exercise.—An epicyclic gear consists of an annular wheel A of 72 teeth, a pinion B, and a spur wheel C of 40 teeth concentric with A. The arm which carries the axis of B makes 30 revolutions per minute. (1) If A be a dead wheel, find the revolutions of C. (2) If C be a dead wheel, find the revolutions of A. *Ans.*, (1) 84; (2) $46\frac{2}{3}$.

Again, if A makes 4 revolutions and C 6 revolutions in the same direction, find the revolutions of the arm. *Ans.*, $4\frac{5}{7}$.

Exercise.—In a horse-gear for driving a chaff-cutter, the bracket that holds the pole supports also a short horizontal shaft carrying a bevel wheel of 31 teeth and a bevel pinion of 16 teeth. The pinion gears into a horizontal bevel ring of 80 teeth that is stationary, and forms part of the framing. The bevel wheel of 31 teeth also gears with a bevel pinion of 22 teeth which is loose on the central vertical axis, and this pinion carries with it a bevel wheel of 60 teeth that gears with a pinion of 16 teeth on the high speed horizontal shaft. Find the number of revolutions of the high speed shaft for each circuit of the horse.

Ans., $30\frac{15}{88}$.

CHAPTER III.

WORK AND ENERGY.

40. Work.—To do work it is necessary to exert a force through a certain distance in the direction of the force. Thus, if we exert a force of 20 lbs. through a distance of 6 feet, we do 20 × 6, or 120 foot-pounds of work. If a body of 5 lbs. weight changes its level by the amount of 10 feet, whether it does this by a direct vertical fall or rise, or is moved up or down an inclined plane or curved surface, so long as there is no friction, the amount of work given out by the body in falling or given to it to make it rise is always the same, 5 × 10, or 50 foot-pounds.

Example.—The weight in a certain clock is 20 lbs., and after being wound up it can fall through a distance of 40 feet. Suppose we wish to alter this height, making it 10 feet; what weight must we use? Evidently the work given out by the new weight in falling 10 feet must be equal to that given out by the old weight, or 800 foot-pounds. In fact, the new weight must be 80 lbs. Of course we must apply this weight to the clock by means of a block and pulleys, or we must reduce the diameter of the drum proportionately; and if in applying it we introduce more friction than there used to be in the clock, we must further increase the weight, so as to be able to overcome this friction.

The work done by a force is well illustrated by the pulling of **a tramcar.** If the pulling force P lbs. is not directly along the track, but makes an angle θ with it, the effective force, or the resolved part of P in the direction of motion is P cos. θ, and this, multiplied by the distance moved through in feet, is the work done in foot-pounds.

When a body is pulled up a curve the work done in overcoming the force of gravity (we are neglecting work spent in overcoming friction) is simply the weight of the body multiplied by the difference in level.

Thus in Fig. 18 the work done in moving a body of weight W from A to B along the curve is simply W y, where y is the difference in level of A and B.

Proof:—Let the co-ordinates of any point P be x, y; and of Q, a point indefinitely near to P, $x + \delta x$, $y + \delta y$.

The weight W resolved in the direction of the tangent at P is W sin. θ, and this multiplied by the distance PQ (we are supposing that the tangent at Q is parallel to the tangent at P, as Q is supposed to be indefinitely near to P), or PQ . W sin. θ is the work done against gravity in pulling the body from P to Q. Therefore the **whole work** done in pulling the body from A to B is the sum

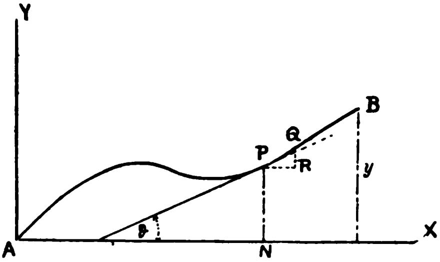

Fig. 18.

of all such terms as PQ . W sin. θ. But PQ sin. θ is QR, or δy, or in the limit dy. Therefore the whole work done is

$$\int_0^y W\,dy \text{ or } W y.$$

Gravity does this work when the body falls. But, both when a body is moved up and down, energy is wasted or converted into heat in overcoming friction. Hence, when a weight, W lbs., is lifted in level h feet, the **useful work** done is Wh, but there is **energy wasted.** Again, when W falls, gravity does the work Wh, but there is energy wasted. If we are depending upon the total store Wh to drive machinery, the useful work done is less than Wh, the difference being wasted, or rather converted into heat, by friction.

41. Horse-power. — One horse-power is the work of 33,000 foot-pounds done in one minute. **Power means** not merely work, but work done in a certain time; the **time rate** of doing work. The work done in one minute by any agent divided by 33,000 is the horse-power of that agent. In a **steam-engine** the piston travels four times the length of the crank in one revolution, and all this time it is being acted upon by the pressure of steam. If the *mean* or average pressure urging the piston is 60 lbs. per square inch, and the area of the piston is 150 square inches, then the total average force urging the piston is 150 × 60, or 9,000 lbs. If the crank, whose length is 0·9 foot, makes 70 revolutions per minute, then the piston travels 4 times 0·9 × 70, or 252 feet per minute, so that the work done in one minute is 9,000 × 252, or 2,268,000 foot-pounds. Dividing this by 33,000, we find the horse-power of the steam-engine to be 68·7. The mean pressure is best found by the use of an **indicator** which draws for us an **indicator diagram**. Measuring the pressures at ten equidistant places on this diagram, adding them together, and dividing by ten, gives the average pressure. Or, measure the area by means of a **planimeter** in square inches; divide by the extreme length of the diagram parallel to the atmospheric line; this gives the average breadth, and therefore the average pressure to scale

As the pressure of steam is usually given per square inch, it is usual to take the diameter of the cylinder in inches, but distances passed through by the piston are evidently to be measured in feet.

Example.—We find by a **spring balance** that some horses or a steam-engine have been pulling a carriage with an average pull of 120 lbs. during one minute, the space passed over in the minute being 500 feet: what is the horse-power expended on the carriage? Here 120 lbs. act through the distance of 500 feet, and the work done in one minute is evidently 500 × 120, or 60,000. Dividing by 33,000, we find the horse-power to be 1·818.

42. **Energy** is the capability of doing work. When a weight is able to fall, it possesses **potential energy** equal to the weight in lbs. multiplied by the change of level in feet through which it can fall. When a body is in motion, it possesses **kinetic energy** equal to half its mass (its weight in London in pounds divided by 32·2 is its **inertia**, which is usually, but we think unwisely, called its mass), multiplied by the square of its velocity in feet per second. (See Art. 190.)

Example.—A body of 60 lbs. is 100 feet above the ground, and has a velocity of 150 feet per second: what is its total amount of mechanical energy?—that is, what energy can it give out before it reaches the ground, and becomes motionless? Here the **potential energy** is 60 × 100, or 6,000 foot-pounds. Its **kinetic** energy is 60 × 150 × 150 ÷ 64·4, or 20,963 foot-pounds. So that the total amount is 26,963 foot-pounds.

Suppose this body to lose no energy through friction with the air, and suppose that, after a time, it is at a height of 20 feet above the ground; find its velocity. Answer: Its *potential* energy is now 60 × 20, or 1,200 foot-pounds, therefore its *kinetic* energy must be 25,763; and evidently this, multiplied by 64·4 and divided by 60, gives 27,652·29, the square of the new velocity. Its velocity is therefore 166·3 feet per second. In such a question we are **not concerned with the direction** in which the body is moving. It may be a cannon-ball, or a falling or rising stone, or the bob of a pendulum. Given its velocity and height at any instant, we can find for any other height what its velocity must be, or for any other velocity what its height must be.

On a **Switchback** railway, the loss of energy (roughly proportional to distance travelled) every journey is represented by the lift of a few feet which is effected by the attendant at each end before making a fresh start. Neglecting this loss, it is easy to calculate the velocity at any place if we know the vertical depth of the place below the starting point. The deepest places on the line are places of greatest velocity; the highest places are those of least velocity. In a bob of a pendulum we see a continual conversion of kinetic into potential and of potential into kinetic energy, the **total store** remaining constant, except in so far as friction is converting mechanical energy into heat. (See Art. 192.)

The **energy of a strained body**—for example, a **strained spring**—is another store of mechanical energy. It is an excellent **laboratory experiment** to measure how much energy a spring can store without getting permanently hurt in shape or broken. This store of energy is called its **resilience**. As the elongation of a spring is proportional to the load, if we gradually increase the load from 0 to w lbs., the elongation increases from 0 to x feet, so that the average force for the whole length being $\frac{1}{2}$ w, the energy stored is $\frac{1}{2}$ w x.

A weight vibrating vertically at the end of a spiral spring gives a good example of the continual conversion of the three kinds of mechanical energy into one another. The total store gradually diminishes, partly by friction in the atmosphere, but greatly also by an internal frictional resistance or **viscosity** in the material of the spring. The student will find it interesting to compare the behaviour of a **spring of steel** and a **spring of indiarubber** in this respect; the viscosity of the indiarubber damps the vibrations with great rapidity. Many hours may be well spent in studying these phenomena, making quantitative measurements.

Air and other fluids in a compressed condition contain stores of energy.

43. A student may employ his leisure in calculating the possible stores of energy in 1 lb. of various materials: 1 lb. of hydrogen, $4{\cdot}7 \times 10^7$ foot-pounds; kerosene, 2×10^7 foot-pounds; coal, 12×10^6 foot-pounds (the weight of oxygen of air for combustion is not counted in); 1 lb. of cast iron in rim of pulley at highest speed to produce greatest working stress, 1,000 foot-pounds; steel spring of the best kind, 270 foot-pounds; usual spiral spring of round wire, 135 foot-pounds.

The heat given to 1 lb. of water to raise it from 0° to 1° C. or from 99° to 100° C. in temperature, is nearly the same. This is a unit of heat energy. Joule showed that this is equivalent to 1,400 foot-pounds.* When energy is in the heat form, a heat-engine may be used to convert part of it into mechanical energy; unfortunately the amount convertible, depends upon how high is the temperature of the stuff containing the heat energy above the temperature of the refrigerator or exhaust. Hence it is that in the very best steam-engines we seldom convert more than one-seventh of the heat energy of the steam into mechanical energy, the other six-sevenths being degraded in temperature, and going to the refrigerator useless for our purposes. In discussing heat-engines we express all energy in foot-pounds.

The energy store is most intense as to volume (we do not include the weight of the air needed for combustion) in a pound of kerosene, being about half as much again as in a pound of coal. We look forward to the time when no heat-engine will be needed in the conversion, and we may then be able to convert over 90 per cent. of the energy of a pound of kerosene into the mechanical form, as at the present time the chemical energy of zinc is convertible in a battery or electric motor, or the chemical energy of oats and other food is convertible in the animal machine, which is probably a gas battery and electric motor.

44. When the engine of the Finsbury College is working mainly for the electric light and possibly one electric motor, the students have sometimes, during a long-continued measurement, been able to trace what becomes of the energy of one pound

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* The latest determination for the *mean* specific heat of water from 0° C. to 100° C. is 1399 foot-pounds, or for 1 gramme it is 0·995 calorie; 1 calorie, the heat to raise 1 gramme of water from 10° C. to 11° C. is 4·2 Joules or $4{\cdot}2 \times 10^7$ ergs. The heat from 20° C. to 21° C. is 1-30th of 1 per cent. less.

of coal. They had previously measured the chemical energy in a pound of coal, and tested all sorts of instruments and ideas used in their measurements.

The energy of 1 lb. of coal is, say, 12,000,000 foot-pounds; of this 4,000,000 go up the chimney or get wasted by radiation as heat, and 8,000,000 foot-pounds of heat energy reach the engine in steam; of this only one-thirteenth, or 600,000 foot-pounds is converted into mechanical energy and given to the piston; the other twelve-thirteenths go off to the condenser and are wasted. Only 500,000 are given out by the engine to the long shaft which drives the dynamo machine, about 400,000 foot-pounds leave the dynamo as electric energy, and part is wasted by conversion into heat in the conductors to the lamps. At the lamps we have about 370,000 foot-pounds of electric energy converted into heat and light. If any of the electric energy or all of it is given to an electric motor, perhaps more than 90 per cent. of it will be converted into mechanical power.

Men who make measurements of this kind get to have very clear ideas as to the various forms of energy and the fact that it is indestructible, and cannot be wasted, although it may alter into forms in which it may not be available for use; and therefore we say that it is wasted.

The very **best steam-engines** use more than 1½ lbs. of coal per hour for each horse-power given out. We cannot hope for much improvement; this is about one useful for nine total. **Gas-engines** using Dowson gas already give out one horse-power for 1 lb. of coal consumed per hour; we hope for considerable improvement. **Oil-engines** give out one horse-power for less than 0·9 lb. of kerosene per hour; we hope for very considerable improvement.

45. Students ought to work many **numerical exercises** on mechanical energy: w lb. of water raised vertically h feet, the energy is wh foot-pounds. If **time** is given, find the work done per minute and divide by 33,000; this is useful horse-power. Divide this by the efficiency of a pump, and we have the actual horse-power which must be supplied to the pump. Or, if the w lbs. of water fall per minute through a turbine, the head available being h feet, we have w$h \div 33{,}000$ as the total horse-power, and the turbine will probably give out 70 per cent. of this usefully, 0·70 or 70 per cent. being the efficiency of a good turbine.

If there were no *friction*, a waggon of weight w requires to be pulled with a force of $W \div s$ up a road which rises 1 foot in every s feet of its length. The frictional resistance to motion of a vehicle on a level road is usually stated as so many pounds per ton weight of the vehicle. If there is also an incline we add the two tractive forces together—one for the incline without friction, the other to overcome friction on the level. The **resistance** in pounds per ton **of a moving train** (including engine) on the level is found roughly by adding 2 to one-quarter of the speed in miles per hour. This is for speeds greater than 20 miles per hour. At less speeds there is quite a different law, which may for some trains and permanent ways be indicated by the following figures :—

Speed in miles per hour.	0	1½	2	5	10
Resistance in pounds per ton.	20	10	7	5	6

A curved line adds 12 per cent. to the resistance on the average English railways. The tractive force of heavy waggons on macadamised roads may be taken as 50 lbs. per ton, on paved roads 30 lbs. per ton, and on gravel roads as 150 lbs. per ton. These rules are good enough for academic exercise work.

If the **pull on a tramcar** is recorded as the ordinate of a diagram, of which the abscissa represents distance along the track, the area of the diagram represents to some scale the work done. The average height of the diagram represents the average pull P. This average pull P, multiplied by the whole length of the track, is the whole work done. In most practical cases the average can only be obtained by making the diagram, finding its area, and dividing by its length, just as we do with an indicator diagram of an engine. But there are cases where we can *calculate* an average.

Example.—A **chain** of length l and weight wl, with a weight W at the end of it, is to be wound up by a capstan ; what work will be done? Obviously the *average* pull is $W +$ half the weight of the chain, or $W + \frac{1}{2}wl$, and the total distance is l, so that the work done is $Wl + \frac{1}{2}wl^2$. As a rule, it is wise to plot the varying pull as the ordinate of a curve on **squared paper**, as, without the aid of the calculus, one is apt to state what *is the average* pull without due thought. Thus, if the above-mentioned chain varies in heaviness per foot, the average pull due to it is not half its weight.

Observe that we ought to call the above the **space average** of a force. The space average of a force, multiplied by the distance, is the work done.

A little consideration will show a student that a **time average** is a very different thing.

46. According to the results of some experiments by Prof. R. H. Smith on the **cutting of metal** in the lathe without water or oil, the force on the tool is not much affected by speed.

For both thin and moderately thick shavings at all speeds, feeds, and depths of cut, we may roughly take it that forged steel takes twice as much power to cut it as does cast iron; wrought iron takes one to one and a half times as much as cast iron.

For broad thin shavings, cast iron required more cutting force than wrought iron.

The force is neither proportional to the breadth of the shaving nor the depth; but it is more nearly proportional to depth than breadth.

It is interesting to note that before these experiments it was usual in books to follow Weisbach in saying that for iron $P = fbd$ where f was 50,000 lbs. per square inch. Smith's experiments show that this rule is not true, and that $\frac{P}{bd}$ varies from 92,000 to 239,000. Possibly when it is discovered that there are other things to be done in college engineering laboratories than to break endless numbers of specimens of metal with a 100- or 200-ton testing-machine, we may have further experimental results on which practical engineers may rely.

Professor Smith tried various depths, d inches, and breadths, b inches, of shaving from cast iron, wrought iron, and forged steel, in every case measuring P, the pressure on the tool in pounds, at the cutting edge. In almost every case we find from his numbers that the energy E usefully spent per cubic inch of metal removed, diminished about 30 per cent. as b was increased from ·03 to ·05, being about its minimum value when b was ·05 : but probably it does not increase much for greater values of b. The minimum values of E for various depths of cut were as follow :—

d	b	P	E	——
·05	·056	288	8,700	Cast iron.
·135	·056	690	7,700	
·03	·056	215	10,700	Wrought iron.
·06	·06	570	13,000	
·14	·056	810	8,700	
·02	·056	250	18,700	Steel.
·04	·056	480	18,000	
·06	·056	705	17,700	

With cast iron if $b = ·05$, and probably for greater values of b, E is much the same for cuts of the depths ·05 and ·135 inch, being about 9,000 foot-pounds per cubic inch of metal removed.

With steel, if $b = ·05$, and probably for greater values of b, E is much the same for cuts of the depths ·02, ·04, and ·06 inch, being about 18,000 foot-pounds per cubic inch of metal removed.

With steel, if $b = ·05$, and probably for greater values of b, E is 10,700 foot-pounds for a cut of ·03 inch depth, 13,000 foot-pounds for a cut of ·06 inch depth, and is practically the same as cast iron for a cut of ·14 inch depth.

EXERCISES.

1. Two tons of rock can fall to a depth of 320 feet; find the work which it may do. *Ans.*, 1,433,600 foot-lbs.

2. In lifting an anchor of $1\frac{1}{2}$ tons from a depth of 15 fathoms in six minutes, what is the useful man-power, if a man-power is defined as 3,500 foot-lbs. per minute? *Ans.*, 14·4.

3. The pull on a tramcar is 200 lbs. at an angle of 25° with the track; what is the component in the direction of the track? What work is done in a distance of 10 feet along the track? If the speed is 4 feet per second, what is the usefully expended horse-power? *Ans.*, 181·26 lbs.; 1812·6 foot-lbs.; 11.18.

4. What horse-power is involved in lowering by 2 feet the level of the surface of a lake 2 square miles in area in 300 hours, the water being lifted to an average height of 5 feet? *Ans.*, 29·23.

5. Taking the average power of a man as $\frac{1}{10}$th of a horse-power, and the efficiency of the pump used as 0·4, in what time will ten men empty a tank of 60 feet × 30 feet × 6 feet filled with water, the lift being an average height of 30 feet? *Ans.*, 21 hours 14 minutes.

6. The diameter of a steam-engine cylinder is 9 inches, the length of crank 9 inches, the number of revolutions per minute 110, and mean effective pressure of the steam 35 lbs. per square inch; find the indicated horse-power. *Ans.*, 22·3.

7. Each of the two cylinders in a locomotive engine is 16 inches diameter, and the length of crank is 1 foot. If the driving-wheels make 105 revolutions per minute, and the mean effective steam-pressure is 85 lbs. per square inch, what is the horse-power? *Ans.*, 218.

8. What would be the indicated horse-power of an Otto gas engine which has a piston 12 inches in diameter and a crank 8 inches long? The engine works at 150 revolutions a minute, and there is an explosion every 2 revolutions, the mean effective pressure in the cylinder during a cycle being 62 lbs. per square inch. *Ans.*, 21.

9. The average breadth of an indicator diagram for one side of the piston is 1·58 inches, and for the other side it is 1·42 inches, and 1 inch represents 32 lbs. per square inch. Piston, 12 inches diameter; crank, 1 foot; 110 revolutions per minute. What is the indicated horse-power? *Ans.*, 72·38.

10. What must be the effective horse-power of a locomotive which moves at the steady speed of 35 miles an hour on level rails, the weight of engine and train being 120 tons, and the resistances 16 lbs. per ton? What additional horse-power would be necessary if the rails were laid along a gradient of 1 in 112? *Ans.*, 179·2; 224.

11. In 10 find in each case how far the train would move after steam was shut off, assuming the above constant resistance and neglecting rotatory motions. Find also the speed of the train after the latter had moved over a distance of 1,000 feet from the point where steam was shut off.

Ans., 5,728 feet; 2,545·8 feet; 31·8 miles per hour; 27·3 miles per hour.

12. A flywheel weighs $2\frac{1}{2}$ tons, and its mean rim has a velocity of 40 feet per second; what is its kinetic energy? If the velocity be reduced 3 per cent., what is the reduction in the kinetic energy? If the kinetic energy be reduced by 10,000 foot-lbs., by how much is the velocity reduced? In estimating the latter, why would it be wrong to subtract from 40 feet per second the velocity which corresponds to 10,000 foot-lbs. of energy in this flywheel?

Ans. 139,130 foot-lbs.; 131,900 foot-lbs.; 1·5 ft. per second.

13. A machine discharges n projectiles per minute, each of w lbs. moving with the velocity of v feet per second; what is the actual horse-power? If the efficiency of the machine is e, and it is driven by a steam-engine which uses w lbs. of steam per hour per horse-power given out by it, what is the total steam per hour? If the engine is governed by throttling, and if the total steam per hour follows the rule:—steam per hour $= a + b \times$ brake horse-power (where a and b are known to us), and if for half an hour n_1 projectiles are discharged per minute, and if for the next half hour n_2 projectiles are discharged per minute, how much steam is used during the hour? *Ans.*, $\frac{n\mathrm{w}v^2}{g \times 66000}$; $\frac{wn\mathrm{w}v^2}{66000\,ge}$; $a + \frac{b(n_1+n_2)\mathrm{w}v^2}{132000\,g}$.

14. A fire-engine pump is provided with a nozzle, the sectional area of which is 1 square inch, and the water is projected through the nozzle with an average normal velocity of 130 feet per second; find (1) the number of cubic feet discharged per second, (2) the weight of water discharged per minute, (3) the kinetic energy of each pound of water as it leaves the nozzle, (4) the horse-power of the engine required to drive the pump, assuming the efficiency to be 70 per cent.

Ans., (1) ·9 cubic feet; (2) 1·51 tons; (3) 262·3 foot-lbs.; (4) 38·3.

15. The section of a *stream is 12* square feet, the average velocity of

the water is 2 feet per second: there is an available fall of 25 feet: what is the horse-power available? A turbine drives a dynamo machine which sends electric power to a motor at a distance. The efficiency of the turbine is 70 per cent.: of the dynamo, 97 per cent.: ten per cent. of the energy from the dynamo is wasted in transmission, and the efficiency of the motor is 72 per cent.: how much power is given out by the motor? The voltage at the dynamo is 102: what is the current in ampères?
Ans., 68: 24·67: 278.

16. Electric lamps giving 1 candle-power for 4 watts: how many 10- or how many 16-candle lamps may be worked per electric horse-power? The combined efficiency of engine, dynamo, and gearing being 70 per cent., what is the candle-power available for every indicated horse-power? *Ans.*, 18: 11: 130·55.

17. On a switchback the carriage is 966 lbs., neglecting friction: find its kinetic energies when it is 5, 10, and 15 feet below its starting-point. And if the starting-point is 20 feet above datum level, write out in two columns its two kinds of energy at each point. If the above points are 0·1, 0·4, and 0·7 of the distance along the track, and if loss of energy by friction is proportional merely to distance along the track, and if the carriage has to be lifted 1·6 feet at the end of each journey, find the correction in the kinetic energy at each place.

Ans., Potential energies, 14,490, 9,660, 4,830 foot-lbs.: kinetic energies, 4,830, 9,660, 14,490 foot-lbs.: corrected kinetic energies, 4,675·4, 9,041·6, 13,407 foot-lbs.

18. The calorific powers of 1 lb. of each of the following fuels are given in centigrade-pound heat units: convert into foot-lbs. if 1 heat unit = 1,400 foot-lbs.:—charcoal, 7,000: coke, 7,000: coal, 8,800 to 7,330; wood, 4,200: and kerosene, 12,200.

19. The pull on a tramcar was registered when the car was at the following distances along the track:—0, 200 lbs.: 10 feet, 150 lbs.; 25 feet, 160 lbs.: 32 feet, 156 lbs.: 41 feet, 163 lbs.: 56 feet, 170 lbs.; 60 feet, 165 lbs.: 73 feet, 160 lbs.; what is the average 'space' pull on the car, and what is the effective work done in pulling the car through the distance of 73 feet? *Ans.*, 161 lbs., 11,800 foot-lbs. about.

20. A chain hanging vertically 520 feet long, weighing 20 lbs. per foot, is wound up: what work is done? *Ans.*, 2,704,000 foot-lbs.

21. When a prismatic column of stone, 20 feet diameter outside, 10 feet inside, 90 feet high, is being built, what actual work is done in lifting the stone from the ground? One cubic foot of stone weighs 125 lbs. *Ans.*, $1 \cdot 193 \times 10^8$.

22. A weight of 3 tons at a depth of 400 feet at the end of a chain which weighs 8 lbs. per foot: what work is done in lifting chain and weight? *Ans.*, 3,328,000 foot-lbs.

23. Four cwt. of material are drawn from a depth of 80 fathoms by a rope weighing 1·15 lbs. per linear foot: how much work is done altogether, and how much per cent. is done, in lifting the rope? What horse-power would be required to raise the material in four and a half minutes?
Ans., 347,520 foot-lbs.: 38; 2·34.

24. The travel of the table of a planing-machine which cuts both ways is 9 feet, if the resistance to be overcome while cutting, be taken at 400 lbs., and the number of double strokes per hour be 80: find the horse-power absorbed in cutting. *Ans.*, ·29.

25. (*a*) A cut of ·06 inch depth is being made on a 4-inch wrought-iron shaft revolving at 10 revolutions per minute; the traverse feed is 0·3 inch per revolution; the pressure on the tool is found to be 435 lbs.; what is the horse-power expended at the tool? How much metal is removed per hour per horse-power?

Ans., ·1381; 98·28 cubic inches.

(*b*) When the traverse feed is ·06 inch per revolution, the pressure on the tool is found to be 570 lbs.; find the horse-power, and the metal removed per hour per horse-power. *Ans.*, ·181; 150 cubic inches.

(*c*) If the above horse-power is called the useful power U, and it is found that the actual horse-power given to the lathe is $0·1 + 1·5\ U$, find actual horse-power and metal removed per hour per actual horse-power in the two cases.

Ans., ·3071, 44·2 cubic inches; ·3715, 73·1 cubic inches.

26. How much work is expended in raising the materials for building a uniform column of brick 66 feet 8 inches high, and 21 feet square? ($w = 112$). *Ans.*, $1·098 \times 10^8$ foot-lbs.

27. A ball weighing 5 ounces, and moving at 1,000 feet per second, pierces a shield, and moves on with a velocity of 400 feet per second; what energy is lost in piercing the shield? *Ans.*, 4,076 foot-lbs.

28. What is the kinetic energy of a tramcar moving at 6 miles per hour, laden with thirty-six passengers, each of the average weight of 11 stones? Weight of car, 2½ tons. What is its momentum? If stopped in two seconds, what is the average force? If the force is constant, this must also be the space average force; so find the distance of stopping if the force is constant.

Ans., 13,400 foot-lbs.; 3045·565; 1522·8 lbs.; 88 feet.

47. Bicycle problems.—When a man says that his bicycle is *geared* to D inches, he means that he advances π D inches for one turn of his pedals. Let the diameter of the pedal circle be d inches. Let W_0 be weight in lbs. of rider, w_0 of machine, $W_0 + w_0 = W$. Let w be the uniform vertical force which the rider applies to each pedal alternately; if w is negative, it means that he is back-pedalling. Let F be the force in lbs. which would pull the bicycle along at the velocity of v miles per hour; $2\ w\ d$ is the work done in inch-pounds by the rider in one revolution, and this is equal to π D F.

Or $w = \frac{\pi D}{2d} F$....(1)

$\frac{F v 5,280}{60 \times 33,000}$, or $F v \div 375$, the horse-power usefully expended....(2)

$\frac{12 \times v \times 5,280}{\pi D \times 60} = \frac{336 v}{D} = n$, the number of revolutions of a pedal per minute....(3).

[It is useful to remember that a machine geared to 53 inches goes at 10 miles an hour when the pedals make one turn per second.]

The vertical force is not by any means constant in practice, nor indeed ought it to be: it is and ought to be greatest when the crank is horizontal. With proper ankle action the force is always somewhat in the direction of the circular path of the pedal, but for exercise work there is no harm in assuming a uniform vertical force P in the down-stroke and no force in the up-stroke. In practice it is difficult to avoid pressing on the pedal in its up-stroke.

If F_1 is the value of F on a level road, then on a rising slope of 1 in s we have

$$F = F_1 + \frac{W}{s} \qquad (4).$$

If it is a descending slope, $\frac{1}{s}$ is negative. F_1 may usually be taken as proportional to W.

The following data were roughly measured by myself. They are good enough for academic problems. They suit my own bicycle on a good road. Students ought to obtain data of their own by careful measurement.

A rider weighing 127 lbs. on a cycle weighing 33 lbs. (or W = 160 lbs.) finds that on a descent of 1 in 80, with his feet off the pedals, he is just able to get on very slowly but steadily; on a long slope of 1 in 40 his steady speed (feet off pedals) was 9¾ miles per hour; on a long slope of 1 in 20 his steady speed (feet off pedals) was 20 miles per hour. In these three cases the values of F_0 were:—

$$\frac{W}{80} \text{ or } 2, \quad \frac{W}{40} \text{ or } 4, \quad \frac{W}{20} \text{ or } 8.$$

We need careful experiments: and it is not of much use *speculating* on the probable law of resistance of a safety bicycle with pneumatic tyres on a certain kind of road. A constant term for quasi solid friction, and a term (the most important at high speeds) proportional to the square of the speed relatively to that of the atmosphere for air resistance: these we ought to have. Resistance due to unevenness of the ground would be constant if a certain kind of unevenness were to repeat itself at intervals so far apart on the road that the vibration due to each had time to die away before the next: but speculation is vague, especially as the kind of vibration will often depend on the velocity. In our present state of knowledge we cannot be far wrong in assuming

$$F_0 = W(a + bv + cv^2).$$

If the above values of F_0 are correct at the given speeds, $v=0$, $9\frac{3}{4}$, and 20 miles per hour, we find

$$F_0 = \frac{W}{80}\left(1 + \frac{v}{20} + \frac{v^2}{200}\right) \ldots\ldots (5)$$

as a formula which represents our experimental data well enough for exercise purposes.

Hence, going up a slope of 1 in s we have

$$F = \frac{W}{80}\left(1 + \frac{v}{20} + \frac{v^2}{200} + \frac{80}{s}\right)$$

for this bicycle. When $W = 160$ lbs.

$$F = 2 + \frac{v}{10} + \frac{v^2}{100} + \frac{160}{s} \quad \ldots\ldots (6).$$

Example 1.—What horse-power is expended in going at 12 miles an hour on a level road? Here $\frac{160}{s} = 0$.

$$F = 2 + \frac{12}{10} + \frac{144}{100} = 4{\cdot}64 \text{ lbs.}$$

The speed is 88 v or 1,056 feet per minute, and the horse-power

$$= \frac{1{,}056 \times 4{\cdot}64}{33{,}000} = 0{\cdot}148.$$

Example 2.—At 12 miles an hour, going up or down an incline of 1 in 60, what is the useful horse-power?

$$F = 4{\cdot}64 \pm \frac{W}{60} \quad = 7{\cdot}31 \text{ or } 1{\cdot}97 \text{ lb.}$$

$$\text{Horse-power up} \quad = \frac{1{,}056 \times 7{\cdot}31}{33{,}000} = 0{\cdot}234.$$

$$\text{Horse-power down} = \frac{1{,}056 \times 1{\cdot}97}{33{,}000} = 0{\cdot}033.$$

EXERCISES.

1. In Examples 1 and 2, what force, w, does the rider exert upon his pedal if his bicycle is geared to $D = 60$ inches and the diameter of the pedal circle is 13 inches?

Ans., On the level, $w = 33{\cdot}6$ lbs.; going up, $w = 53$ lbs.; going down, $w = 14{\cdot}3$ lbs.

2. On what slope downward would the velocity of 12 miles an hour be steadily maintained, feet off pedals? *Ans.*, 1 in $34\frac{1}{2}$.

3. Going down a slope of 1 in 30 at 8 miles per hour, what is the force on the pedal? *Ans.*, $-12{\cdot}30$ lbs.

The minus sign means that the rider must back-pedal.

4. If a rider whose weight is 157 lbs., back-pedals on this same bicycle with a force of only 10 lbs., down a slope of 1 in 25, what is his velocity?

Ans., 13·6 miles per hour.

The following example is for students who can integrate :—

The resistance to motion being $F = a + bv + cv^2$ on a road which does not alter in character for a whole journey, compare the work done in going over a certain distance, l—first, at a constant speed, v_0 ; second, at the same average speed, but varying according to the law $v = v_0 + f \sin. qt$. In both cases the time is the same if the journey is just so long as to be finished by the rider in the same state as to speed, etc., with which he starts.

$$\int F ds = \int F \frac{ds}{dt} dt = \int F v dt = \int (av + bv^2 + cv^3) dt,$$

the work done. If we calculate this for a time, T, where $q = 2\pi T$, it is just as good as for any number of such periods. The value divided by T gives the average work per second, or

$$av_0 + bv_0^2 + cv_0^3 + \frac{1}{2} bf + \frac{3}{2} cv_0 f^2;$$

whereas, if f is 0 and the speed is constant, we have the average work per second $av_0 + bv_0^2 + cv_0^3$. Hence the fractional increase of work is $(bf + 3cv_0 f^2) / 2 (av_0 + bv_0^2 + cv_0^3)$. Thus, taking our values of Art. 47 for a weight of 160 lbs.,

$$F = 2 + \frac{1}{10} v + \frac{1}{100} v^2 + \frac{160}{s}.$$

If the road has everywhere an upward slope, 1 in s, we take

$$a = 2 + \frac{160}{s1};$$

but we had better take a level road, so that $a = 2$. In any case,

$$b = \cdot 1, \quad c = \cdot 01.$$

Taking an average speed of 10 miles per hour and

$$v = 10 + 3 \sin. qt,$$

so that the speed fluctuates between 13 and 7 miles per hour, we have a fractional waste of power

$$\frac{3}{20} \frac{b + 90c}{a + 10b + 100c} \text{ or } \frac{3}{80};$$

that is, the power at constant speed being 80, the average power at varying speed is 83.

48. Much of what we have given may be said to be mere exercise work in the **use of formulæ**. But we hope that it is much more than that, and that students are getting to understand how the formulæ are derived. Much of this book is devoted to the explanation of how these formulæ are derived, and somewhat similar exercises will be given later. Our object has been to familiarise students with the notion of the quantitative transformation of energy. The subject of this book is almost altogether the **study of energy and momentum**. We close this introductory part of our subject with a few problems on the **hydraulic transmission** of power and the **propulsion of ships**.

For academic exercises it is sufficiently correct to say that the **energy wasted per pound of water flowing in a pipe** is experimentally found to be k times its kinetic energy, where k has the following values :—In l feet of pipe of diameter d feet, $k = \cdot 0232\ l/d$; entrance or exit by cylindric pipe to or from a reservoir, $k = 0\cdot 5$; bend in a pipe, $k = \left\{ \cdot 131 + 1\cdot 847 \left(\frac{d}{\text{D}} \right)^{7/2} \right\} a$, where a is the fraction of two right angles through which the bend extends, D the diameter of the circle of which the centre line of the pipe is part and d the pipe's diameter. Probably only the first of these values of k is fairly correct.

Exercise.—Prove that if the horse-power, H, enters a straight pipe as pressure water, the waste power, W, is $\cdot 00374\ l\,\text{H}^3/p^3 d^5$, where p is the pressure at the entering end in pounds per square inch. Notice that the fractional loss of power is the fractional loss of pressure.

If V is the voltage and A the current in ampères, the horse-power delivered electrically to a conductor is V A watts (746 watts are equal to 1 horse-power). The loss in the conductor is $\text{A}^2\,\text{R}$ watts, if R is its resistance in ohms. Hence, if H is the horse-power sent in, the wasted power is $\text{W} = 746\ \text{H}^2\,\text{R}/\text{V}^2$.

As $\text{R} = \frac{n}{a} \times \cdot 044$ if the copper conductor is n miles long (going and coming, so that the distance is $\frac{1}{2} n$ miles) and a square inches in cross-section, $\text{W} = 32\cdot 7\, \frac{\text{H}^2}{\text{V}^2}\, \frac{n}{a}$. Notice that the fractional loss of power is the fractional loss of voltage.

Exercise.—Prove that at an entering pressure of 700 lbs. per square inch, if we admit the following amounts of hydraulic power, we have the following amounts of waste power. Also find the values in the table for a conductor.

H	At pressure of 700 lbs. per square inch. Horse-power lost in one mile.		At 700 volts. Horse-power lost in one mile.	
The Power sent in.	6-inch pipe.	3-inch pipe.	Conductor 0·25 square inch in cross-section.	Conductor 0·125 square inch in cross-section.
20	...	0·9	...	0·34
50	0·23	7·4	0·67	1·34
100	1·84	59	2·67	5·34
200	14·72	...	10·68	21·36
300	117·8	...	24·03	48·06

Up to the highest usual speeds of commercial ships we may assume without great error that, for vessels not dissimilar

in form and character and going at the usual speeds, the indicated horse-power is $H = D^{2\cdot3} V^3 \div c$, where D is the displacement in tons and V is the speed in knots and c is a constant, which for many classes of vessel may be taken as not very different from 240.

Exercise 1.—What is the indicated horse-power of a vessel of 1,330 tons moving at a speed of 12 knots, if it obeys the above rule? *Ans.*, 871.

Exercise 2.—If a vessel of 1,720 tons moves at 10 knots when its indicated horse-power is 2,620, what is the value of c in such a class of vessel? *Ans.*, 54·8.

The resistance to the motion of a ship is considered to be made up of two parts. 1. **The skin friction** in pounds, $S = f A V^n$, where V is the speed in knots, n is 1·83 for varnished or painted wooden models or clean iron ships, A is the wetted area in square feet, f is ·009 for ships of over 200 feet long, and ·012, ·0106, ·0096 for ship lengths of 8, 20, and 50 feet. At speeds of 6 to 8 knots in ordinary vessels this skin resistance is about 80 or 90 per cent. of the whole: at high speeds it is about half the whole.

2. A **residuary resistance** due to the fact that **eddies** (the smaller part) and **waves** are produced. Eddy resistance is thought not to be more than 8 per cent. of the skin resistance even at high speeds. It is mainly caused by bluntness of the stern of a vessel. In two perfectly similar ships, similarly loaded, of lengths l and L, at speeds v and $v\sqrt{L/l}$, which are said to be the *corresponding speeds*, the residuary resistances are proportional to l^3 and L^3.

The skin resistances S_1 and s_1 of the ship and its model can be calculated from Froude's numbers given above. Hence, if R is the resistance in pounds of a ship L feet long, A its wetted area in square feet, V its speed in knots, and if r and l are the resistance and length of a model which is exactly similar and of similar draught when the model is drawn at the corresponding speed v knots, where $V : v :: \sqrt{L} : \sqrt{l}$, prove that it follows from the above that

$$R = \frac{L^3}{l^3} r - \cdot009 \; A V^{1\cdot83} \left(1\cdot75 \; \frac{L^{1\cdot12}}{l^{1\cdot5}} - 1\right)$$

if the ship is more than 200 feet long, and the model is from 8 to 30 feet long.

Example.—Before building a vessel 400 feet long, of wetted surface 26,000 square feet, we wish to know R, its resistance, at $V = 12$ knots. A model is made 10 feet long, it is drawn at a speed of $12\sqrt{40}$ or 1·9 knots in the tank, and its resistance r is found to be 0·9 lb. We find R to be 39,720 lbs.

Prove that R in pounds × V in knots ÷ 325 = utilised horse-power. In this case we find 1,463 horse-power. The indicated power will probably be more than 3,000.

The vagueness of our knowledge as to the probable loss of power by friction, makes any attempt to calculate R for the above purpose rather useless, and the better use of the tank would therefore seem to lie in helping to improve a particular class of vessel.

The following great **simplification** has recently been tried by **Colonel English.** Suppose an existing vessel to be run at various speeds and its indicated horse-power noted. Now, assume that the effective horse-power in a new ship will be the same fraction of the indicated that we take it to be in the existing ship—say one-half. Find the resistance of the existing ship at the speed V_1. We wish to know the resistance of the new ship at the speed V_2. We only need to compare the wave and eddy resistances, which we shall call W_1 and W_2. **Make two models,** one of the existing ship and one of the ship being designed. Let the values of V, D, S L, W for the two ships and the two models be indicated by capital and small letters, the existing ship and its model having the affixes $_1$. S is skin friction; D is displacement, which in similar ships is proportional to the cubes of the lengths.

Let $v_1 = V_1 \left(\frac{d_1}{D_1}\right)^{1/6}$, $v_2 = V_2 \left(\frac{d_2}{D_2}\right)^{1/6}$, and let $v_1 = v_2$; that is, make the models of such sizes that V_2 and v_2 as well as V_1 and v_1, are "corresponding speeds," and yet that the speeds of the two models shall be the same. In fact, $\frac{d_2}{d_1} = \frac{D_2}{D_1}\left(\frac{V_1}{V_2}\right)^6$. Now let the two models be towed from the two arms of a lever whose fulcrum may be adjusted and **the ratio of the resistances,** n, of the second to the first may be measured. Note that we need only find this ratio—a much easier thing to do than to find either resistance. Show that the total resistance of the new ship is

$$S_2 + \left(\frac{V_2}{V_1}\right)^6 \left\{ nW_1 + \frac{D_1}{d_1}(ns_1 - s_2) \right\}.$$

CHAPTER IV

FRICTION

44. We have said that the mechanical principles which must be studied by the young engineer are few in number, but they must be very familiar to him. It is not well to say that any one method of study is more important than another, the fact being that a student must not only study in the workshops and drawing-office, but he must read, work numerical exercises, and make a great many quantitative laboratory experiments to illustrate these principles. Our aim is to get students to think, and it is astonishing how difficult it is to effect this object. We cannot easily get students to wrangle over these subjects. We have few pretty lecture experiments. Even in chemistry and experimental physics pretty lecture experiments are not very effective in causing students to think. Students will think about things that they do. Hence it is that boys should be allowed to chip and file metals, and to pare and cut wood. Merely in learning how to hold a mortising-chisel or in setting a plane iron, a student must think about the properties of materials and forces. Country boys who make their own things have a great advantage over town boys who buy their things in shops. In the mechanical laboratory, I find that even the dullest student begins to think for himself if he is not too much spoon-fed; and if his difficulties are not cleared away by some wretched routine system of laboratory work being adopted by cheap laboratory instructors, the fundamental principles of mechanics will become part of his mental machinery.

It is not necessary to illustrate everything: a few things carefully done in the laboratory are better than many. For example, let the **triangle of forces** be illustrated in its simplest form. **The principle is:**—If three forces act on a small body and just keep it at rest, then if we draw on a sheet of paper three straight lines parallel to the directions of the three forces, and let them form a triangle in such a way that arrow-heads representing the directions of the forces go round the triangle circuitally, it will be found that the lengths of the sides of the triangle are proportional to the amounts of the forces. *Fig.* 19 shows how the strings, pulleys, and the smooth ring P are used. I am in the habit of using three scale-pans with weights in them, and I measure beforehand the weights of the scale-pans themselves. Put almost any weights in at random (only you will find that any two must be greater than the third), and let P find its position of equilibrium, and you will find the rule to be nearly true every time. Also for the same set of weights you will find that there is a small region within which, anywhere, the centre of the ring P may be placed without disturbing the state of equilibrium; this is owing to the friction of the pulleys. The **polygon of forces** is also easily illustrated. (See Fig. 20.)

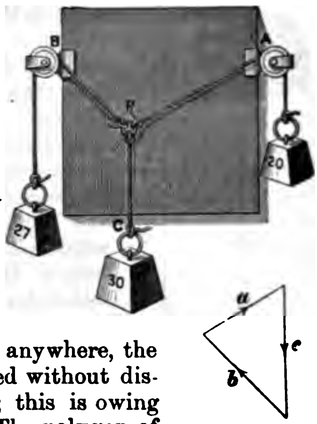

Fig. 19.

50. Our one Theory insufficient.—No man can make these trials without finding that there is a great deal to be observed beyond what his teacher or his book has taught him. A force has been represented by the pull in a string passing over a little pulley with a weight at its end. He finds that as his pulley works more easily, and as its pivots are better oiled, his illustration of the law is better and better; in fact, he finds that the pull in a string is not exactly the

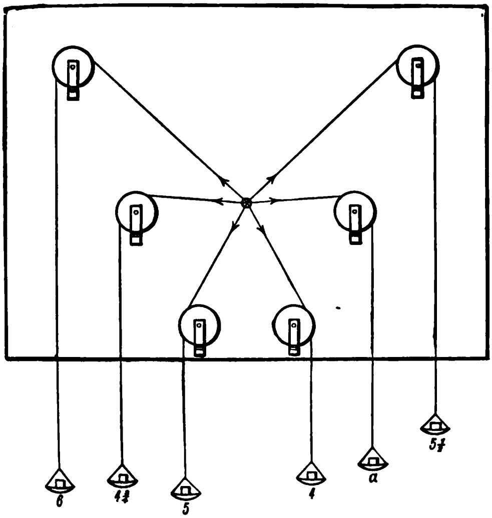

Fig. 20.

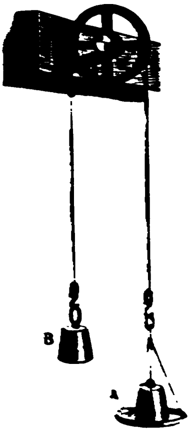

Fig. 21.

same on the two sides of a pulley. If he takes one pulley and one string and two weights, called A and B, Fig. 21, at its ends, he will find that there is equilibrium even when the two weights are not exactly equal. If A is slightly greater than B, and he further increases the weight of A till it is just able to overcome B, then the difference between the weights represents, in some fashion, what may be called the **friction** of the pulley. If now he increases both the weights which he uses, he will find that the friction is proportionately increased, and he will get to understand why we so generally find in machinery that there is a law, "**friction is proportional to load.**" This law is not quite true, but it is sufficiently true to be of great value to engineers. Again, he sees that this friction, which is a resistance experienced in the *rubbing* together of any two surfaces, is a force which always opposes motion, always acts against the stronger influence. Suppose, for example, that he found that a weight of 5·1 ounces was just able to overcome a weight of 5 ounces: he will find that a weight of about 4·9 ounces will just be overcome by a weight of 5 ounces, and that there is equilibrium with 5 ounces and any weight varying from 5·1 to 4·9. Friction always helps the weaker forces to produce a balance.

51. Law of Work.—Take any machine, from a simple pulley to the most complicated mechanism. Let a weight, A, hung from a cord round a grooved pulley or axle in one part of the mechanism, balance another weight, B, hung from a cord round another axle or pulley somewhere else. In Fig. 22 we have imagined that the **mechanism is enclosed in a box**, and only the two

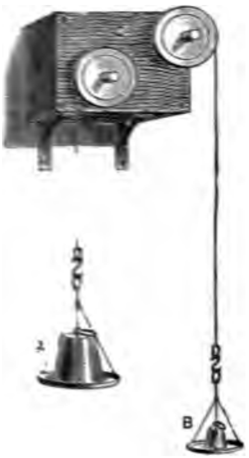

Fig. 22.

axles in question make their appearance. Now move the mechanism so that A falls and B rises steadily. Suppose that when A falls 1 foot B rises 20 feet, then if there were no friction in the machine a weight at A is exactly balanced by one-twentieth of this weight at B. This is the law which you will find proved in books on mechanics. The reason why it is true is this. The work or mechanical energy given out by a body in falling is measured by the weight of the body multiplied into the distance through which it falls. It is in this way that we get the energy derivable from the fall of a certain quantity of water down a **waterfall**, and it is in this way that we find out whether a certain waterfall gives out enough power to drive a **mill**. Similarly the energy given to a body when we raise it, is measured by the weight of the body multiplied by the vertical height through which it is raised. Now, every experiment we can make shows that **energy is indestructible**, and consequently, if I give energy to a machine, and find that none remains in it, that there are no means there of converting mechanical energy into heat by friction or into any other form such as electrical energy, or *vice versâ*, and no storage or unstorage of energy, as by lifted weights or the coiling of springs, or by increasing or diminishing the kinetic energy of anything (this is why I pre-suppose uniform velocity in A and B), then all the energy given to the machine must be given out by it. This is often what people really mean when they say that their machine is supposed to have no friction.

Therefore the energy given out by A in falling must be equal to the energy received by B in rising; and as A falls 1 foot when B rises 20 feet, the weight of A must be twenty times the weight of B. If, then, there were no friction in the machine, and if a weight of 20 lbs. were hung at A and a weight of 1 lb. at B, we should find that if we start A downwards or upwards there will be a **steady motion** produced. Any excess at A will cause it to overcome B, the weights moving more and more quickly as the motion continues.

Now, in our machine, Fig. 22, we can always find by trial what is the **velocity ratio**—that is, the speed of B as compared with the speed of A—and this is usually called the **mechanical advantage** when there is no friction. I have chosen a machine in which I suppose that if A has a uniform motion, so has B. But if when A is uniform, B is not uniform in its motion, then the velocity ratio for any particular position must be measured during exceedingly small motions, as, after a little motion everything alters. Let us continue to suppose that the velocity ratio does not alter. Now, when we try to balance a weight at B by a weight at A, we find that the above relation is quite **untrue**. Hang a weight of 1 lb. at B, hang a weight of 20 lbs. at A, there is certainly a balance; but when we have somewhat less or more than 20 lbs. at A there still is balance. The reason for this is that there is friction in the mechanism, and this friction always tends to resist motion, always acts against the stronger influence.

52. Effect of Friction.—We proceed to find out in what way friction modifies the law given in books. You must make **actual experiments** with some machine if you are to get any good from your reading. Hang on a weight, B, and find the weight, A, which will just cause a slow, steady motion. Do this every time when a number of different weights are placed at B. Now suppose you have measured the velocity ratio—that is, suppose you find that B rises *four* times as rapidly as A falls. Then, according to the books, there would be an exact balance if A were four times the weight of B. On actual trial, however, I find in a special case the following table of values:—

A overcomes B when

A is	23·4	ounces	and B is	5	ounces
"	44·7	"	"	10	"
"	65·4	"	"	15	"
"	86·8	"	"	20	"
"	107·5	"	"	25	"
"	128·8	"	"	30	"
"	149·6	"	"	35	"
"	171·0	"	"	40	"

But if there had been no friction in the first experiment, A would have been 20 ounces instead of 23·4, hence the friction is represented by this 3·4 ounces. For every experiment let this be done, subtract four times B from A and call this difference the friction. Now how shall we compare this friction with the corresponding load?

53. The Use of Squared Paper.—And here we come to a matter of the greatest importance to the practical man. How do we practically compare two things whose values depend on one another? How do we find out the **law of their dependence**? It is a strange fact that there should be a class in

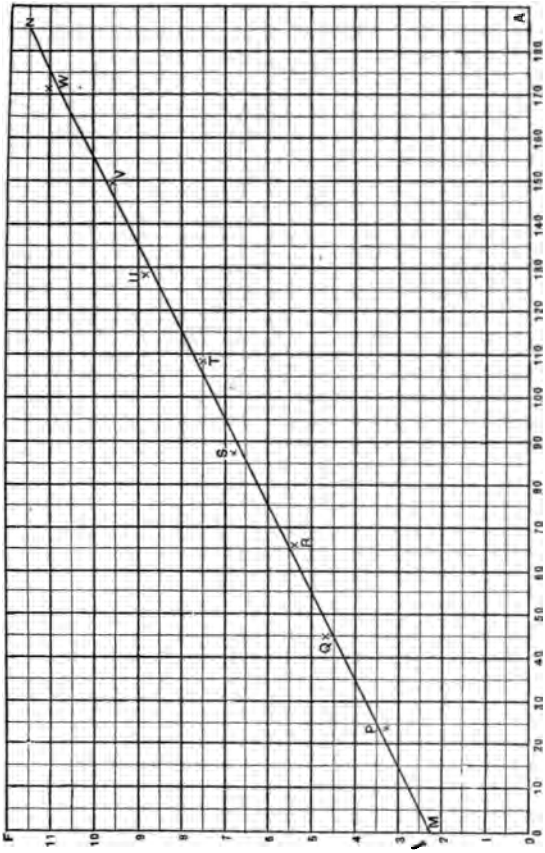

Fig. 23.

the community who have a little difficulty in manipulating decimals in arithmetic, but it is almost a stranger evidence of **neglected education** that so many people should be ignorant of the great uses to which a sheet of squared paper may be put.

A sheet of squared paper can be bought very cheaply. It has a great number of *horizontal lines* at equal distances apart, and these are crossed by a great number of vertical lines of the same kind, so that the sheet is covered with little squares. This sheet will enable me first of all to correct for **errors of observation** in the above series of experiments; and, secondly, to discover the **law** which I am in search of. A miniature drawing is shown in Fig. 23, many lines being left out because of the difficulties of wood-cutting. At the bottom left-hand corner I place the figure 0, and I write 10, 20, etc., to indicate the number of squares along the line, 0 A. Instead of 10, 20, etc., I might write 1, 2, etc., or 100, 200, etc., according to the **scale** I am going to use. Indeed, on account of the friction being so much less than the weight A with which it is to be compared, I number the squares along the vertical line 0 F by 1, 2, etc., instead of 10, 20, etc. **We can employ any scale we please** in representing any of the things to be compared, and it is usual to multiply all the numbers of one kind by some number, so as to represent all our experiments on one sheet of paper, and on as much of this sheet as possible. Having subtracted four times B from A, I find the following numbers :—

A.	Friction.	A.	Friction.
23·4	3·4	107·5	7·5
44·7	4 7	128·8	8·8
65·4	5·4	149·6	9·6
86·8	6·8	171·0	11·0

I now find on my sheet of paper the point P, which is 23·4 horizontally and 3·4 vertically, and mark it with a cross in pencil. Q is 44·7 horizontally and 4·7 vertically, and so for the others. The last point, W, is 171 horizontally and 11·0 vertically. We guess at the decimal part of a small square. The point P represents the two numbers of my first experiment, and every other point represents the two observations made in one experiment. Now we are certain that if there is any **simple law** connecting load and friction, the points P, Q, to W, lie in a simple curve or in a **straight line.** You see that in this case no curve is needed to suit the points; we assume that they would lie in a straight line, only that we made some errors of observation. You must now find what straight line lies most evenly among all the points; this you can do by means of a fine stretched **string**, and the line M N seems to me to answer best. It tells me, for instance, that when A is 44·7 the friction is *really 4·5, instead* of 4·7. Take any point in the line, its vertical measurement gives me the true friction corresponding to a load represented by its horizontal measurement. Thus, for instance, you see that friction 5·0 corresponds to load 55.

This is a simple way of correcting errors made in experiments, but you cannot hope to understand much about it till you actually make experiments, and use the squared paper. You will find the matter all very simple when you try for yourself: my description of it is as complicated as if I were teaching you by mere words to walk, or to bicycle.

If at any time you make a number of measurements of **two variable things** which have some relation to one another, plot them on a sheet of squared paper, and correct by using a flexible **strip of wood** or a ruler, to draw an easy curve or a straight line so that it passes nearly through all the points. If the line is straight, the *law connecting the two things* will prove to be a very simple one. In the present case it means that any increase in the load is accompanied by a proportionate increase in the amount of friction. Thus, when the load is 0, the friction is 2·3; when the load is 100, the friction is 7·2. That is, when the load increases by 100, the friction increases by 4·9, so that the increased friction is always the fraction ·049, of the increased load. In fact, it is evident that we can calculate the friction at any time from the rule

$$\text{Friction} = 2{\cdot}3 + {\cdot}049\,L.$$

That is, multiply the load L in pounds by ·049 and add 2·3; the answer is the friction.

54. Law of Friction.—Our result is that the friction is equal to the friction 2·3 of the unloaded machine, together with a constant fraction, ·049, of the load. Now, when a similar series of experiments is made on any machine, whether it be a watch or clock, or be it a great steam-engine, we always find this sort of simple law.

If you clean all the bearings or parts, or if you use a different kind of lubricant, you will get other values for the two numbers in the above rule, but the law will remain of the same simple kind. I find it nearly impossible to get my pupils to believe that rough and rusty old machines, such as screw-jacks or hydraulic jacks, which have been long in use, are far more instructive to study than beautiful, specially made, frictionless machines. In my laboratory, now, at Finsbury, there is an ideal screw-jack; the weight P (Fig. 40) is not a single weight, but two equal ones at the ends of two cords which pass round two equal grooves and produce a **true couple** in turning the screw. I prefer the rough old thing previously in use, just as I prefer the cheap **Attwood's machine** (Fig. 163), which used to be employed in my laboratory, to either of the two elaborate machines which are now in use. The pulley of Fig. 21, which I use, is out of balance and badly made; it gives ever so much more instruction than if it were so expensively and correctly made that the friction which one wants to measure had almost disappeared. My laboratory crane is a model crane, much too carefully constructed; but my hydraulic jack and differential pulley block are the real things, made "for human nature's daily use." **Our object** is not to find out how to make a machine with the most frictionless bearings; else we should find it instructive enough to work experimentally with ball bearings, such as are used in cycles (Art. 70), and with friction wheel bearings. Again, a student is told to **judge with his eye** as to whether a weight A is falling steadily, with a uniform velocity. Let him find out for himself how much of the steadiness is due to irregularities in the rubbing surfaces of the machine, and how much can be altered by altering the weight. Do not spoon-feed him. Let it be a discovery of his own that his eye is somewhat defective as a speed-measurer; he will be led to suggest plans for more accurate working if you refrain from forcing upon his attention your elaborate plans. Indeed, your electric and other contrivances for measuring velocity may be so elaborate as to hide altogether from a student the main object of his experiments; just as when a young student works with a 200-ton testing-machine, it is almost impossible for him to think of the little specimen of material which is being tested, the testing-machine itself takes up so much space.

55. Force of Friction.—I have in all this used the term "friction," or the term "effect of friction," to mean the difference between the weight which would **balance** another through the mechanism if there were *no* resistance to the rubbing of surfaces, and the weight which will just **overcome** the other when there *is* such resistance. Observe that there is a great difference between the cases, A overcoming B (Fig. 21), and B overcoming A. What we have called friction is due to the *rubbing at all sorts* of surfaces in all sorts of directions, at all

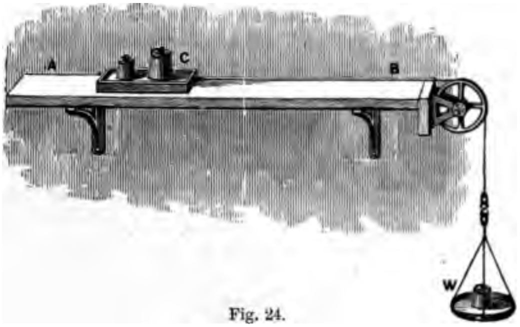

Fig. 24.

sorts of velocities under all sorts of pressures, and we are led to study it in its simplest form, where at one part of a pair of surfaces the rubbing is exactly of the same kind and in the same direction as at another part; so that we may speak of the **resultant force which resists motion** as the force of friction. Experiments may be made upon the apparatus shown in Fig. 24, where A B represents a table, the upper level surface of which is wood, iron, brass, or other material to be experimented upon. We usually experiment on smooth surfaces. C is a little **slide** made of any material whose **coefficient of friction** with the table we wish to find. Different weights may be placed on it. The weight of the slide, together with the weight lying upon it, is the total force (R) pressing the two surfaces together. C is pulled by the weight, W, hung from a string, passing over a pulley working on very frictionless pivots. The weight, W, which will just cause the slide to keep up a steady motion on the table, is taken as a measure of the friction. Of course, however, it really includes the resistance of the pulley, but this is usually neglected, as we know from previous experiment that it is small. It is found necessary to start the slide by giving a little **jerk** to the arrangement, as the friction when the slide is motionless is found to be somewhat **greater** than when it is moving. This is one of the most instructive experiments which can be made in mechanics, and I hope that every reader will make a series of observations. Let him correct his results by means of squared paper, and he will find it nearly true that the friction is a constant fraction of the force pressing the surfaces together. This fraction is called the **coefficient of friction** and usually denoted by μ.

give its value for a few surfaces, but a student had better depend upon the values which he himself arrives at; they will not be the same; they may differ greatly from these.

Oak on oak, fibres parallel to direction of motion	0·48
,, ,, perpendicular ,, ,,	0·34
,, ,, endwise ,, ,,	0·19
Metals on oak ,, parallel ,, ,,	0·5 to 0·6
Wrought iron on wrought iron, wrought iron on cast iron	0·18
Cast iron on cast iron	0·15

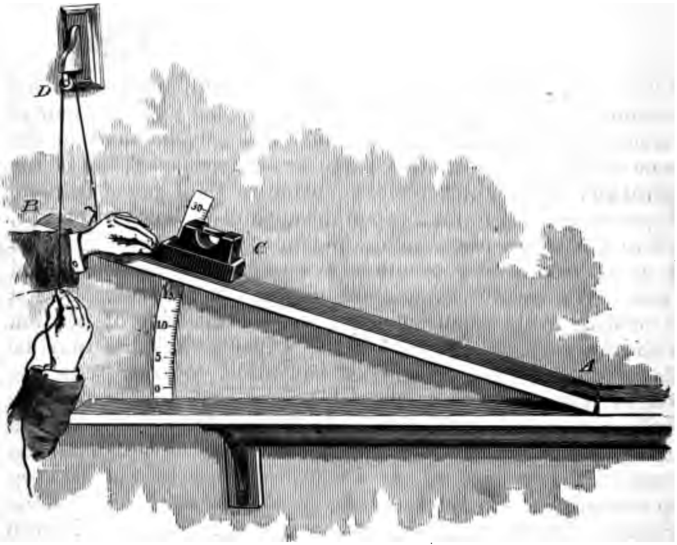

Fig. 25.

A layer of oil or other lubricant between the surfaces will greatly reduce the friction. Figures given for the coefficient of friction when a lubricant is used are, however, very greatly misleading, but for student's exercise work μ may be taken anything between ·04 and ·01 for sperm oil, between not too generously but continuously lubricated surfaces, being twice as much for greases as for sperm oil.

It is very interesting, after determining the coefficient in *the case* of a certain pair of materials, to diminish the size of the slide. **You will find** that unless you diminish it so much that the pressure actually alters the surfaces in contact from being quite plane you will get pretty much the same result. You will also find that, whether the motion of the slide is quick or slow, if your weight maintains the motion steady when it is slow it will also maintain it steady when quick.

If, instead of using a cord and weight, we move the slide by **tilting** the table more and more from the horizontal (see Fig. 25), the slide getting an occasional shove to start it, let the inclination of the table be found in degrees when the weight of the body itself is just able to keep up a steady motion. The **tangent of the angle of inclination** of the table when this occurs can be found in a book of mathematical tables; it proves to be equal to the coefficient of friction. This method of experimenting is much easier and is more exact than the other, but it is not so instructive for a beginner. (See Art. 90.)

56. Loss of Energy due to Friction.—In the simple case with which we began Art. 50, the difference of pull in a cord on the two sides of a pulley was what we called the friction of the arrangement, whereas we see that the friction takes place at every point where rubbing occurs, not only at the pivot but even in the fibres of the cord itself, and the force at one point may be very different from the force at another point. Again, any force acting on the cord has a greater leverage about the axis than any of the forces of friction has. **The real connection** between the two things is, then, this: what we have generally called "the effect of friction," or "the friction of the arrangement," multiplied by the velocity of the cord on which it is measured, is equal to the sum of all such products as the friction at any point in a rubbing surface, multiplied by the velocity of rubbing. In fact, if the weight A in falling causes the weight B to rise, the work done by A is greater than the work done on B by an amount which is called the **work lost in friction**, and this is the work done against the forces of friction at all the rubbing surfaces.

If we know the force of friction at any place, *in pounds*, and the distance, *in feet*, through which this force is overcome—that is, the distance through which rubbing has occurred—the **product of force** by **distance** measures the work or energy spent in overcoming friction, in *foot-pounds*. This energy is all wasted, or rather, it is all changed into heat and does not come *out* of the machine as mechanical work, the shape in which it was when we put it *into* the machine. And inasmuch as no machine can be constructed which will move without friction, we never get out of a machine as much mechanical work as we put into it.

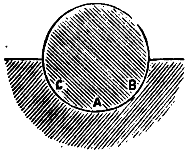

Fig. 26.

57. Friction at Bearings of Shafts.—At almost every rubbing surface which you can consider, the force of friction is different at every point of the surface, and it is generally acting in different directions at different points. Consider, for example, a horizontal shaft and its bearing (Fig. 26). The force of friction at C, per square inch of area of rubbing surface, is probably not the same as at A. A very little difference in the sizes of the **journal** and **step** will cause a considerable difference in the pressure per square inch at C or at A. Now the force of friction at C, multiplied by the velocity of rubbing, gives the work or energy lost per second in friction at C; and this, added to the energy lost at every other place where rubbing occurs, gives the total loss of energy per second at all the points. It is not, then, a simple matter to investigate the force of friction at every point of such a bearing; and the rigidity of the metal, and a number of other important matters, not to speak of the nature of the lubricant, must be taken into account in investigating the force of friction everywhere when the shaft is transmitting different amounts of **power**. As we have already seen, however, **experiment shows** that the energy lost in friction for a certain amount of motion increases proportionately with the energy actually transmitted by the shaft. Keeping in mind, then, the general law—"the force of friction is proportional to load"—it is easy to see how to **reduce the frictional loss** in any machine. For instance, when a wheel is transmitting power, the load on the rubbing surfaces of its bearings or pivots depends on the power transmitted. Now, the actual **force of friction** at the rubbing surface is about the same, whatever be the size of the bearing; but the **distance through which rubbing occurs** when the wheel makes one revolution is less as we have a less diameter of bearing; in fact, the force of friction, multiplied by the circumference of a cylindric bearing, is the energy in foot-pounds lost in one revolution. Our rule is, then, to make this diameter as **small *as possible***, consistently with sufficient **strength**. The wheel of a carriage is made large, and the axle, where rubbing occurs, is made as small as possible, because in this way the carriage moves over a great distance for a small amount of rubbing. There is another reason, however, for the use of large wheels in carriages on common roads—namely, their requiring a less tractive force to get over obstacles, such as stones. In some machines, where it is important that there should be very little friction at the bearings of axles, the axles are made to lie at each end in the angle formed by two wheels with plain rims. The main axle rolls on these wheels, and it is only at the axles of the wheels that there is rubbing. This rubbing is a very slow motion, and as the *force* of friction is but little increased in consequence of the weights of the **friction wheels**, the energy lost in friction may be made very small in this way. Every curious student is aware of the way in which *rolling* takes the place of *sliding* in the **ball-bearings** of cycles. This kind of bearing will probably be greatly used in ordinary machinery. The resistance is as if we had a co-efficient of friction inversely proportional to the diameter of the ball or friction roller, because of indentation of the rolling surfaces.

If you compare Watt's **parallel motion** which is still used in some pumping engines to cause the piston-rod to move in a straight line, with the **slide** which is now so common, you will see that there is very much less loss of energy by friction when the parallel motion is employed, because, whereas in the slide the rubbing motion is as much as the motion of the piston, in the parallel motion rubbing only occurs at the pins of the arrangement. Unfortunately, this arrangement does not allow the piston-rod end to move *exactly* in a straight line, and produces some friction between the piston and its cylinder, and between the piston-rod and stuffing-box; and it is also much more costly and less compact than slides. Hence slides are now in general use.

58. In a **journal** of length l and diameter d, if w is the load and μw the force of friction, at n turns per minute, the velocity being proportional to nd, the rate per second at which **heat** is developed per square inch of area is proportional to $\frac{\mu w n d}{l d}$. Calling $\frac{w}{l d}$ the pressure p, we see that pnd ought to be constant in all journals if they are to have about the same **rise of temperature** above surrounding objects, because the giving out of heat by a surface per square inch may be taken as proportional to this rise of temperature. *This rule is found to* be somewhat misleading for

lubricated bearings, because μ is not by any means constant. The values of pni found in practice are 1,000,000 to 1,500,000 in locomotive crank pins, calculated from full pressure and speed; 250,000 in marine-engine crank pins; 60,000 to 200,000 in stationary engine crank pins; crank shaft bearings, 36,000; railway carriage axles, 300,000. When what is understood to be a constant in one's theory varies between 30,000 and 1,500,000, it strains one's sense of humour to maintain the gravity necessary in the writer of a text book. It is evident that we must pay more attention to mere pressure; and it may be said that in practice it is the rule not to greatly exceed 200 lbs. per square inch in shafting, unless there is bath lubrication, and then the limit is 500 lbs. per square inch; 600 lbs. per square inch in crank pins; 1,000 lbs. per square inch in crosshead pins.

The practical engineer has by processes of **success** and **failure** arrived at dimensions in **machine design** which we have always the desire to see reasons for in our theory. It is, however, sometimes forgotten that a complete theory must be a very complicated one, and attempts to deduce (find reasons for) certain very useful rules (sometimes impertinently called rules of thumb) from very imperfect theory, do not always succeed.

It would be interesting to find out why it is the universal practice of good engineers to make the ratio of length l of a bearing to its diameter d increase nearly in proportion to the number of revolutions per minute. It has usually been lost sight of that these bearings never occur in long lengths of shafting—only in separate machines like **fans, centrifugal pumps,** and **dynamo machines.** In long lengths of shafting, bearings, as their name implies, are mainly used as mere supports; but in separate machines they not only carry weight—their function is, especially in light machines running at great speeds, to keep the shaft **fixed in direction.** If then there is any bending moment M in the spindle, due to centrifugal force through want of balance, or due to other causes, it is easy to show that the pressure per square inch of bearing, besides what is due to steady load, is proportional to M/l^2d, and this multiplied by n is supposed to be kept constant. If M is proportional to the twisting moment, or to the horse-power, or d^3n, we have a rule $l/d \propto n$ which agrees with the practical one. I have myself worked out such a rule for M as a very likely one in certain kinds of dynamo machines. In all probability the rule for any quick-speed machine would turn out to be this: that if questions of cost of construction and space did not intervene, the ratio of l to d ought not only to increase with n, but also with L, the distance between the main bearings of the machine. Where there is a possibility of error in the allineation of the two bearings, we have a reason for the ratio of l to d increasing as the square root of the speed.

Exercises.—1. Find the horse-power necessary to turn a shaft 9″ diameter, and making 75 revolutions per minute, if the total load on it is 12 tons and ϕ, the angle of friction, is such that sin. $\phi = \cdot015$. *Remember that tan. $\phi = \mu$.* *Ans.*, 2·16.

2. Find the horse-power absorbed in overcoming the friction of a foot-step bearing 4″ diameter, the total load being 1½ tons, the number of revolutions 100 per minute, and the average co-efficient of friction ·07. *Ans.*, 0·5 nearly.

3. If it be assumed that the power wasted between the end of a flat pivot and its step is proportional at each point to the product of the velocity and pressure, what horse-power will be absorbed by such a pivot, 3″ external diameter and 1″ internal diameter, when running at 120 revolutions per minute, the load on the pivot being 2¼ tons, and the average co-efficient of friction ·06? Integration gives the total energy wasted per second as $\frac{2}{3} \alpha \mu W R$, where α is the angular velocity in radians per second, W is the total load, R is outside radius of pivot, and μ is the coefficient of friction. *Ans.*, 1·535.

4. The length of a journal is 9″ and its diameter 6″; it carries a load of 3 tons. What horse-power is absorbed in friction when making 100 revolutions per minute, the average co-efficient of friction being ·015? What number of thermal units per minute will be conducted away per square inch of the brass? *Ans.*, 0·48; 0·2 centigrade heat units.

5. A shaft makes 50 revolutions per minute. If the load on the bearing be 8 tons, and the diameter of the bearing 7 inches, at what rate is heat being generated, the average co-efficient of friction being ·05?

If 3 thermal units escape per minute when the temperature of the bearing is 1° C. higher than that of surrounding objects, what will be the increase in temperature caused by the heat produced at the bearing? *Ans.*, 19°·6 C.

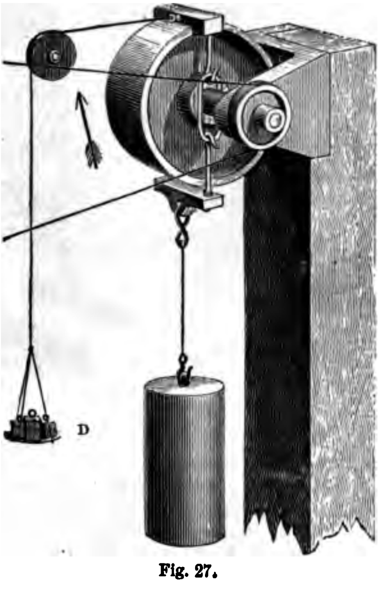

Fig. 27.

59. Friction and Speed—You will find it instructive to experiment with such a piece of apparatus as is represented in Fig. 28, designed to measure the friction between sliders of different materials and the cast-iron surface P. Here we have a pulley with a broad, smooth outer surface. On this surface lies a slide made slightly concave, to fit the rim of the pulley. On this slide we can hang different loads W by the arrangement shown in the figure, and the slide can only move a small distance in any direction on account of stops. There is a fly-wheel to give steadiness of motion when the apparatus is worked by hand. Suppose, now, that P rotates in the direction of the arrow. Friction causes the slide to move in the direction of motion until it is brought up by a stop. Now let weights be placed in the scale-pan *w* until the slide is held in a position half way between the stops. Evidently the force of friction between the slide and P is just balanced by the weight in the scale-pan. With this apparatus you can not only find the co-efficient of friction for two rubbing surfaces easily at any speed, but you can very quickly vary your experiments. S is a speed counter.

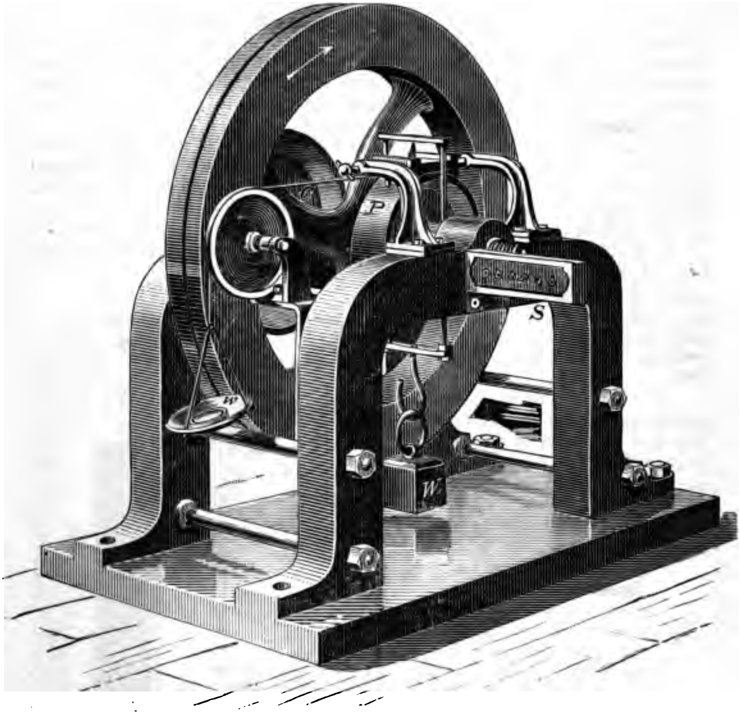

Fig. 28.

To what extent I ought to be ashamed of the following facts I don't know. They are instructive. Successive generations of *students* at Finsbury obtained results from an apparatus like Fig. 27.

It was arranged to be driven by the college engine at many different speeds; and there was a speed counter. The slide had a much longer arc of contact than is shown in the figure, because a rocking motion, instead of sliding, was apt to be set up. The load was applied at a single point in the centre of the top of the slider. In every case when the load was kept constant the friction was **greatest at low speeds**; it got less and less as the speed increased, and reached a minimum value; after which it increased steadily as the speed increased, the rate of increase of friction with speed getting less towards our highest speeds. I thought these curves obtained by students well worthy of special study, and several times projected an investigation of my own, and urged others to take it up; but I was too busy with other matters to give it much attention. Now, from **Prof. Osborne Reynolds's** explanation of the results of **Mr. Beauchamp Tower's** experiments one sees how very different the phenomenon is from what occurs between flat surfaces. Air is being pumped by friction into the space between the slider and pulley, and the pressure underneath the slider is greater than the atmospheric pressure, and varies from point to point, as we have proved by inserting little pressure-gauges.

We are about to use now another piece of apparatus (Fig. 29), the slider being flat, and lying on a circular, flat horizontal plate, which may be kept rotating at any speed. The slider is prevented from moving much by stops, and friction is balanced, as in the above case, by a scale-pan. The apparatus is only now (July, 1896) being run for the first time. What sorts of results will be obtained from it I do not know.

60. Although, on the whole, in any machine the average forces of friction do not seem to depend much upon speed, and they increase in proportion to the load, and in other ways seem to follow the laws set down in Art. 55, when we make experiments on the friction at any one place in a machine we obtain inconsistent results. So long as our theory of an action is wrong, our experiments give rise to what are called inconsistent results. On the hypothesis of Art. 55 the mathematicians have built a science, and thousands of examples and exercises have been invented to illustrate it. The exercises and examples are valuable to the engineer; but he must remember that they have been invented by mathematicians for the training of mathematicians, and he must exercise caution in using the results. I give some examples in Art. 58. Two substances, when they really touch, get welded together; and this **seizing** seems to occur in some **journals** and **footsteps** under heavy loads. Bodies said to touch or rub on one another are really separated by a layer of air or other fluid. A slider like C (Fig. 24) is separated from the table AB by a layer of air; and the greater the load, the less is the thickness of air.

Students will find it very interesting to study the friction between two **scraped surfaces** in the workshop. If one plate is laid down on the other, there is usually very little friction, because there is a thick layer of air separating the surfaces. By putting on a load, and giving small sliding motions, we can make the layer of air very thin—so thin, indeed, that when the top plate is lifted the bottom one sticks to it, and lifts also, partly because there is a partial vacuum between them, partly also, probably, because of molecular attraction seeing that it occurs even in a good vacuum. Now, when, through the one plate having lain on the other a considerable time, or through pressure, we get the layer of air very thin, it is found that there is considerable resistance to sliding. In fact in this case, where we might expect to find the phenomena of friction assuming their simplest form, we find what seem to be the most inconsistent results. When c (Fig. 24) slides, it seems as if fresh air were being carried into the space between the surfaces, keeping them apart, and that the greater the velocity of rubbing the more air is carried in, so that the separation between the surfaces is proportional to the velocity of rubbing. When we come to discuss fluid friction, and reflect that all the friction we know of is really fluid friction, we shall not be astonished that the laboratory results from the rubbing of solids are sometimes inconsistent-looking—we shall wonder greatly that the above-mentioned law should be even approximately true. But I cannot think the effect merely one of the fluid, there is also molecular attraction. In any lubricated bearing which is not kept flooded with oil, the rules called the laws of solid friction are found to be approximately true—that is, we may take the load on the bearing in pounds, multiplied by a coefficient μ, as representing a force of friction; and this, multiplied by the distance of rubbing in feet, is the mechanical energy converted into heat by friction. μ *is less as* the temperature is greater, partly because the viscosity

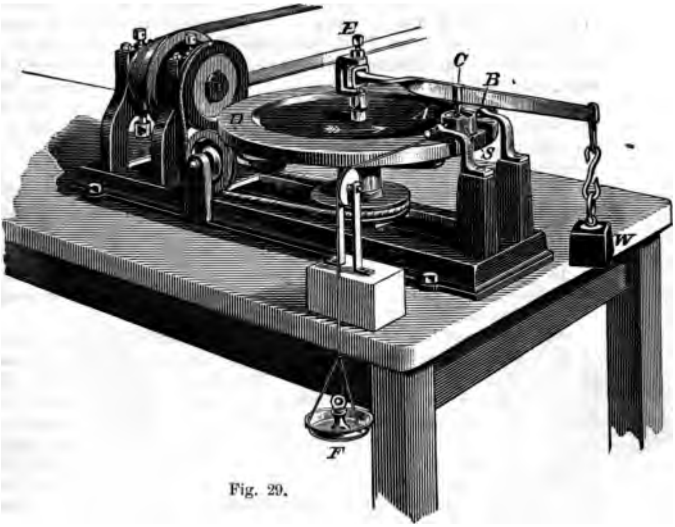

Fig. 29.

of a fluid diminishes with temperature (see Art. 63); but it is not only because of this, for the **body of the lubricant** also alters with temperature—that is, it tends more to get squeezed out of place. Of course the friction depends greatly upon the nature of the lubricant, and the phenomena are really so very complicated that in the present state of our knowledge the reader must perforce be satisfied with the rough general law that I have mentioned. That the **supply of oil**, however small, shall be **continuous**, and not intermittent, is regarded as the most important condition in the lubrication of bearings. The lubricant ought to suit the nature of the load. Thus great body is necessary when the loads are great, so that the oil may not be squeezed out; and greases and solid lubricants, such as soapstone and plumbago, must be used for very heavy loads. It is also usual to cast plugs of **white metal** and other soft alloys in recesses of the step, and in some cases to line the whole step with such a soft alloy. As for the detailed construction of pedestals, hangers, A frames, and other supports for shafting, this is to be learnt in the drawing-office and shops, and it would be useless to refer to it here.

61. As it has been found that with some kinds of material the **statical friction**—that is, the friction which resists motion from rest—*is somewhat greater than the friction of the surfaces when actually moving*, **experiments** have been made to determine whether, at **very small velocities** indeed, with such materials, there is not a gradual increase in the friction. It is known that at ordinary velocities the friction is much the same as at a velocity of ·01 foot per second. We have reason to believe that with **metals on metals** and air between, there is the same friction at all velocities, even down to one-five-thousandth of a foot per second, whereas with **metals on wood** the friction increases gradually as the velocity diminishes, until when the velocity is 0, the friction is what we call **static friction**. Again, **at very high velocities** it has been found that there is a very decided diminution of the coefficient of friction between a cast iron *railway brake* and the wrought iron tyre of a wheel. The coefficient was ·33 for very slow motion, ·19 for a speed of 29 feet per second, and ·127 for a speed of 66 feet per second. It has also been observed in these railway brake experiments that when a certain pressure is applied for a short space of time the friction diminishes. All such results as these, however interesting they may be to the railway engineer, tell us nothing about what I have hitherto called friction, because I have supposed the rubbing surfaces to **remain unaltered**, whereas these railway brakes are rapidly worn away, and the *effects of abrasion* and polishing are of an utterly different kind from the effects of friction of which I have hitherto been speaking.

62. We must remember that although friction leads to waste of energy, all the energy spent in overcoming friction being converted into another form of energy called *heat*, still the force of friction is very useful. The weight resting on the **driving-wheels of a locomotive engine** multiplied by the coefficient of friction between the wheels and rails represents the greatest pull which the engine can exert upon a train. Suppose the weight on the driving-wheels to be 15 tons, and that the coefficient of friction of wrought iron on wrought iron is about 0·2, the greatest pull which the locomotive can exert is 15 × 0·2, or 3 tons. If the train, including the locomotive itself, resists with a greater force than this, the driving-wheels must slip; if the train resists with a less force than this, there is no slipping, the wheels simply roll on the rails. Again, it is the friction between the soles of our feet and the ground that enables us to walk; friction enables us to handle objects; friction enables a nail to remain in wood; friction keeps mountains from rolling down.

63. **Fluid Friction.**—I have been considering the friction between solid bodies only. The friction between liquids and solids or between liquids and liquids is of a very different kind. If a man attempts to dive into water unskilfully, and falls prone, you know that the water offers a very considerable resistance to a change of shape. Now this is mainly the resistance that any body offers to being rapidly set in motion. If you came colliding against the end of the most frictionless carriage, you would also experience its resistance to being **suddenly set in motion**; whereas the constant **steady** resistance to motion which the carriage experiences when moving with a uniform velocity is called friction. What I wish rather to refer to is **the resistance to the motion of water in a pipe**, the resistance to the steady motion of a ship.

In nearly all ordinary cases the motion is complicated and difficult to study. The simplest motion is in plane parallel layers. Imagine two infinite plane parallel boundaries with the fluid between: one of the boundaries at rest, the other moving with uniform velocity v in its own plane. Imagine the fluid to stick to each boundary. If b is the distance between them, the tangential force per unit area required to keep up the motion is $\mu \mathrm{v} \div b$ if μ is the **coefficient of *viscosity*.** *Theory* shows that μ ought to be constant if the motion is truly in plane layers. As we cannot experiment with infinite surfaces, I thought that I could approach the condition most nearly with the apparatus shown in Fig. 30. F is a hollow cylindric body supported so that it cannot move sidewise, and yet so that its only resistance to turning is due to the twist it would give the suspension wire, A. C C is water or other liquid filling the annular space between the cylindric surfaces D D and E E, and wetting both sides of F. When the vessel D D, E E is rotated, the water moving past the surfaces of F tends to make F turn round, and this frictional **torque** is resisted by the twist which is given to the wire. The amount of twist in the wire gives us, then, a measurement of the **viscosity** of liquids, and investigations may be made under very different conditions.

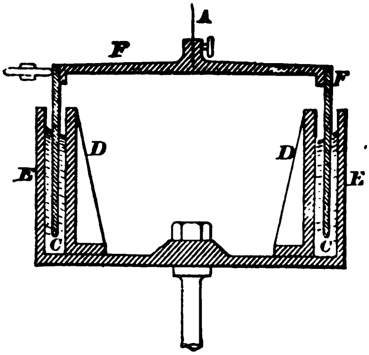

Fig. 30.

The above apparatus was designed and partly constructed in Japan in 1876. Experiments made with it by Finsbury students on olive oil are described in the Proceedings of the Physical Society of London, March, 1893. At constant temperature below a certain **critical speed**, I found that the friction was **proportional to the velocity**, so that μ could be found. At that critical speed I found that there was a sudden change in the law, and above that speed the friction is proportional to a higher power of the speed than 1. We know that above the critical speed the plane motion which I described above would become unstable, and eddies would be formed. From a theoretical point of view it is curious* that the phenomenon should have been so marked between my cylindric surfaces, because even at slow speeds cylindric motion ought to be unstable inside a fixed cylindric surface—that is, in the inner part of my trough.

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* Some experiments of Mr. D. Baxandall, not yet published, show that the forces of friction—that is, the resultant forces applied to the solid bodies which form the boundaries of a mass of fluid to maintain relative motion—are strictly proportional to the relative velocity at small speeds, and there is always a critical speed above which the friction is proportional to a higher power of the velocity. We have tried surfaces arranged like those of churns, with paddles and curiously shaped vanes, and the law is always true. With mixtures of glycerine and water, the higher power of the velocity above referred to depends on the *proportions of* glycerine and water.

At speeds below the critical, I measured μ at many different temperatures, and noted the rapid decrease in it as the temperature increased.

Very interesting observations may be made at small speeds by immersing similar and equal heavy **discs of brass in air, water,** and **oil,** suspending them by fine steel wires. (See Fig. 30.) When the suspension wires are twisted and let go, the bodies vibrate like the balance of a watch. But it is only the one which vibrates in air that goes on vibrating for a long time; the one in water keeps up its motion longer, however, than the one in oil, showing that there is more frictional resistance in oil than in water, and more in water than in air. The *rates of diminution of swing* or the *stilling of the vibrations* tell us the relative viscosities of the fluids. If, by means of a pointer or mirror attached to the wire, you observe the various angular displacements, noting the time for each, and then plot your observations on squared paper (as in Art. 53), you will find what is very nearly a **curve of sines** for the vibrations in air; and for the different liquids **damping curves**, which show the effect of friction in the liquids. Similarly, the rate of diminution of swing of the vibrating fluids in **U** tubes, one containing water and the other oil, tells us about the relative co-efficients of viscosity of the liquids.

64. The motions in these cases are not so simple as in the case which I considered experimentally. The question of the resistance to the passage of **fluids through pipes** is one which has attracted much attention, and the results of experiments seemed very inconsistent until Professor Osborne Reynolds considered the problem. It was known that the pressure difference at the ends of a level uniform pipe necessary to produce a certain flow was proportional to the length of the pipe, and it was usual to say that the force of friction was proportional, as in all other cases of fluid friction, to the wetted area; it is quite independent of the pressure; it is proportional to the velocity of the water when the velocity is small, but at high speeds it increases much more quickly than the speed. Thus, as I said in the first edition of this book, of water flowing in a certain *pipe, "at the* velocities of 1, 2, 3, etc. inches per second, the friction is proportional to the numbers 1, 2, 3, etc., whereas at the velocities of 1, 2, 3 yards per second the friction is proportional to the numbers 1, 4, 9, etc." At small velocities, three times the speed means three times the friction; whereas at great velocities, such as those of ships, three times the speed means nine or more times the friction. We see, then, that friction in fluids is proportional to the speed when the speed is small, to the square of the speed when the speed is greater, and at still greater speeds the friction increases more rapidly than the square of the speed. The resistance to motion of a **rifle bullet** is proportional to the square root of the fifth power of the speed; that is, a bullet going at four times the velocity meets with thirty-two times the frictional resistance from the atmosphere. Again, it has been found that the friction is much the same **whatever be the pressure**. Thus it is found that when the disc and liquid apparatus is placed in a partial vacuum or under considerable pressure, there is exactly the same stilling of the vibrations.

This fact is illustrated by the apparatus, Fig. 31. Water tends to flow from vessel A to vessel B, through the long tube. Whether the tube is in the position shown in Fig. 31, or in the position Fig. 32, or is acting as a **syphon**, we find the same flow through it; the same quantity of water passes through it per second, although the pressure of the water in the tube in the position Fig. 32 is very much greater than in the position Fig. 31, or again when the tube is a syphon. In the apparatus actually used by me, there is a stopcock in the middle of the tube, and by nearly closing it one is sure that the friction occurs at the place where the pressure is

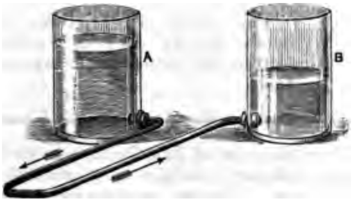

Fig. 31.

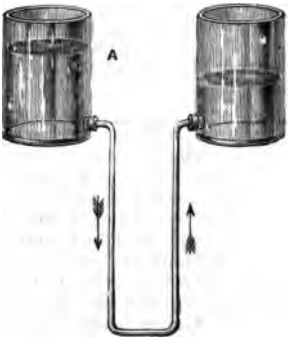

Fig. 32.

greatest in Fig. 32. The comparison is most readily made by observing how long it takes for a certain change of levels to take place in the two vessels, repeating this several times with the tube in various positions, beginning and ending each experiment with the same difference of levels. Again, fluid friction, for even considerable velocities, does not seem to depend much on the roughness of the solid boundary. This seems to be due to the fact that a layer of fluid adheres to the solid surface and moves with it. Even when the disc of Art. 63 is indented, or when large grooves are cut in it we find practically the same frictional resistance.

Comparison of the Laws of Fluid and Solid Friction.

Friction between Solids.	Fluid Friction.
1. The force of friction does not much depend on the velocity, but is certainly greatest at slow speeds.	1. The force of friction very much depends on the velocity, and is indefinitely small when the speed is very slow.
2. The force of friction is proportional to the total pressure between two surfaces.	2. The force of friction does not depend on the pressure.
3. The force of friction is independent of the areas of the rubbing surfaces.	3. The force of friction is proportional to the area of the wetted surface.
4. The force of friction depends very much on the nature of the rubbing surfaces, their roughness, etc.	4. The force of friction at moderate speeds does not much depend on the nature of the wetted surfaces.

65. Molecular theory gives us the cause of fluid friction in such a fluid as air. Layers of fluid at different velocities are continually interchanging molecules by ordinary diffusion; consequently, the relative motion is being destroyed, the rate of loss of momentum by one layer and gain of it by the other enabling us to state that the force required to maintain the motion is proportional to the surface of contact and to the relative velocity. In regard to friction in gases, the explanation is complete, the greater diffusivity at higher temperatures causing the viscosity to be greater also. Indeed, viscosity is proportional to the square of the absolute temperature. In the same way, if two trains were passing one another and the same number of passengers jumped from each train to the other, the trains would become more equal in speed; there would seem *to be a* mutual frictional force between them proportional to the rate of loss or gain of momentum per second. But in liquids there is less viscosity at higher temperatures, although there is greater diffusivity. This is probably due to the fact that mere diffusivity is all-important in gases, the molecules of which exert no forces upon one another except by collision; whereas in liquids, although the greater diffusivity at higher temperatures would tend to make them behave like gases, in regard to viscosity, forces are always acting between the molecules which resist shearing strain, and these forces get less as the temperature increases.

Now, in any case of relative motion between the bounding surfaces of a fluid, beyond a certain velocity, motion in plane layers becomes unstable and sinuous motion sets in. This means that the surfaces across which interchange of momentum by diffusion may take place become greater in area; so that above a certain critical speed I take it that we may expect almost any law connecting friction and speed. Theory shows that for any given shape of surface the critical speed will be less as the density of the fluid is greater, and it is less as μ is less. A very frictionless fluid is very unstable.

I believe that all friction said to be between solid surfaces is to be regarded as taking place in the fluid which always separates such surfaces. This statement seems a mere truism. It is like many another yet to be made by discoverers in applied physics. As a matter of fact, the above table showing the utter difference in character between the phenomena of solid and fluid friction quite hid from everybody's view the fact that all friction must be a fluid friction, until **Professor O. Reynolds** opened our eyes. He has given us in his lectures at the Royal Institution and in his paper published in the Transactions of the Royal Society the suggestion that it is to some extent in the solution of hydrodynamic problems we must look for an explanation of the curious phenomena of solid friction. I have already mentioned a curious phenomenon often brought to my notice in connection with the use of the apparatus shown in Fig. 27. Let me now describe some experiments made on the friction of journals. Probably everybody has been occasionally interested in curious results obtained when testing oils with the **Thurston oil-tester** (Fig. 33). Many of these will be found published in Mr. Thurston's book on "The Materials of Engineering," Part I.; but every mechanical laboratory ought to be provided with the apparatus, that students may study the phenomena for themselves. In Hirn's experiments, made in 1855, he found that the force of friction was proportional to the square root of the product of load and velocity. In the experiments of **Mr. Beauchamp Tower** upon a steel journal with a gun-metal cap only, the cap being loaded, the lubrication being practically an oil bath, the friction was found to be practically independent of the load for loads so excessive as from 100 to 520 lbs. to the square inch (the diameter being 4 inches and length 6 inches, the pressure is taken as the whole load divided by 24), and in all cases to be practically proportional to the square root of the velocity. If, instead of such excessive lubrication as we have in an oil bath, there was only the lubrication due to an oily pad pressed against

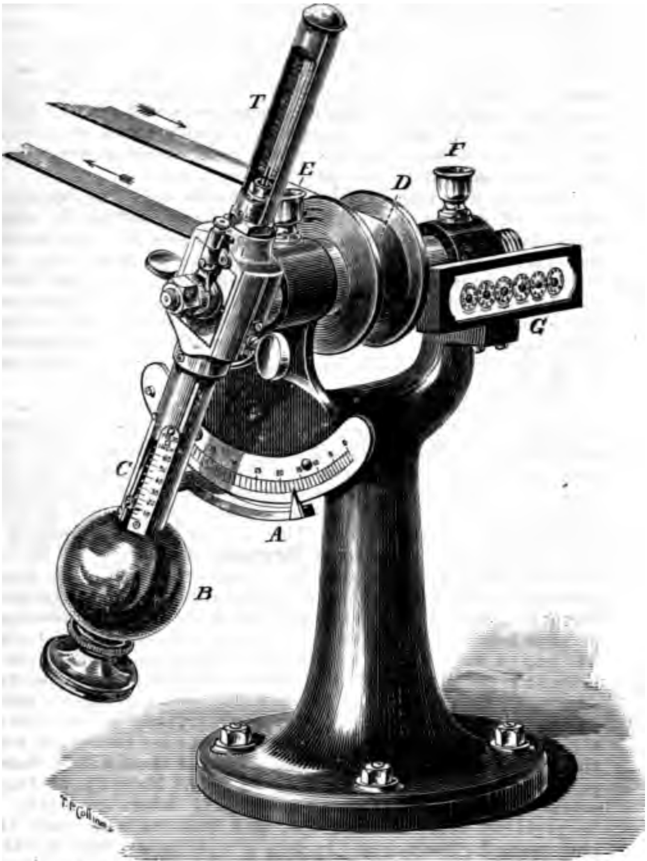

Fig. 33.

the journal below, the ordinary law assumed for the friction of solids was found to be approximately followed. In collar bearings, as in the thrust bearings of a **propeller shaft,** he again found that the ordinary laws of solid friction are fairly well followed, and that only very much less pressures (75 lbs. per square inch at high speeds and 90 lbs. per square inch at low speeds) were possible without seizing. These curious phenomena have been completely *explained by* Professor O. Reynolds. They depend upon the carrying of the lubricant into the space between the step and the journal, and if there is not an oil bath and the motion is all in one direction, the lubricant leaves the place where it is most wanted and the journal seizes at comparatively low pressures; whereas if there is such an irregularity of motion or reversal of motion as helps the lubricant to maintain its place, very great pressures may be employed. Thus in crank pins and **railway axles** pressures as high as 400 lbs. per square inch with sperm oil and 600 lbs. with mineral grease have been used; and, indeed, in slow-moving steam engines nearly double these pressures have been used. It is well to remember that at the place where the journal most nearly approaches the step the pressure in the oil becomes very great, and if there is an opening there the oil is forced out. It is now quite common to employ a **force-pump** to pump oil at great pressure into these parts of the bearings, and in consequence much higher loads on bearings are possible than used to be the case. In any case, we must try to understand the distribution of pressure in the bearing, so that in our endeavours to utilise an ordinary **syphon-lubricator** for example, we shall not be attempting impossible things.*

66. It is in the drawing-office and shops that students will become acquainted with the methods in actual use for supporting horizontal shafts. Footsteps for vertical shafts, which give endless trouble in many high factories, are now in many cases water- or oil-borne, being converted into the rams of hydraulic presses having only a very small range of vertical motion, or, rather, so arranged that the lifting force of the fluid shall always be less than the weight of the shaft.

67. It is when great forces have to be overcome slowly, and particularly with a long translational motion, that **water-pressure machinery** shows itself most greatly superior to other machinery, for friction seems to be nearly independent of pressure. But if in any place the water is set in rapid motion there is internal friction and waste of energy. Where fluids move so slowly that the friction is proportional to the velocity, we seldom consider it in our engineering work. At the valves of pumps and in pipes it is usually, on the whole, economical to let energy be wasted in water friction. There is a certain relationship between velocity, v, and diameter, d, of pipe for a particular fluid which causes a certain v to be critical. Below that value of v the water flows in straight streams; at that critical value the beautiful straight lines which Professor Reynolds shows coloured with aniline dye suddenly break up into confused, smoke-like eddying cloud. Below the **critical velocity** the total pressure difference—or friction, as we may call it—required to keep up the flow is proportional to the velocity. Above that critical velocity the friction is proportional to a power of the velocity which varies from 1·7 to 2, depending upon the nature of the material of the pipe.

68. The mathematical investigation of the resistance to the passage of a body through a viscous fluid is so difficult that we have almost no results which may be relied upon. Without viscosity there would be no resistance to steady motion, whatever the shape of the object. It is difficult to imagine that there would be no propelling force on a sailing-boat if the air were frictionless, and yet this is so. Even in the case of a ship, experiments on which have been going on continuously since ships were first built, our knowledge is very incomplete. Roughly, we may take it that resistance is generally proportional to square of speed. In the case of shot this law holds, probably up to speeds of 300 feet per second; from 400 to 1,000 feet per second the resistance is possibly proportional to the $2\frac{1}{2}$ power of the speed. Beyond 1,100 feet per second we may take, F being in pounds, d the diameter of a shot in feet, v the velocity in feet per second, $F = fd^2 (v - 800)$, where $f = 3$ for spherical and 2 for elongated shots with ogee-shaped heads. The velocity is greater than that of sound, and probably it is to this that the change of law is due. The fact that even in the steadiest winds there is pulsation, causes scientific speculation about wind pressure to be difficult.

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* *See p.* 534.

69. Reynolds has deduced from hydrodynamics the rational formula

$$L\,B^n\ P^{2-n}\ D^{n-3}\ v^n\ /\ A \quad \ldots\ldots \quad (1)$$

as the loss of energy per pound of **fluid** passing through a **pipe** of length L feet and diameter D feet at v feet per second. (I have reduced his numbers to suit the foot as the unit of length.) The index n is 1 below the critical velocity, $v_c = \cdot 039\ P/D$, and n varies from 1·7 to 2 at higher velocities than the critical. P is proportional to the co-efficient of viscosity, which changes with temperature. In the case of water he takes

$$P = 1 \div (1 - \cdot 0336\,\theta + \cdot 000221\,\theta^2) \quad \ldots\ldots \quad (2),$$

where θ is temperature Centigrade.

$$A = 1\cdot 917 \times 10^6; \qquad B = 36\cdot 8.$$

Note that the **critical velocity** depends upon the temperature and size of pipe. Thus, for a tube $\frac{1}{10}$th of an inch in diameter, the critical velocity is 4·65 feet per second; for a pipe 1 inch in diameter the critical velocity is ·465 feet per second; for a 6-inch pipe ($d = 0\cdot 5$) the critical velocity is ·077 feet per second. In all practical hydraulic cases the critical velocity is exceeded, and for a general rule, with cast-iron pipes in actual use, we usually take $n = 2$. In this case, in (1), the influence of P, the temperature term, is unfelt; that is, in practical hydraulic work, temperature has no important influence. The formula now becomes

$$L\,B^2 v^2 / A\,D \text{ or } \cdot 0007\ L\,v^2/D \quad \ldots\ldots \quad (3).$$

As a mnemonic for this simple formula, let the student imagine that a solid prism of water of length L is moved along a pipe rubbing all round its perimeter, the friction being proportional to the square of the velocity and to the area of the rubbing surface. Thus, if s is the wetted perimeter, Ls is the area and the force of friction is Lsv^2; that is, if p is the pressure difference which produces the motion and A is the area of cross-section, $pA \propto Lsv^2$. The loss of energy per pound being proportional to p, this $\propto Lv^2 \frac{s}{A}$. Now, A/s is called the **hydraulic mean depth**, m, of any channel, and we find loss of energy per pound $= c\text{L}v^2/m$ . . . . (4).

In the case of a round channel full of water

$$m = \frac{\pi}{4}\text{D}^2 \div \pi\text{D} = \frac{1}{4}\text{D}.$$

Comparing (4) with (3), the formula of Osborne Reynolds corresponds to $c = \cdot 000175$.

The constant in the formula (3) agrees with that of **D'Arcy** for small pipes. Reynolds has compared (1) with D'Arcy's experiments as well as with his own, from the smallest sizes of pipes to 20 inches diameter, and finds that there is practical agreement. D'Arcy's pipes had joints which somewhat vitiated the results. Reynolds gives for n the values:—Lead-jointed pipes, 1·79; varnished, 1·82; glass, 1·79; new cast-iron, 1·88; incrusted pipe, 2·0; cleaned pipe, 1·91. This formula of Reynolds is rational, and suits every imaginable size of pipe, and I prefer it. But that of D'Arcy is more commonly used for pipes of from 3 inches to 2 feet in diameter, and it is often used in academic exercises, in which, indeed, almost any loss of energy per pound of water (usually called loss of *head*) is expressed as $f \times$ the kinetic energy per pound of water, or $f \times \frac{v^2}{2g}$. D'Arcy gives

$$f = \cdot 02\left(1 + \frac{1}{12\text{D}}\right)\frac{\text{L}}{\text{D}}$$

for a straight pipe of length L feet and diameter D feet.

70. Resistance to Rolling.—When one wheel or cylindric body rolls upon another there is some conversion of mechanical energy into heat. The power lost seems, roughly, to be proportional to the force pressing the two bodies together, to the velocity of rolling, and to the curvature of the smaller of the two. We have very little experimental knowledge of the subject. In all probability the power wasted is proportional to the **strain energy** per second stored in the material, the waste being due to viscosity. When the velocity is very great, as at the driving-wheels of locomotives, secondary effects are produced, waves of compression and extension travelling in the rim of the wheel and in the rail, with very curious results. In some experiments which I have made with great pressures between hard cast iron wheels rolling upon one another, there seems to have been much local heating just at the surfaces. At the end of some months of work a quantity of black dust had been produced, and each particle, when examined by the microscope, looked like a piece of slag.

Besides energy wasted by changing strain in the material, there is slipping at the surfaces in contact. A student who remembers that when a **strut** is compressed it swells, and when a tie bar is lengthened it gets thinner, can study the "**creep**" which occurs both here and in **belting**, for himself. Imagine points one inch apart upon the rim of an iron wheel, and another set upon an unstrained plane indiarubber surface. Now draw the wheel as it indents the surface. As in Fig. 34, points 1, 2, 3, and 4 are further apart, and points 6, 7, 8, 9 are nearer together than in the unstrained condition, and hence the metal and indiarubber surfaces slide upon one another. No ordinary material coating seems to have much effect in preventing the sliding. A cast iron wheel on planes of cast iron, boxwood, and on indiarubber, seems to have frictional resistances to rolling in the proportion of 1 : 2 : 8.

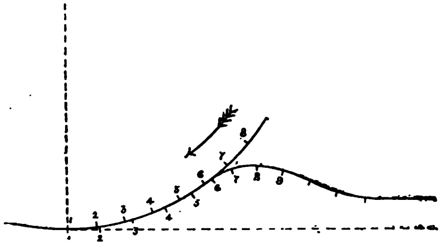

Fig. 34.

The loss of energy in belting is partly due to this, partly due to energy wasted in bending and unbending the belt. Both the lessened distance of rolling and the **slip** of a belt seem to be proportional to the power transmitted. M. Raffard has actually used a **dynamometer** on this principle. He transmits his power, to be measured, by means of a thick indiarubber belt through two equal pulleys; the difference of speed of these pulleys is taken to be a measure of the power. The slip is quite noticeable when speed cones are used in driving machines at various speeds with variable power, for the actual speed has to be carefully measured; calculation from the known sizes of the steps of the cones giving inaccurate results.

When pressures are not too great, as in the ball bearings of cycles and some machine tools, there can be no doubt whatever of the ease of running. Fig. 35 shows the ordinary adjustable **ball bearing** used in bicycles. D is the fork and H the hub of a wheel. The spindle A is fixed to the fork D. One of the hard steel cones C is tight against a shoulder V; the other C′ is tightened just enough to let the wheel revolve easily, and *then it is locked* by the lock-nut K. The linings of the shaped ends of C are hardened steel, and a number of hard steel balls are placed between. Fig. 36 is an enlarged drawing of the ball and the linings, showing that the radii of curvature of the ball, cone, and cup are different; the friction of the bearing will be much less than if the radii of curvature were nearly the same. A very little oil getting inside the hub finds its way to the balls.

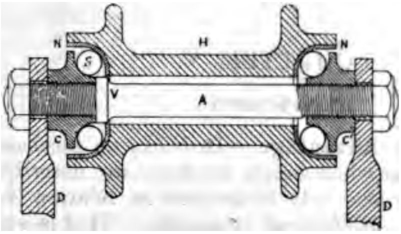

Fig. 35.

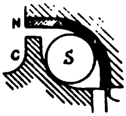

Fig. 36.

CHAPTER V.

EFFICIENCY.

71. Mechanical Advantage.—In books on mechanics you will usually find that when simple machines are described, they are only considered in relation to their *Mechanical Advantage.* That is, suppose a small weight E, now usually called the **effort**, is able by means of the mechanism to cause a larger *weight*, R, usually called the **resistance**, to rise, the ratio of R to E is called the mechanical advantage. Now, in nearly all cases you will find that, when there is a mathematical investigation of a machine, the assumption is made that there is **no friction**. I have already shown you that the problem of taking friction into account is a very difficult one. But, as we have seen, a practical man can experiment on the effect of friction; and, happily for us, he obtains results which are generally very simple. Let the reader make a few experiments himself, or let him by means of squared paper find the relation between E and R from the following results, taken from a crane, Fig. 37, whose gearing was well oiled, and whose handle was replaced by a grooved *wheel*, round which was a cord supporting E :—

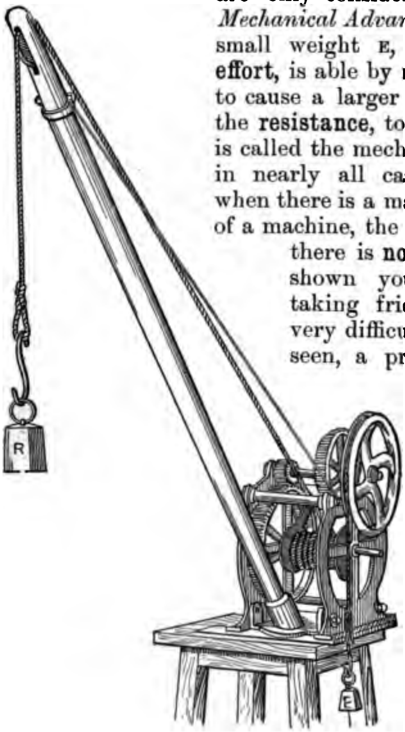

Fig. 37.

R. Resistance just Overcome.	E. Effort just able to Overcome Resistance.
100 lbs.	8·5 lbs.
200 „	12·8 „
300 „	17·0 „
400 „	21·4 „
500 „	25·6 „
600 „	29·9 „
700 „	34·2 „
800 „	38·5 „

We found that E fell forty times as rapidily as R rose, and you may have imagined that the **mechanical advantage** was forty, or that a weight, E, could lift a weight, R, forty times as great as itself. This would be true if there were no friction; but we see that in practice it is not the case. Plot the above values of E and R on squared paper, and you will find that, if the weight R is increased 1 lb., E must be increased ·0429 lb.; and also that when R is 0, an effort E of 4·21 lbs. is needed to cause a slow motion of the crane; so that the law is

$$E = 4{\cdot}21 + {\cdot}0429\ R.$$

Namely, multiply the resistance R in pounds by the fraction ·0429, and add 4·21: the answer is the effort required to lift R. When you have worked out this rule, employ it in finding how much effort, E, is required to lift a ton with such a crane. —*Answer*, 100·3 lbs.

The word **power** is generally, but very unscientifically, used as the name of the force which I have called E, the effort. Power may, of course, be used in many senses by newspaper writers, but when used by the engineer it is a technical term, meaning the **rate of doing work**. If a weight of 1,000 lbs. falls 100 feet in two minutes, it does 1,000 × 100 or 100,000 foot-pounds of work in two minutes, or 50,000 foot-pounds of work in one minute. Now, 33,000 foot-pounds of work done in *one minute* is called **a horse-power**, and hence our falling weight gives out 50,000 ÷ 33,000 or 1·5 horse-power. Ten horse-power means ten times 33,000 foot-pounds of work done in one minute. The idea, then, of *power* is an idea of work done in a certain time.

72. Economical Efficiency.—Take any pair of numbers from the above table, say E = 8·5 lbs., when R = 100 lbs. Let us suppose that E is moving at the rate of forty feet per second, then we know *that R is rising* at the rate of one foot per second. E is giving out the power $8 \cdot 5 \times 40$, or 340 foot-pounds per second; R is receiving 100 foot-pounds per second. The ratio of the power usefully employed to the power given to the machine is called the *efficiency* of the machine, so that our crane has an efficiency $100 \div 340$, or ·294. Sometimes the efficiency is put in the form of a fraction; sometimes we say that it is 29·4 per cent., meaning that the machine employs usefully 29·4 per cent. of the energy given to it.

Now take another pair of numbers, say $E = 38 \cdot 5$, $R = 800$, and let E fall forty feet in one second, as before. We now get as our answer ·519—that is, more than half, or 51·9 per cent., of the power given to the crane is usefully employed. We see, then, that as the power given to the crane is greater, the efficiency is also greater. This arises from the fact that the friction of the unloaded crane is always entering into the calculation; and if we take the case where no resistance, R, is being overcome, and E must be 4·21 lbs., we shall find an efficiency, 0, because work is being given to the crane, and none is coming out usefully. You will always find that the power usefully given out is a certain fixed fraction of the total power given to the machine, minus the power required to drive the crane at the given speed when it is unloaded. Choose some speed, say that E falls forty feet per second; find the total power or 40 E; find the usefully employed power $1 \times R$ for every case of the above table. Plot your answers on **squared paper**, and you will find this rule: if P_0 is the power required to drive the crane at the same speed when unloaded, if U is the useful power, and T is the total power supplied:—

$$T = 1 \cdot 761 \; U + P_0$$

73. In some machines there have been attempts to apportion the loss of power to various parts. As a rule, these are very speculative. Roughly, we may say that the frictional loss in a steam-engine may be divided in the following proportions:—Crank-shaft bearings and eccentric sleeves, 1; valve, if unbalanced, ·6; valve, if balanced, ·05; piston and rod, ·4; crosshead and slides, ·2; crank pin, ·14; total loss because of air pump, 0·3 to 0·5. It is fairly obvious that on account of the great weight of the fly-wheel and other parts much of the energy loss in a steam-engine is the same whether the engine is giving out much or little power, and in many cases for *purposes of* calculation this assumption may be made.

Example 1.—From the following results of experiment with a gas-engine, show, by plotting on squared paper and correcting for errors of observation, that if I is the indicated horse-power, B the brake horse-power, G the cubic feet of gas per hour, including what is used for ignition, then

$$G = 20{\cdot}3\,I + 8,\quad G = 20{\cdot}4\,B + 45,\quad B = I - 1{\cdot}8.$$

I	B	G	G/I	G/B	Efficiency.
13·4	11·6	280	20·9	24·1	·166
10·2	8·4	216	21·2	25·7	·155
7·3	5·4	156	21·4	28·9	·123
4·6	2·9	104	22·6	37·9	·112
1·8	0	45	25·0	–	0

When we plot the values of G and I and of G and B on squared paper, we find points lying very nearly in straight lines. Assuming that they ought to lie in straight lines, we find the above laws satisfied.

Let the student fill in the columns showing G/I and G/B. Also from the brake horse-power, and knowing that one cubic foot of gas per hour means an actual supply of energy of a quarter horse-power, let him fill in the column of efficiencies. [The calorific power of one cubic foot of average coal gas may be taken to be from 500,000 to 540,000 foot-pounds.]

74. *Example* 2.—Again, taking the following experimental results from an oil-engine (one pound of oil being taken to give out 11,700 centigrade heat units in burning), I being the indicated, B the brake horse-power, and O the pounds of oil used per hour.

I	B	O	O/I	O/B	Efficiency.
7·41	6·77	6·4	·86	·95	·128
8·33	6·88	6·8	·82	·99	·122
4·71	3·62	5·0	1·06	1·38	·088
0·89	0	3·1	3·48	–	0

Let the student show that $O = 0·505\ I + 2·62$, $O = 0·52\ B + 3·1$, $B = 0·98\ I - ·89$. Let him also fill in the columns O/I and O/B. Prove that 1 lb. of oil per hour means 8·27 horse-power actually supplied, and fill in the column of efficiency.

75. Accept the following measurements. A steam-engine employed in driving a dynamo machine delivering electric energy to customers, each load being kept steady for four hours, each measurement being the average of the results obtained during the four hours. I is indicated horse-power; B the brake horse-power measured by a transmission dynamometer; the electrical horse-power E is obtained by multiplying ampères and volts to get the power in watts and dividing by 746; C is the coal used per hour; and W is the weight of steam used by the engine per hour. The governor acted upon the throttle-valve, and not upon the cut-off.

I	B	Ampères.	Volts.	E	W	C
190	163	1,050	100	143	4805	730
142	115	730	100	96	3770	544
108	86	506	100	69	3080	387
65	43	219	100	29	2155	218
19	0	–	–	–	1220	–

First plot the values of I and W, I and B, E and B, I and C on squared paper. It will be found that there is approximately a linear law in every case. See if you get some such laws as

$$
\begin{aligned}
W &= 800 + 21\ I \\
B &= ·95\ I - 18 \\
E &= ·93\ B - 10 \\
C &= 4·2\ I - 62
\end{aligned}
$$

Now produce a few more columns of numbers and study them. Give $W \div I$ and $C \div I$. Give $W \div B$ and $C \div B$. Give $W \div E$ and $C \div E$. Also give $W \div C$.

76. Students may compare the above results with the following average measurements made at an electric supply station in 1891:

	I	E	W	F
Average for 7 hours. 11 a.m to 6 p.m.	80·3	57·1	3,268	552
Average for 6 hours. 6 p.m. to midnight.	227·7	163·2	7,122	742
Average for 11 hours. Midnight to 11 a.m.	37	23·64	2,143	232
24 hours. 11 a.m. to 11 a.m.	97·3	68·3	3,718	453

Here it will be found that even when the averages for the 24 hours are taken with the others, we have linear laws between I, E and W, $\text{W} = 1150 + 26{\cdot}25\,\text{I}$, and $\text{E} = {\cdot}72\,\text{I} - 2$. But F does not follow a linear law with the others. The reason lies in the fact that a spare boiler was used during part of the time, and there is consequently a greater consumption of fuel than if one or two boilers had been used the whole time. Since we have considered fuel consumption in the above exercises, it may not be out of place to introduce here some figures from the testing of a water-tube boiler:—

Steam per hour from and at 100° C. per lb. of coal.	Coal per sq. ft. of grate per hour.	Water evaporated per sq. ft. of total boiler heating surface per hour. This is not reduced to 100° C.	w.
13·40	7·74	1·24	103
12·48	18·6	3·20	233
12·00	29·8	4·70	357
10·29	66·8	8·50	686

If w = steam per hour per square foot of grate, f = fuel per hour per square foot of grate. Plotting w and f on squared paper, we find a fair approach to a linear law,

$$w = 45 + 9{\cdot}78f$$

$$\frac{w}{f} = \frac{45}{f} + 9{\cdot}78.$$

77. If c is the total cost per hour when the useful horse-power P is being sent out by an hydraulic or electric or other supply company (c includes interest and depreciation on first cost, rent, taxes, repairs, wages, stores, coal, water, office expenses and management), and if it is found that there is a simple law like $c = a P + b$, where a and b are constants, prove that the average cost per hour is calculated from the average power in exactly the same way as the real cost c in any hour is from the P during that hour. For if t is time in hours, and P is some function of the time, then the cost during a year is, if T is the number of hours in a year,

$$\int_0^T c\,.dt = \int_0^T (a P + b)\,dt = a\int_0^T P\,dt + b\,T.$$

Hence the average cost per hour is $\frac{a}{T}\int_0^T P.\,dt + b.$

Now $\frac{1}{T}\int_0^T P\,.dt$ is the average power, call it P_m, and we see that the average cost is $a\,P_m + b$. The average power delivered in a day or year, divided by the maximum power, is called the daily or yearly **load factor.** If f is the load factor and P_1 is the maximum power, then average cost per hour $= a\,P_1 \div f + b$.

78. An electrical company has arranged for a maximum output of 1,000 horse-power. It is found that the total cost per hour in pence c is $c = 0·8\,P + 350$. If p pence is charged for every **horse-power hour** sent out, what is the yearly profit when the average power sent out day and night is P_m?

Ans.—Subtract from $p\,P_m$ the average cost per hour to the company which is $0·8\,P_m + 350$, and the profit per hour is $(p - 0·8)\,P_m - 350$. The profit per annum in pounds is therefore this divided by 36·5.

Exercise.—What charge per horse-power hour will give just no profit?

Ans.—$x = 0·8 + 350/P_m$.

Thus if P_m has the following values, we have the values of x

P_m	1,000	500	200	100
x	1·15	1·50	2·55	4·30

79. We have seen that if a force E lb., acting through e feet, overcomes a force R lb., acting through r feet, instead of the non-frictional law $E = \frac{r\,R}{e}$ . . . . (1), we find experimentally some such law as

$$E = a\,R + A.$$

It is evident that the constant A is a frictional resistance from the mere weight of the parts of the machine; a is always found to be greater than $\frac{r}{e}$. The work done by E in the distance e feet is Ee or $ea\mathrm{R} + \mathrm{A}e$, and the useful work is $r\mathrm{R}$, so that the

$$\text{Efficiency } x = \frac{r\mathrm{R}}{ea\mathrm{R} + \mathrm{A}e} = \frac{\frac{r}{ea}}{1 + \frac{\mathrm{A}}{a\mathrm{R}}}.$$

It will be observed that the larger R is, the smaller is $\frac{\mathrm{A}}{a\mathrm{R}}$, or the more insignificant is the term due to weight of parts of machine, and the more nearly does the denominator approach unity. However great R may become, the efficiency cannot exceed $\frac{r}{e} \cdot \frac{1}{a}$; and as a is always greater than $\frac{r}{e}$, the efficiency is always less than unity, as was to be expected.

Denote a by $\frac{1}{k} \cdot \frac{r}{e}$, where k is always less than 1. Then the efficiency can never be greater than k. If e and r are feet per second, then $\frac{\mathrm{E}e}{550}$ and $\frac{\mathrm{R}r}{550}$ represent the horse-powers. Let us call $\frac{\mathrm{E}e}{550}$ the total horse-power T, and $\frac{\mathrm{R}r}{550}$ the useful horse-power U, and we shall call T—U the lost or wasted horse-power L. Then as $\frac{\mathrm{E}e}{550} = \frac{ae}{r} \cdot \frac{\mathrm{R}r}{550} + \frac{\mathrm{A}e}{550}$, $\mathrm{T} = \frac{1}{k}\mathrm{U} + \frac{\mathrm{A}e}{550}$ and $\mathrm{U} = k\mathrm{T} - \frac{\mathrm{A}ke}{550}$ . . . . (1).

80. It is interesting to know, from such examples as we have studied in Arts. 73-76, that if T is the **indicated horse-power** of a **steam-engine**, and U is the **brake horse-power**, or what is given out usefully by the crank shaft, and if this shaft drives a dynamo machine, and E is the **electrical horse-power** given out; or if the shaft drives a **pump**, and E is the effective horse-power of the pump, we always find (probably with only approximate accuracy) a linear law connecting T and U, and U and E, and therefore also connecting T and E. Furthermore, in a steam-engine which does not vary its period of cut-off—that is, the regulation is by throttling or the boiler-pressure changes—if W is the **poundage of steam** per hour, there is a linear law connecting any two of the quantities W, T, U, and E.

If we write (1) as

$$\mathrm{U} = k\mathrm{T} - a, \text{ or } \mathrm{T} = \frac{1}{k}\mathrm{U} + \frac{1}{k}a,$$

it is curious that in so many of our machines in which the transmission of power is by mere shafting k should be so much less than unity as it is.

This occurs because a mere **torque** is very seldom applied to a shaft when power is being given or taken. When the power is supplied or taken off by a belt, the power is proportional to the difference of pulls in the belt, whereas the loads on the bearings and the friction are nearly constant. Such kinds of driving tend rather to increase a than to diminish k. All **spur or bevel gearings** tend to diminish k, and only by their mere weight do they tend to increase a. The poor efficiency of ordinary lines of shafting is a matter to which few people seem to have paid any attention; and this is, I think, altogether due to the absence of **dynamometer couplings** or other means of forcing upon the attention the actual amounts of power which are being transmitted at each place. Electrical engineers have very accurate methods of measuring the power given out by their dynamos, and hence they are concerned as to the actual mechanical power supplied to them. It was almost altogether due to this that direct driving of dynamos from engines took the place of indirect methods; and, indeed, I may go further, and say that it was due to this that the great **improvements** have taken place in steam-engine manufacture and working during the last fifteen years. If even the practical engineer is beginning, therefore, to think of economy, it is greatly due to the fact that electricity is paid for at so much per unit of energy; although, no doubt, much is also due to the fact that **economy of coal** is very important in ocean-going steamships.

81. In any machine in which a small effort, E, overcomes a great resistance, R, we usually find that the frictional loss of energy in the machine is almost altogether dependent upon R. Thus in the **inclined plane** (Fig. 25), with E parallel to the plane, the friction is independent of E. When in the inclined plane of small inclination E is horizontal, the friction is almost altogether dependent on the load to be lifted; and it is so in a screw-jack and in the differential pulley-block. It is not so much the case in cranes, unless such gearing as worm-gearing is employed.

Let us, then, suppose that the loss of energy in the machine is altogether due to R. Then in the direct use of the machine—that is, E overcoming R—if the efficiency is only 50 per cent., the loss of energy is equal to $R\,r$ (if R is lifted r feet), $R\,r$ being also the useful energy. Consequently, if we try by increasing R to R^1 to make the machine reverse, the loss is always R^1r; or the whole energy would be wasted if the machine could be supposed to move. If, then, the direct efficiency is equal to or less than 50 per cent., the machine **cannot reverse or overhaul**.

In any machine there is some loss due to E, and also to the weight of the parts of the machine, as well as to R, and consequently the direct efficiency must always be less than 50 per cent. if the machine is not to **overhaul**. On the assumption that the lost energy, L, is

$$L = m\,R\,r + n\,E\,e + A,$$

where A is a constant, due to the weight of parts of the machine, and m and n are fractions, we have, in direct working,

$$E\,e - R\,r = L,$$

$$E\,e - R\,r = m\,R\,r + n\,E\,e + A,$$

$$E\,e = \frac{(m+1)\,R\,r + A}{1 - n}.$$

Hence the direct efficiency x is $\dfrac{(1-n)\,R\,r}{(m+1)\,R\,r + A}$ . . . . (1), or $\frac{1-n}{m+1+A/Rr}$. Again, in reversed working

$$R^1r - Ee = L.$$

$$R^1r - Ee = mR^1r + nEe + A.$$

$$Ee = \frac{R^1r\,(1-m) - A}{1+n}.$$

The reversed efficiency $y = \frac{Ee}{R^1r} = \frac{(1-m)\,R^1r - A}{(1+n)\,R^1r} \quad \ldots\ldots (2).$

This is 0 or negative when

$$\frac{A}{R^1r} + m = \text{or} > 1.$$

As R^1 may be made very great, our condition of non-reversibility must be $m = 1$ or > 1. Hence the direct efficiency must be equal to or less than $\frac{1-n}{2+A/Rr}$. We therefore see that there is no general rule such as many people seem to believe in—the result of misleading mathematics. We can hardly call it a general rule to say that, if m is equal to 1, R cannot overcome E, because $m = 1$ means that all R's energy would be wasted.

CHAPTER VI.

MACHINES. SPECIAL CASES.

82. Blocks and Tackle.—It is very good to have a general law telling us about machines in which there is no friction. This law you now know. *The mechanical work given to a machine is equal to the work given out by it*, unless it is stored up in the machine itself by the coiling of a spring or in some other way. But, besides knowing the law itself, it is well to know what it leads to in certain special cases. Take, for instance, a pulley-block, Fig. 38. It is evident here that if we have three pulleys in the block B, if the effort P acts through six inches, W will only rise *one* inch, and therefore P will balance six times its weight at W if there is no friction. The mechanical advantage is therefore six.

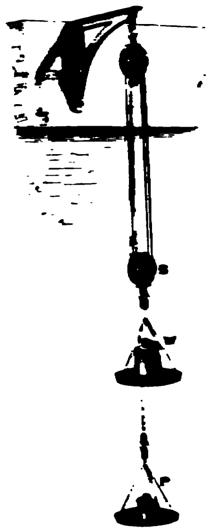

Fig. 38.

This is a case in which it is less difficult than usual to trace the loss of energy due to friction. If, when the pull of a cord is P_0 on the overcoming side of a pulley, the tension on the other side is given by $P_1 = a\,P_0 - b$, where a is less than 1, and b does not depend much on the weight of a sheave, but rather on the viscosity of the rope, we may take it that $P_2 = a\,P_1 - b$, and so on. I shall leave it to students to work out an algebraic expression connecting W, the sum of P_1, P_2, P_3, etc., and P_0. W will include the weight of the lower block. But it is good to take a numerical example. For instance, let $P_1 = 0{\cdot}9\,P_0 - 3$, the forces being in pounds. Then

$$\begin{aligned}
P_2 &= 0{\cdot}81\ P_0 - 2{\cdot}7 - 3 = 0{\cdot}81\ P_0 - 5{\cdot}7,\\
P_3 &= 0{\cdot}729\ P_0 - 5{\cdot}13 - 3 = 0{\cdot}729\ P_0 - 8{\cdot}13,\\
P_4 &= {\cdot}6561\ P_0 - 7{\cdot}317 - 3 = {\cdot}6561\ P_0 - 10{\cdot}32,\\
P_5 &= {\cdot}5905\ P_0 - 9{\cdot}288 - 3 = {\cdot}5905\ P_0 - 12{\cdot}29,\\
P_6 &= {\cdot}5315\ P_0 - 11{\cdot}06 - 3 = {\cdot}5315\ P_0 - 14{\cdot}06\,;
\end{aligned}$$

Hence $W = 4{\cdot}2171\ P_0 - 53{\cdot}50$, or $P_0 = 0{\cdot}237\ W + 12{\cdot}7$.

The efficiency $$x = \frac{\cdot 167\ \text{w}}{\cdot 237\ \text{w} + 12\cdot 7} = \frac{\cdot 705}{1 + \frac{53\cdot 6}{\text{w}}}$$

Hence, however great w may be, the efficiency is less than 70·5 per cent. When w = 53·6 the efficiency is only half this.

83. This is a fairly good example of the **cumulative effect** of friction. If we give power P_0 to a machine whose efficiency is e_0, if this machine gives power to another whose efficiency is e_1 and so on, the power given out by the last of a series of machines is $P_0 \times e_0 \times e_1 \times e_2 \times$ etc. If, as in transmitting power for a great distance by means of ropes, all the contrivances are the same, we have the **compound interest law** of diminution of power. (See Exercise, page 232.) Students may find it interesting to study a machine or method of transmission of power in a manner allied to what follows in Art. 91 of this chapter.

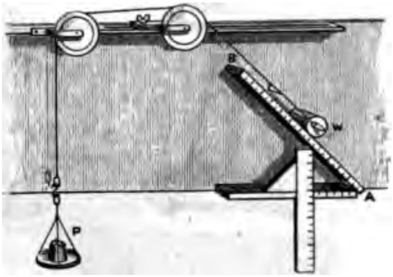

Fig. 39.

84. Inclined Plane.—Again, take the inclined plane, Fig. 39 ; w is a weight which may roll down the plane without friction, let us suppose; P is the pull in a cord which just prevents w from falling. The cord is parallel to the plane. Evidently when w rises from level A to level B the cord is pulled the distance A B; that is, w multiplied by the height of the plane is equal to P multiplied by the length of the plane. Thus, if w is 1,000 lbs., and the length of the plane 10 feet for a rise of 2 feet, then ten times P is equal to 2,000, or P is 200 lbs.

Barrels and boilers are often raised along an inclined plane, ropes or chains held fast at the top of the plane passing round the cylindric object and back towards the top of the plane where force is applied to them. In this case P is evidently only half what is stated above. As the weight rolls on the double rope or chain there is not much friction. In Art. 90 we shall consider the effect of friction in the inclined plane.

85. The Screw.—Again, suppose there is no friction in the screw A B, Fig. 40; if it rises it lifts a weight say of 3,000 lbs. Now, if the screw make one turn it rises by a distance equal to its **pitch**; that is, *the distance* (measured parallel to the axis) *between two threads.* Say that the pitch is ·02 foot, then when the screw makes one turn it does work on the weight 3,000 × ·02, or 60 foot-pounds. But to do this, P must fall through a distance equal to the circumference of the pulley A, about which I suppose the cord to be wound. Suppose the circumference of the pulley to be 6 feet, then P multiplied by 6 must be 60, or P is 10 lbs. The rule, then, for a screw is this—*effort multiplied by circumference of the pulley equals resistance multiplied by pitch of screw.* It is not usual to have a pulley and a cord working a screw; it is more usual to have a handle, and to push or pull at right angles to the handle. Instead of the circumference of the pulley, we should take, then, the circumference of the circle described by the point where the effort is applied to the handle.

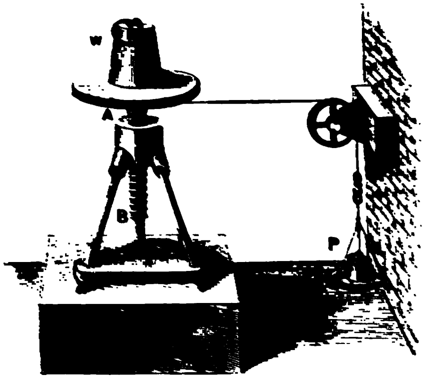

Fig. 40.

Example. **A steam-engine** gives to a **propeller shaft** in one revolution 60,000 foot-pounds of work: the pitch of the screw is 12 feet. What is the resistance to the motion of the vessel? Answer: The resistance in pounds multiplied by 12 gives the work done in overcoming this resistance, and this work must be equal to 60,000 foot-pounds: hence the resistance to the motion of the vessel is 5,000 lbs. [It would be more correct to say that this is the work done per foot-travel of the vessel, assuming no slip of the screw.]

Screws are used for many purposes. When used, as in bolts to fasten things, the threads are triangular in section. The **Whitworth** thread is shown in Fig. 41. A B is the pitch; G H is ·96 of the pitch, the angle H B J being 50° ; the corners are rounded, so that I E is ·64 of the pitch. The **Sellers** thread used in America has an angle of 60°. Fig. 42 shows a square thread. There is less friction and less wear with this form of screw, and it is used when accuracy of motion is important. As there is only half the amount of material resisting shearing, the square thread has only half the strength of a triangular thread. The **Buttress** thread of Fig. 43 has strength and accuracy as to motion; it is used when the important motion is in one direction only.

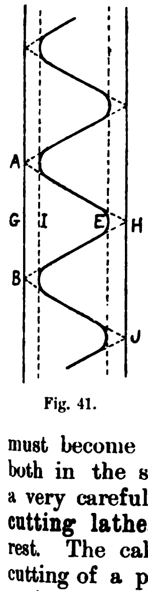

Fig. 41.

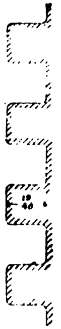

Fig. 42.

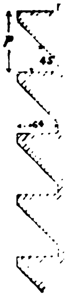

Fig. 43.

A student has plenty of opportunity of examining examples of the use of screws. To fasten things together we have many kinds of bolts and many forms of heads and nuts, and many ways of locking nuts. Other fastenings also, such as **pins** and **keys** and **cottars**, come before his eyes every day in the workshops, and he must become familiar with them and their usual proportions both in the shops and the drawing-office. He must make a very careful examination of such a tool as a good **screw-cutting lathe**, and especially of the mechanism of the slide rest. The calculation of the proper **change wheels** for the cutting of a particular screw is a very simple matter; but all so simple as it is, he must work out some examples.

He must note the **shapes of threads** of screws for wood, for sand or mud, and for water; the use of screw piles as the supports of structures, the forms of screw-propeller for steamships and wind-mills, and screws used as fans for blowing air.

86. Wheel and Axle.—If A and B, Fig. 44, are two pulleys or drums on the same axis and having cords round them, a small weight, P, hung from A, will balance a larger weight, W, hung from B. For, suppose that one complete turn is given to the axis, P falls a distance equal to the circumference of A whilst W is rising a distance equal to the circumference of B. Hence

$$P \times \text{circumference of } A = W \times \text{circumference of } B,$$

or, what really comes to the same thing,

$$P \times \text{diameter of } A = W \times \text{diameter of } B \; . \; . \; . \; . \; (1),$$

or $P \times \text{radius of } A = W \times \text{radius of } B$.

In practice A is often a handle and B a sort of barrel on which a chain may be wound. Or the **axle** may be **compound**, consisting of two parts, the diameters D and *d* being nearly equal, the rope being coiled round them in opposite directions so as to form a loop, upon which hangs a pulley. In this case (1) becomes

$$P \times \text{diameter of } A = W \times \frac{D - d}{2}$$

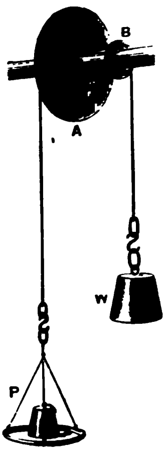

Fig. 44.

Or, again, A may not be on the same shaft as the barrel B, but may gear with it through **spur and bevel-gearing**. Thus, in a hand **crane**, the handle may turn many times for one turn of the barrel. Also there may be a **snatch-block**, so that when two feet of chain are coiled on the barrel, the weight rises only one foot. There is no mystery about tooth gearing, and anyone who has looked at a crane knows how by merely measuring the length of the handle, or, rather, the circumference of the circle described by the handle, and the amount of chain coiled on in one revolution of the barrel (this is larger than the circumference of the barrel itself), and counting the number of teeth of the driving and driven wheels, what is the **velocity ratio**, and, therefore, the hypothetical **mechanical advantage**. I write for men who go about among machinery, and the most illiterate workman knows well how speeds of shafts depend upon numbers of teeth. Also, all my readers have seen combinations of barrels, snatch-blocks, blocks and tackle and crab-winches. A worm and **worm wheel**, or other kinds of screw gearing may also be imagined.

Example.—The handle of a **crab** or crane is 18 inches long; 20 inches of chain are wound on in one revolution of the barrel. The barrel is driven from the handle by a train of wheels—driver 15 teeth, follower 64 teeth; driver 16 teeth, *follower* 60 teeth (the **value** of this **train** is said to be

$\frac{15 \times 16}{64 \times 60}$, or $\frac{1}{16}$); so that the handle makes 16 revolutions for one of the barrel; the chain lifts the weight through the agency of a block and tackle, with 3 sheaves below.

When the barrel turns once, 20 inches of chain are coiled on, and therefore the weight rises $\frac{20}{2 \times 3}$ inches or $3\frac{1}{3}$ inches. The handle turns 16 times, and the hand moves through $16 \times 36\,\pi$ inches. Hence the velocity ratio of hand to weight is $\frac{16 \times 36\,\pi}{3\frac{1}{3}} = 543$. Often we have a means provided of disengaging the spur wheels, driver 16, follower 60, and so we can decrease the velocity ratio to $543 \times \frac{16}{60}$ or about 145.

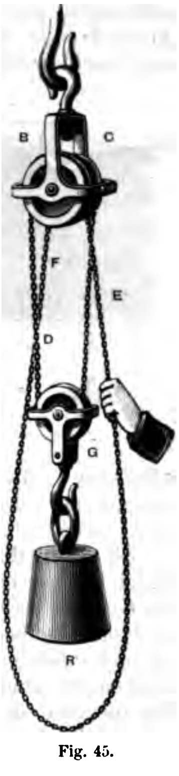

Fig. 45.

87. A differential pulley-block is shown in Fig. 45. When the chain E is pulled, it turns the two pulleys, or rather one pulley with two grooves, B and C. Now C is a little smaller than B, so that, although at D the chain is lifted, it is lowered at F. If the circumference of B is 2 feet and that of C is 1·95 feet, then, when E is pulled 2 feet, D is lifted 2 feet, but F is lowered 1·95 feet, so that there is 0·05 foot of chain less than before in the parts D and F, and the pulley G rises the half of this, or ·025 foot. If R is 2,000 lbs., then 2,000 × ·025, or 50, must be equal to the pull E multiplied by 2, hence E is 25 lbs., or an effort of 25 lbs. is able to overcome a resistance of 2,000 lbs. The general rule, then, for the differential pulley-block is, effort E *multiplied by circumference of larger groove* B *is equal to resistance* R *multiplied by half the difference between the circumference of the two grooves* B *and* C. You will find that this rule comes to the same thing—*effort multiplied by diameter of* B *is equal to resistance multiplied by half the difference between the diameters* *of* B *and* C. The grooves are furnished with ridges to catch the links of the chain, so that there shall be no slipping.

It is only when we experimentally measure the effort E which will slowly overcome the resistance R in this pulley-block as actually made, that we see how great is the frictional waste of energy. It is in consequence of this that, however great the resistance R may be, it will not fall, even when there is no force exerted at E. This property of not "**overhauling**" makes the differential pulley-block the very useful implement which we know it to be in a machine shop. It is the characteristic of any machine which has a very great velocity ratio that if its **efficiency** is less than half, it will not overhaul (see Art. 81), and lifting machines which do not overhaul are often very convenient.

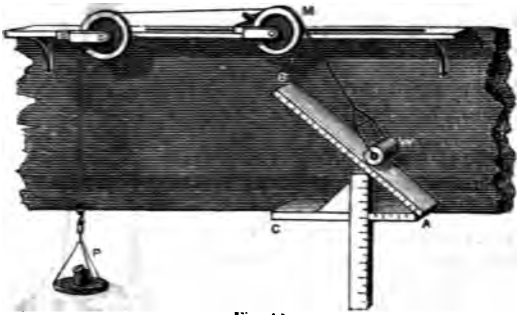

Fig. 46

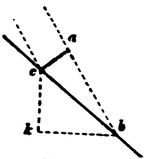

Fig. 47.

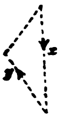

Fig 48.

88. Equilibrium in one Position.—In all the machines which we have hitherto considered, we could give motion without altering the balance of the forces, but there are many machines in which the mechanical advantage alters when motion is given. In such cases you will employ your general principle, but you must make your calculation from a very small motion indeed. For instance, in the inclined plane, if the cord which prevents the weight from falling is not parallel to the plane—say that it is like M, Fig. 46—you will find that the necessary pull depends on the angle the cord makes with the plane. Now, suppose that the cord pulls the carriage from *b* to *c*, evidently the angle of the cord alters. The question is, what is P, that *it may support* W in the position shown in the figure? We know that it will be different after a little motion, but what is it now? Imagine such a very small motion from *b* to *c* to occur that the angle of the cord does not alter perceptibly, and now make a magnified drawing, Fig. 47. P has not fallen so much as the distance *b c*, it has only fallen the distance *b a* (*c a* is perpendicular to *b a*). In the meantime the weight W has been lifted the distance *k c*. Hence,

$$\text{W} \times k\,c \text{ ought to be equal to } \text{P} \times b\,a.$$

Thus, if you measure *k c* and *b a* on your magnified drawing to any scale you will find the relation between P and W. Another way of finding the same relationship is this. We know that the weight of W acting downwards, the pull in the cord, and a force acting at right angles to the plane, are the three forces which keep W where it is. Draw a triangle whose three sides are parallel to the directions of these three forces, Fig. 48, with arrow-heads circuital; then x and y are in the proportion of W and P. Here we have used the principle called "**the triangle of forces**" to find P.

EXERCISES.

1. Find the force parallel to the plane required to draw a weight of 2 cwt. up a smooth inclined plane. Height of plane, 3; length, 5.
Ans., 1·2 cwts.

2. In a screw-jack the pitch of the screw is $\frac{3}{8}$ inch; radius of circle described by hand, 19 inches; find the velocity ratio. It is found by experiment that a force A at the handle of 30 lbs. will overcome a weight of 2,300 lbs., and a force of 10 lbs. will overcome a weight of 500 lbs.: what law connects A and W? When W is 3,000 lbs., what is A? And what is the efficiency?

Ans., 318·47; $\text{A} = \frac{\text{W}}{90} + 4\frac{4}{9}$; $37\frac{7}{9}$ lbs.; 25 per cent.

3. The handle of a lifting-jack measures 24 inches in length; the pitch of the screw is $\frac{3}{8}$ inch; what force applied at the end of the handle would be required to raise a load of 22 cwt., the effect of friction being neglected? *Ans.*, 6·125 lbs.

4. Pitch of screw-propeller, 18 feet; slip, 10 per cent.; speed of ship, 15 knots; find the revolutions per minute. What is the thrust if the actual horse-power spent by the propeller is 2,000, and the waste by surface friction is 30 per cent.? What is the torque in the shaft? *Ans.*, 93·827.

5. The British Association rules for the pitch and diameter and index number of screw-threads for instrument work are

$$d = 6\,p^{\frac{6}{5}},\ p = (0{\cdot}9)^{s}.$$

Taking the values of 0, 1, 2, 3, ... 12 for *s*, calculate *p* and *d*, and keep for reference in a table. Try to what extent the rule $p = {\cdot}08\,d + {\cdot}04$ for the pitch and outside diameter of triangular screws agrees with the well-known Whitworth table.

The grooves are furnished with ridges to catc the links of the chain so that there shall be no slipping.

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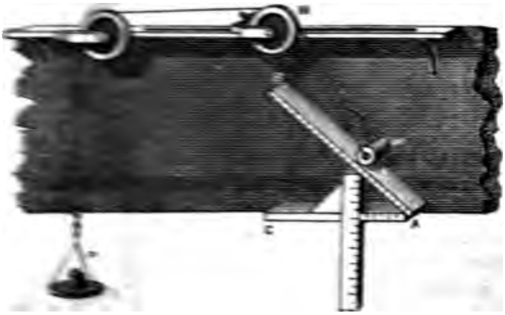

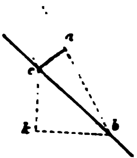

Fig. 47.

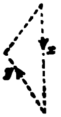

Fig 48.

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6. What must be the difference in the diameters of a compound wheel and axle so that the velocity of E may be eighty times that of R, the length of the handle being 2 feet? *Ans.*, 1·2 inches.

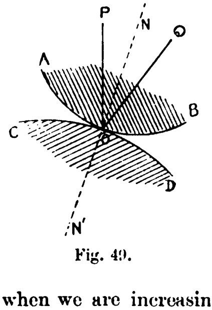
7. The weight on a crane is carried by a snatch-block; the chain goes to a barrel on which 17 inches are wrapped in one revolution. The barrel is driven by wheels, which give it 1 revolution for 15 revolutions of the handle, and the hand describes a circle of 21 inches radius; what is the velocity ratio? *Ans.*, 232·8.

8. In a single-purchase crab the pinion has 12 teeth and the wheel has 78 teeth, the diameter of the barrel (or rather of the chain on the barrel) being 7 inches, and the length of the lever-handle 14 inches. It is found that the application of a force of 15 lbs. at the end of the handle suffices to raise a weight of 280 lbs.: find the efficiency of the machine. *Ans.*, 0·718.

9. In the differential pulley, if the weight is to be raised at the rate of 5 feet per minute, and the diameters of the pulleys of the compound sheave are 7 and 8 inches, at what rate must the chain be hauled? *Ans.*, 80 feet per minute.

10. In a differential pulley in which the velocity ratio of fall to lift is 30, a pull of 7 lbs. will just raise a load of 24 lbs., and a pull of 25 lbs. a load of 240 lbs. Find the pull required to lift 600 lbs., and the efficiency of the machine when such a weight is being raised. *Ans.*, 55 lbs.; 0·36.

89. When one body touches another, and there is equilibrium, if there is no friction, this means that there is no **tangential** force at the surface. The force with which either body acts on the other has no tangential component. If, however, there is friction, and we assume that the friction is μ R where R is the force **normal** to the rubbing surface, it is evident that the total force at the place makes an angle ϕ with the normal if tan. $\phi = \mu$. If the total force makes an angle less than ϕ, its tangential component is less than the friction, and there can be no motion. If, then, two bodies A O B and D O C touch at O, and N O N^1 is their common normal at O, so long as the direction of the force acting between the bodies lies inside the cone Q O P, whose axis is O N, the angle P O N or Q O N being ϕ, there will be no sliding motion or rubbing at O. Thus, when the block lies on the inclined plane (Fig. 50), the total force transmitted between block and plane is W, the vertical weight of the block. So long as the angle is less than ϕ, there is no sliding. But the angle is the angle of the plane; hence sliding is about to begin when we are increasing the angle of the plane, and it has become as much as ϕ.



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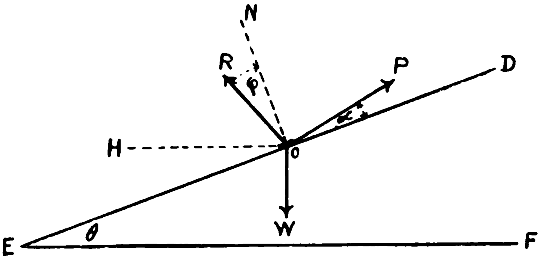

Fig. 50.

about an axis in the tangent plane, and an angular velocity of spinning γ sin. θ about the common normal to the two planes.

If in Fig. 49 the surfaces are cylindric and the rolling angular velocity is ω, and if r is the radius of curvature of A O B at O, and if R is the radius of curvature at O of the fixed surface D O C, and if v is the linear velocity of the point of contact, it is easy to show that

$$\omega = \left(\frac{1}{r} + \frac{1}{R}\right) v.$$

The centres of curvature are supposed to be on different sides of the tangent plane.

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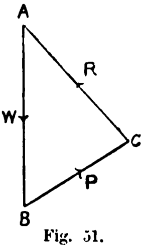

Fig. 51.

Analytically.—Resolve all the forces *horizontally* and *vertically*, and we have (see Art. 31), if the angle P O D is α, as P makes an angle $\alpha + \theta$ with the horizontal, and R makes an angle R O H, which is the complement of $(\phi + \theta)$,

$$\text{P cos. } (\alpha + \theta) = \text{R sin. } (\phi + \theta) \; . \; . \; . \; . \; (1)$$
$$\text{P sin. } (\alpha + \theta) + \text{R cos. } \phi + \theta) = \text{W} \; . \; . \; . \; . \; (2).$$

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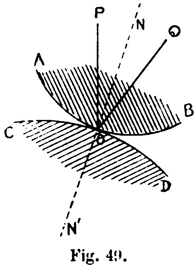

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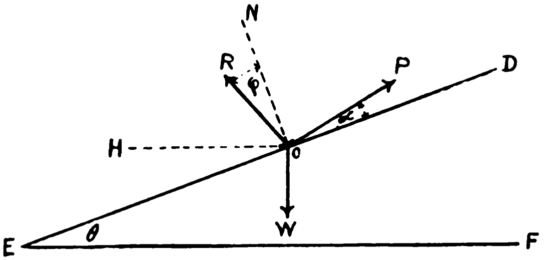

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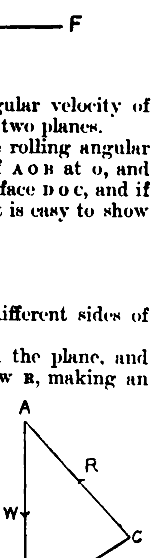

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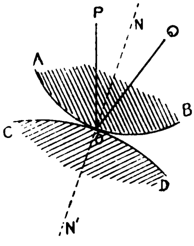

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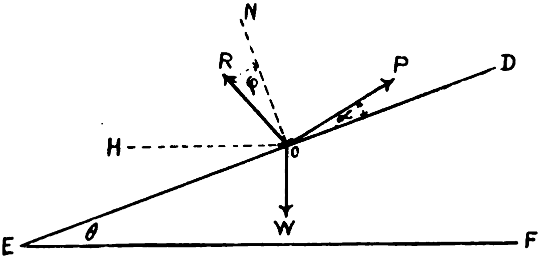

Fig. 50.

about an axis in the tangent plane, and an angular velocity of spinning γ sin. θ about the common normal to the two planes.

If in Fig. 49 the surfaces are cylindric and the rolling angular velocity is ω, and if r is the radius of curvature of A O B at O, and if R is the radius of curvature at O of the fixed surface D O C, and if v is the linear velocity of the point of contact, it is easy to show that

$$\omega = \left(\frac{1}{r} + \frac{1}{R}\right) v.$$

The centres of curvature are supposed to be on different sides of the tangent plane.

If in Fig. 50 there is friction between W and the plane, and motion up the plane is steadily taking place, draw R, making an angle N O R = ϕ, with O N the normal. Let P show the direction of the pulling force, and W of the weight. Knowing only W to calculate P and R, we have simply to use the triangle of forces. We are given one side and the angles to find the other sides. Thus draw A B representing W to scale; and draw B C and A C parallel to the two unknown forces, with the arrow-heads going circuitally. Measure B C and C A, and the forces P and R are known.

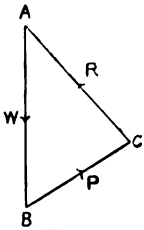

Fig. 51.

Analytically.—Resolve all the forces *horizontally* and *vertically*, and we have (see Art. 31), if the angle P O D is α, as P makes an angle $\alpha + \theta$ with the horizontal, and R makes an angle R O H, which is the complement of $(\phi + \theta)$,

$$P \cos. (\alpha + \theta) = R \sin. (\phi + \theta) \; . \; . \; . \; . \; (1)$$
$$P \sin. (\alpha + \theta) + R \cos. \; \phi + \theta) = W \; . \; . \; . \; . \; (2).$$

As we *really do not wish to know* R, use its value from (1) in (2),

6. What must be the difference in the diameters of a compound wheel and axle so that the velocity of E may be eighty times that of R, the length of the handle being 2 feet? *Ans.*, 1·2 inches.

7. The weight on a crane is carried by a snatch-block; the chain goes to a barrel on which 17 inches are wrapped in one revolution. The barrel is driven by wheels, which give it 1 revolution for 15 revolutions of the handle, and the hand describes a circle of 21 inches radius; what is the velocity ratio? *Ans.*, 232·8.

8. In a single-purchase crab the pinion has 12 teeth and the wheel has 78 teeth, the diameter of the barrel (or rather of the chain on the barrel) being 7 inches, and the length of the lever-handle 14 inches. It is found that the application of a force of 15 lbs. at the end of the handle suffices to raise a weight of 280 lbs.; find the efficiency of the machine. *Ans.*, 0·718.

9. In the differential pulley, if the weight is to be raised at the rate of 5 feet per minute, and the diameters of the pulleys of the compound sheave are 7 and 8 inches, at what rate must the chain be hauled? *Ans.*, 80 feet per minute.

10. In a differential pulley in which the velocity ratio of fall to lift is 30, a pull of 7 lbs. will just raise a load of 24 lbs., and a pull of 25 lbs. a load of 240 lbs. Find the pull required to lift 600 lbs., and the efficiency of the machine when such a weight is being raised. *Ans.*, 55 lbs.; 0·36.

89. When one body touches another, and there is equilibrium, if there is no friction, this means that there is no **tangential** force at the surface. The force with which either body acts on the other has no tangential component. If, however, there is friction, and we assume that the friction is μR where R is the force **normal** to the rubbing surface, it is evident that the total force at the place makes an angle ϕ with the normal if $\tan. \phi = \mu$. If the total force makes an angle less than ϕ, its tangential component is less than the friction, and there can be no motion. If, then, two bodies A O B and D O C touch at O, and N O N^1 is their common normal at O, so long as the direction of the force acting between the bodies lies inside the cone Q O P, whose axis is O N, the angle P O N or Q O N being ϕ, there will be no sliding motion or rubbing at O. Thus, when the block lies on the inclined plane (Fig. 50), the total force transmitted between block and plane is W, the vertical weight of the block. So long as the angle is less than ϕ, there is no sliding. But the angle is the angle of the plane; hence sliding is about to begin when we are increasing the angle of the plane, and it has become as much as ϕ.

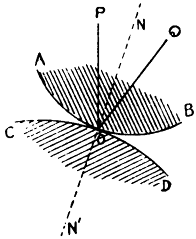

Fig. 49.

If the body D O C is at rest, and A O B moves, touching the first, the motion is one of **sliding, rolling,** or **spinning,** or combinations of these three. If there is no sliding, so that O is momentarily at *rest*, the **instantaneous** motion of A O B must be an angular velocity

γ about some axis through O. If this axis makes an angle θ with the tangent plane, there is an angular velocity of rolling $\gamma \cos. \theta$

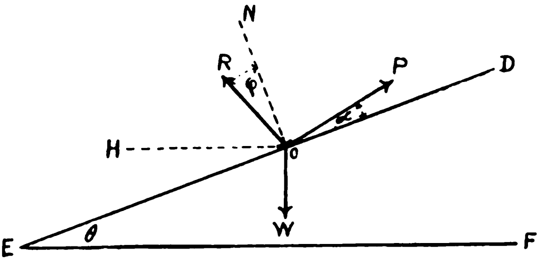

Fig. 50.

about an axis in the tangent plane, and an angular velocity of spinning $\gamma \sin. \theta$ about the common normal to the two planes.

If in Fig. 49 the surfaces are cylindric and the rolling angular velocity is ω, and if r is the radius of curvature of A O B at O, and if R is the radius of curvature at O of the fixed surface D O C, and if v is the linear velocity of the point of contact, it is easy to show that

$$\omega = \left(\frac{1}{r} + \frac{1}{R}\right) v.$$

The centres of curvature are supposed to be on different sides of the tangent plane.

If in Fig. 50 there is friction between W and the plane, and motion up the plane is steadily taking place, draw R, making an angle N O R $= \phi$, with O N the normal. Let P show the direction of the pulling force, and W of the weight. Knowing only W to calculate P and R, we have simply to use the triangle of forces. We are given one side and the angles to find the other sides. Thus draw A B representing W to scale; and draw B C and A C parallel to the two unknown forces, with the arrow-heads going circuitally. Measure B C and C A, and the forces P and R are known.

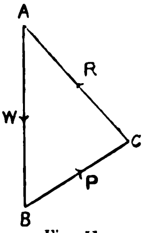

Fig. 51.

Analytically.—Resolve all the forces *horizontally* and *vertically*, and we have (see Art. 31), if the angle P O D is α, as P makes an angle $\alpha + \theta$ with the horizontal, and R makes an angle R O H, which is the complement of $(\phi + \theta)$,

$$P \cos. (\alpha + \theta) = R \sin. (\phi + \theta) \dots . (1)$$
$$P \sin. (\alpha + \theta) + R \cos. \phi + \theta) = W \dots . (2).$$

As we *really do not wish to know* R, use its value from (1) in (2),

of B *and* C. The grooves are furnished with ridges to catch the links of the chain, so that there shall be no slipping.

It is only when we experimentally measure the effort E which will slowly overcome the resistance R in this pulley-block as actually made, that we see how great is the frictional waste of energy. It is in consequence of this that, however great the resistance R may be, it will not fall, even when there is no force exerted at E. This property of not "**overhauling**" makes the differential pulley-block the very useful implement which we know it to be in a machine shop. It is the characteristic of any machine which has a very great velocity ratio that if its **efficiency** is less than half, it will not overhaul (see Art. 81), and lifting machines which do not overhaul are often very convenient.

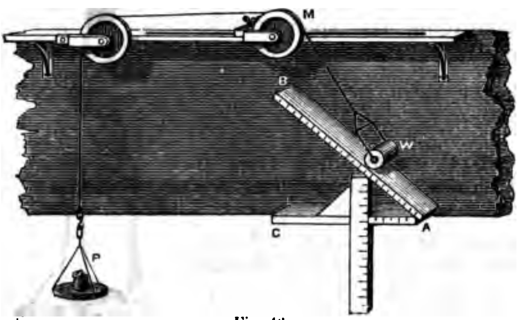

Fig. 46

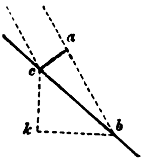

Fig. 47.

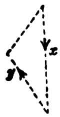

Fig 48.

88. Equilibrium in one Position.—In all the machines which we have hitherto considered, we could give motion without altering the balance of the forces, but there are many machines in which the mechanical advantage alters when motion is given. In such cases you will employ your general principle, but you must make your calculation from a very small motion indeed. For instance, in the inclined plane, if the cord which prevents the weight from falling is not parallel to the plane—say that it is like M, Fig. 46—you will find that the necessary pull depends on the angle the cord makes with the plane. Now, suppose that the cord pulls the carriage from *b* to *c*, evidently the angle of the cord alters. The question is, what is P, that *it may support* W in the position shown in the figure? We know that it will be different after a little motion, but what is it now? Imagine such a very small motion from b to c to occur that the angle of the cord does not alter perceptibly, and now make a magnified drawing, Fig. 47. P has not fallen so much as the distance $b\,c$, it has only fallen the distance $b\,a$ ($c\,a$ is perpendicular to $b\,a$). In the meantime the weight W has been lifted the distance $k\,c$. Hence,

$$\text{W} \times k\,c \text{ ought to be equal to } \text{P} \times b\,a.$$

Thus, if you measure $k\,c$ and $b\,a$ on your magnified drawing to any scale you will find the relation between P and W. Another way of finding the same relationship is this. We know that the weight of W acting downwards, the pull in the cord, and a force acting at right angles to the plane, are the three forces which keep W where it is. Draw a triangle whose three sides are parallel to the directions of these three forces, Fig. 48, with arrow-heads circuital; then x and y are in the proportion of W and P. Here we have used the principle called "**the triangle of forces**" to find P.

EXERCISES.

1. Find the force parallel to the plane required to draw a weight of 2 cwt. up a smooth inclined plane. Height of plane, 3; length, 5.

Ans., 1·2 cwts.

2. In a screw-jack the pitch of the screw is $\frac{3}{8}$ inch; radius of circle described by hand, 19 inches; find the velocity ratio. It is found by experiment that a force A at the handle of 30 lbs. will overcome a weight of 2,300 lbs., and a force of 10 lbs. will overcome a weight of 500 lbs.: what law connects A and W? When W is 3,000 lbs., what is A? And what is the efficiency?

Ans., 318·47; $\text{A} = \frac{\text{W}}{90} + 4\frac{4}{9}$; $37\frac{7}{9}$ lbs.; 25 per cent.

3. The handle of a lifting-jack measures 24 inches in length; the pitch of the screw is $\frac{3}{8}$ inch; what force applied at the end of the handle would be required to raise a load of 22 cwt., the effect of friction being neglected? *Ans.*, 6·125 lbs.

4. Pitch of screw-propeller, 18 feet; slip, 10 per cent.; speed of ship, 15 knots; find the revolutions per minute. What is the thrust if the actual horse-power spent by the propeller is 2,000, and the waste by surface friction is 30 per cent.? What is the torque in the shaft? *Ans.*, 93·827.

5. The British Association rules for the pitch and diameter and index number of screw-threads for instrument work are

$$d = 6\,p^{\frac{6}{5}},\ p = (0{\cdot}9)^{s}.$$

Taking the values of 0, 1, 2, 3, ... 12 for s, calculate p and d, and keep for reference in a table. Try to what extent the rule $p = {\cdot}08\,d + {\cdot}04$ for the pitch and outside diameter of triangular screws agrees with the well-known Whitworth table.

6. What must be the difference in the diameters of a compound wheel and axle so that the velocity of E may be eighty times that of R, the length of the handle being 2 feet? *Ans.*, 1·2 inches.

7. The weight on a crane is carried by a snatch-block; the chain goes to a barrel on which 17 inches are wrapped in one revolution. The barrel is driven by wheels, which give it 1 revolution for 15 revolutions of the handle, and the hand describes a circle of 21 inches radius; what is the velocity ratio? *Ans.*, 232·8.

8. In a single-purchase crab the pinion has 12 teeth and the wheel has 78 teeth, the diameter of the barrel (or rather of the chain on the barrel) being 7 inches, and the length of the lever-handle 14 inches. It is found that the application of a force of 15 lbs. at the end of the handle suffices to raise a weight of 280 lbs.; find the efficiency of the machine. *Ans.*, 0·718.

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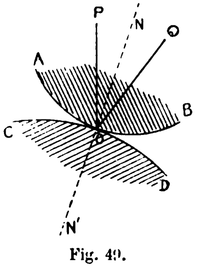

Fig. 49.

If the body DOC is at rest, and AOB moves, touching the first, the motion is one of **sliding, rolling,** or **spinning,** or combinations of these three. If there is no sliding, so that O is momentarily at *rest, the* **instantaneous** motion of AOB must be an angular velocity

γ about some axis through O. If this axis makes an angle θ with the tangent plane, there is an angular velocity of rolling γ cos. θ

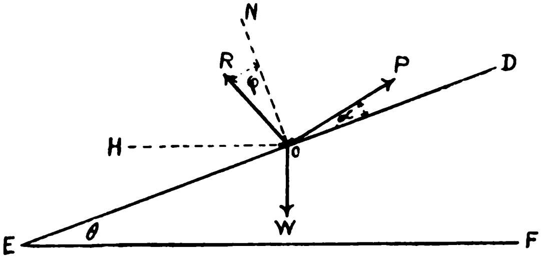

Fig. 50.

about an axis in the tangent plane, and an angular velocity of spinning γ sin. θ about the common normal to the two planes.

If in Fig. 49 the surfaces are cylindric and the rolling angular velocity is ω, and if r is the radius of curvature of A O B at O, and if R is the radius of curvature at O of the fixed surface D O C, and if v is the linear velocity of the point of contact, it is easy to show that

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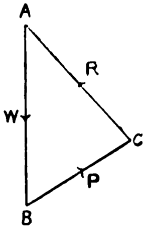


Fig. 51.

Analytically.—Resolve all the forces *horizontally* and *vertically*, and we have (see Art. 31), if the angle P O D is α, as P makes an angle $\alpha + \theta$ with the horizontal, and R makes an angle R O H, which is the complement of $(\phi + \theta)$,

$$\text{P cos. } (\alpha + \theta) = \text{R sin. } (\phi + \theta) \text{ . . . . } (1)$$
$$\text{P sin. } (\alpha + \theta) + \text{R cos. } \phi + \theta) = \text{W . . . . } (2).$$

As we really *do not wish to know* R, use its value from (1) in (2), and we find $\text{P} \sin.(\alpha + \theta) + \cos.(\phi + \theta)\dfrac{\text{P} \cos.(\alpha + \theta)}{\sin.(\phi + \theta)} = \text{W}$,

$$\text{P} = \frac{\text{W} \sin.(\phi + \theta)}{\cos.(\phi + \theta)\cos.(\alpha + \theta) + \sin.(\phi + \theta)\sin.(\alpha + \theta)} =$$

$$\frac{\text{W} \sin.(\phi + \theta)}{\cos.\left\{(\alpha + \theta) - (\phi + \theta)\right\}} = \mathbf{W}\frac{\sin.(\theta + \phi)}{\cos.(\alpha - \phi)} \text{ . . . . . . (3).}$$

If P is the force required to *just prevent* the body sliding *down* the plane, it is necessary to draw the angle N O R upwards, instead of downwards, and to take $-\phi$ for ϕ in the formula. Then

$$\mathbf{P} = \mathbf{W}\frac{\sin.(\theta - \phi)}{\cos.(\alpha + \phi)} \text{ . . . . . . (4).}$$

Example 1.—If $\alpha = 0$, $\text{P} = \text{W}\dfrac{\sin.(\phi + \theta)}{\cos.\phi}$ for *upward* motion; $\text{P} = \text{W}\dfrac{\sin.(\theta - \phi)}{\cos.\phi}$ for *downward* motion. P is 0 for downward motion when $\theta = \phi$. Under these circumstances the body will just be about sliding down under the action of its own weight.

When $\theta = \phi$, it takes a force $\text{P} = 2\text{W} \sin.\phi$ to drag the body up the plane.

Notice that when the body is pulled up the plane, if θ is a small angle, expanding $\sin.(\phi + \theta)$, and dividing, we find $\text{P} = \text{W}(\tan.\phi \cos.\theta + \sin.\theta)$.

Taking $\cos.\theta = 1$, we have, since $\tan.\phi = \mu$, $\text{P} = \mu\text{W} + \text{W} \sin.\theta$.

Now μW is the force that must be exerted if we have no inclination θ, and $\text{W} \sin.\theta$ is the force that must be exerted if we have inclination but no friction. Hence the rule usually employed in calculating the pulling force on a **vehicle** (Art. 47)—namely, the total pull when the inclination is 1 in n $\left(\text{or } \sin.\theta = \dfrac{1}{n}\right)$—is equal to the pull necessary on a level road plus $\dfrac{1}{n}$th of the weight. It is only true when θ is small.

Example 2.—If $\alpha = -\theta$, so that P acts horizontally, $\text{P} = \text{W}\dfrac{\sin.(\phi + \theta)}{\cos.(\phi + \theta)} = \text{W} \tan.(\theta + \phi)$ for *upward* motion, $\text{P} = \text{W}\dfrac{\sin.(\theta - \phi)}{\cos.(\theta - \phi)} = \text{W}.\tan.(\theta - \phi)$ for *downward* motion.

Example 3.—If $\phi = 0$, so that there is no friction, $\text{P} = \text{W}\dfrac{\sin.\theta}{\cos.\alpha}$.

Example 4.—Find α so that P for *upward* motion may be a **minimum**. That is, what value of α will cause $\dfrac{\sin.(\theta + \phi)}{\cos.(\alpha - \phi)}$ to be a minimum? That is, what value of α will make $\cos.(\alpha - \phi)$ a maximum? Evidently it is a maximum when $\alpha - \phi = 0$ or $\alpha = \phi$, and then $\text{P} = \text{W} \sin.(\phi + \theta)$. This **important result** seems to be completely ignored by men who deal practically with **traction** problems.

Example 5.—If there is no force P, what is the **acceleration** down the plane? We must first find what value of P acting parallel to the plane would just be overcome. This is given in Example 1 as $\mathrm{W}\,\frac{\sin.\,(\theta - \phi)}{\cos.\,\phi}$. Now, this force acts upon the **mass** $\frac{\mathrm{W}}{g}$, and acceleration is force ÷ mass: so that the acceleration is $g\,\frac{\sin.\,(\theta - \phi)}{\cos.\,\phi}$. Of course, when $\phi = 0$, this becomes the well-known $g \sin.\,\theta$.

Example 6.—When the **nut** of a **square-threaded screw** is turned, the surface of the thread is like an inclined plane, the tangent of whose inclination, θ, is $\frac{\text{pitch}}{\pi d}$, if d is the mean diameter of the thread (or the diameter of the pitch cylinder of the screw). The nut presses upon the inclined plane exactly as in Example 2, and hence $\mathrm{P} = \mathrm{W} \tan.\,(\theta + \phi) \ldots . (1)$ if P overcomes W, the total weight to be lifted when the nut is turned; or $\mathrm{P} = \mathrm{W} \tan.\,(\theta - \phi) \ldots . (2)$ if W overcomes P. If the length of the handle to which the real force A is applied be l, then $l\mathrm{A} = \frac{1}{2}\mathrm{P}d$ or $\mathrm{A} = \frac{1}{2}\frac{\mathrm{P}d}{l}$; so that (1) and (2) become

$$\mathrm{A} = \tfrac{1}{2}\frac{d}{l}\,\mathrm{W} \tan.\,(\theta + \phi) \ldots . (1),$$

$$\mathrm{A} = \tfrac{1}{2}\frac{d}{l}\,\mathrm{W} \tan.\,(\theta - \phi) \ldots . (2).$$

(1) and (2) are equal, of course, if ϕ is 0, and then

$$\mathrm{A} = \tfrac{1}{2}\frac{d}{l}\,\mathrm{W} \tan.\,\theta.$$

In this case, as there is *no friction*, the **efficiency** is 1, and the **mechanical advantage** is $\frac{\mathrm{W}}{\mathrm{A}} = \frac{2\,l}{d \tan.\,\theta} \ldots . (3)^*$. The mechanical advantage from (1) when there is friction is $\frac{2\,l}{d \tan.\,(\theta + \phi)} \ldots . (3)$.

Hence the **efficiency** when there *is* friction is $e = \frac{\tan.\,\theta}{\tan.(\theta + \phi)} \ldots . (4)$.

When W overcomes A the efficiency is similarly $e = \frac{\tan.\,(\theta - \phi)}{\tan.\,\theta} \,..\, (5)$.

Both (4) and (5) become 1 when there is no friction.

It is an easy exercise in the **calculus** to show that (4) is a **maximum** when $\theta = \frac{\pi}{4} - \frac{1}{2}\phi \ldots . (6)$, and that (5) is a maximum when $\theta = \frac{\pi}{4} + \frac{1}{2}\phi \ldots . (7)$.

If, then, a screw is to work as much in one way as the other, it seems reasonable to use $\theta = \frac{\pi}{4}$ or 45° as the angle of its **mean spiral**.

Note that when θ is as small as ϕ, W cannot overcome A, and the screw will not **overhaul** or reverse. In this case, if A overcomes W, $e = \frac{\tan.\ \phi}{\tan.\ 2\phi}$. When a **lubricant** is used we cannot be sure that the here assumed law of friction holds; that is, μ is not a constant. When a lubricant is not used it is safe to say that in no actual **screw-jack** does one find θ approaching ϕ in value.

Note that when there is no friction the mechanical advantage is $\frac{W}{A} = \frac{2l}{d \tan.\ \theta}$, where $\tan.\ \theta = \frac{\text{pitch}}{\pi d}$; so that

$$\frac{W}{A} = \frac{2l\pi d}{d \times \text{pitch}} = \frac{2\pi l}{\text{pitch}},$$

or *circumference described by the end of the handle ÷ pitch*, the rule given already.

I have not considered the very considerable loss of efficiency due to friction because the load which is lifted is being kept from turning. Even the carefully-constructed and well-lubricated square-threaded **screw-jack** in my laboratory has an **efficiency** of only about ·25 even at the highest loads. The ordinary jack used in workshops has a very much smaller efficiency than this. When the screw has a **triangular thread** we may assume that with the same kinds of rubbing surface the co-efficient of friction has **greatly** increased; in reality it is the **normal pressure** on the thread which has increased.

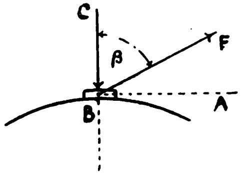


Fig. 52.

90. When a *pulling* force F, making an angle β with the normal, is applied to move a block which is being pressed against a surface, the **tangential component** which overcomes **friction** is $F \sin.\ \beta$. The **normal component** $F \cos.\ \beta$ diminishes R, so that the friction is $\mu(R - F\cos.\ \beta)$. Hence, when sliding occurs,

$$F \sin.\ \beta = \mu(R - F\cos.\ \beta),$$

$$\text{or} \quad F = \frac{\mu R}{\sin.\ \beta + \mu \cos.\ \beta} \quad \ldots\ldots (1).$$

For any given value of μ it is easy to find the value of β which will make F a **minimum**. In fact, the denominator of (1) is a maximum when $\cos.\ \beta = \mu \sin.\ \beta$, or $\mu \tan.\ \beta = 1$, or β is the complement of ϕ, the **angle of repose**.

But if F is a *pushing* force, we have $F \sin.\ \beta = \mu(R + F\cos.\ \beta)$; so that $F = \frac{\mu R}{\sin.\ \beta - \mu \cos.\ \beta}$. In this case F becomes **infinity** if β is less than the complement of ϕ. For an angle β which is less than 90° the *pushing* F must be greater than the *pulling* F. When one piece of machinery drives another at a **sliding contact** this great **distinction** between pushing and pulling must be remembered.

EXERCISES.

1. A weight of 5 cwt. resting on a horizontal plane requires a horizontal force of 100 lbs. to move it against friction. What, in that case, is the value of the co-efficient of friction? *Ans.*, 0·18.

2. A weight of 50 lbs. is supported by friction alone on an inclined plane: what is the force of friction? Angle of plane, $\sin.^{-1}\frac{2}{5}$. *Ans.*, 20 lbs.

3. A body placed on a horizontal plane requires a horizontal force equal to one-half its own weight to overcome the friction. If the plane be gradually tilted, at what angle will the body begin to slide? *Ans.* 26° 26′.

4. Find the force parallel to the plane required to draw a weight of 40 lbs. up a rough inclined plane if $\mu = \frac{1}{4}$, the inclination of the plane being such that a force of 12 lbs. acting at an angle of 15° to the plane would support the weight if the plane were smooth. *Ans.*, 24·35 lbs.

5. On a rough inclined plane it is found that a body is just supported on it by a horizontal force equal to three-quarters the weight of the body. Find the co-efficient of friction. Angle of plane, $\sin.^{-1}\frac{3}{5}$. *Ans.*, $\frac{3}{16}$.

6. Two unequal weights, w_1 and w_2, of the same material on a rough inclined plane are connected by a string which passes over a fixed pulley in plane. Find inclination of plane when the system is in equilibrium.

$$\textit{Ans.},\ \tan.^{-1}\frac{\mu\,(w_1 + w_2)}{w_1 - w_2}.$$

7. Two rough bodies, w_1 and w_2, rest on an inclined plane and are connected by a string, the inclination of string to the horizontal being the same as the plane. If the co-efficients of friction are μ_1 and μ_2 respectively, find the greatest inclination of the plane when the system is in equilibrium.

$$\textit{Ans.},\ \frac{\mu_1 w_1 + \mu_2 w_2}{w_1 + w_2}.$$

8. A plane is inclined to the horizontal at 24°: there is a load of 1,200 lbs. on it, the co-efficient of friction being ·18. Determine (1) the force required to just draw the load up the plane; (2) the force required to just prevent it slipping down; (3) the force required to support it if there be no friction, the direction of the line of action of the force being in each case 23° with the plane, and above it. Determine the corresponding results if the direction of the force be parallel to the plane, and again if the force be horizontal. Obtain also the least force, in magnitude and direction, which is necessary in each of the three cases first referred to: that is, for raising, lowering, and supporting without friction. There are twelve results in all.

Ans.,

	Effort E in pounds.			
	E making 23° with plane.	E parallel to plane.	E horizontal.	E a minimum.
Upward motion.	692	685	816	674
Downward motion.	342	291	295	286
No friction.	530	488	534	488

9. A load of 1,200 lbs. has to be pulled along a horizontal plane. Determine in magnitude and direction the least force necessary to do this, the co-efficient of friction being taken 0·4. *Ans.*, 446 ; 21° 48′.

10. Determine the efficiency in the case of a screw 2½ inches diameter, in which there are four threads to the inch, taking the co-efficient of friction ·04. *Ans.*, 0·442.

11. Taking the mean diameter of the threads of a 1 inch bolt to be ·92 inch, number of threads per inch 8, the co-efficient of friction reduced to an equivalent square thread ·17, find the turning couple required to overcome an axial force of 2½ tons. *Ans.*, ·20 : 46·1 pound-feet.

JOINTS WITH FRICTION.

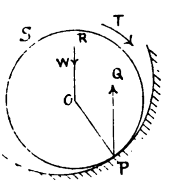


Fig. 53.

91. When we know the resultant of the loads or forces on a **pin** or **journal**, and the law of friction, we can calculate the resultant force on the **eye** or **step**. Thus, if R O represents the resultant load W upon the journal S P, and there is also a twisting couple turning the journal in the direction of the arrow T, the journal rides up in the step till P is the point of contact, and until O P Q = ϕ is the angle of repose, P Q being a vertical force equal to W, and this is the resultant of the forces with which the step acts on the journal; μ, the coefficient of friction, is tan. O P Q.

It is usually stated in books that the resultant force between an eye and pin must act through some point such as P in the surface of contact. To show that this is a mistake, imagine A B and B C two pieces with a pin joint at B. These letters indicate the centres of pins at A, B, and C. If we neglect the weights of the pieces, imagine frictionless pins at A and C, then any force communicated from A to C through the frictional joint B is in the straight line A C; for the resultant force of the pin A upon the piece A B acts through the centre A, and the resultant force of the piece B C on the pin C acts through the centre C. Imagine the pin at B to be rigidly fixed to B C. The force between A B and C B at the joint B must be in the direction A C; and we can have so much friction at B that A C does not necessarily act through any point of the

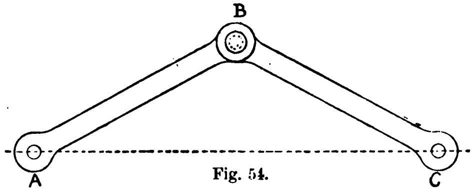
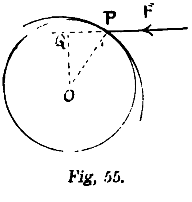


Fig. 54.



Fig, 55.

surface of contact. We cannot, however, imagine this happening if the pin and eye at B meet only at one point or line at right angles to the paper; and it is therefore important that pins and eyes should be of slack fit, and we shall in future imagine them to be so. We avoid unnecessary complication if we imagine that when two pieces act through a pin, the pin is rigidly attached to one of them. Let C D and E F be two frictional pins acting on the piece G H. Let A and B be their centres. Let the coefficients of friction be a and b, or the angles of repose ϕ and ϕ^1. Let the radii of the pins be A and B. Describe about A a circle with the radius A sin. ϕ, or A $\frac{a}{\sqrt{1+a^2}}$, and about B a circle with the radius B $\frac{b}{\sqrt{1+b^2}}$. These may be called the **friction circles** of the pins.

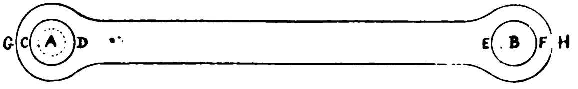


Fig. 56.

If there were no friction, the equal and opposite forces at A and B acting on the pin would be in the direction A B. They are really in the direction of a common tangent to the two circles. There are four positions of this common tangent depending upon the direction of relative sliding of pin and piece at each place.

There are certain cases in which we see the direction without trouble. Thus: if A (Fig. 57) is the centre of the **crosshead** of a **steam-engine**, and C of the crank shaft. If the resultant force due to the piston-rod, etc., is F at A, with no friction, we know that F sec. B A C is the pushing force in the connecting-rod A B, and the moment of this about C is F sec. B A C × C D cos. B A C = F × C D. If, then, we had no friction at the slide, but only in the pins A and B, as we see that both frictions diminish the turning moment, the **line of resistance** touches the two friction circles drawn at A and B in the manner shown in the figure, and the turning moment on the crank shaft is F × C E. Happily, in steam-engines that are well attended to the friction circles are so small that we may practically neglect them, and consider the force between a rod and pin to be truly through the centre of the pin. Should we, however, as in the case of the hinged joint of an **arch**, fear friction at the joint, it is worth while remembering that whatever be the loads on a piece,

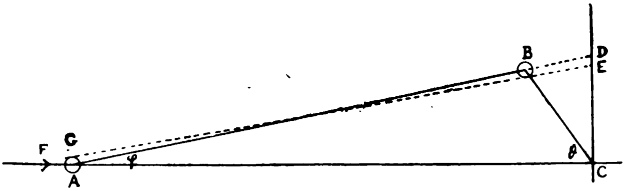


Fig. 57.

the necessary equilibrating forces at the pins are to be found tangential to the friction circles there. Thus, if A and B (Fig. 58) are the centres of the hinges of the arch A C B, draw the friction circles at A and B. It is not enough to find the line of resistance A C B, but also other possible ones not passing through the centres A and B, but touching the friction circles there. If α is the inclination of the centre line of the arch at A, and r is the radius of the friction circle there, there will be no great error in adopting this rule;—The line of resistance may start from any point between A' and A'' in the vertical through A, if $A'A = A\,A'' = r/\sin.\ \alpha$.

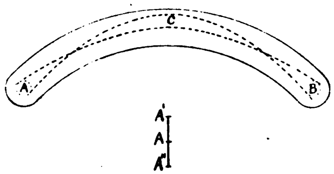


Fig. 58.

The smaller, therefore, the radius of the cylindric surface or hinge of a **hinged arch** the better. When an arch ring abuts at a cylindric surface of large radius, I cannot see much distinction between it and an arch fixed at the ends, except that there may be no tension in the joint, or rather that the **line of resistance** may not pass outside the joint.

92. Body turning about an Axis.—In Fig. 59 we have a body which can move about an axis. It is acted on by a number of cords exerting forces which just balance one another. Now, if you make this experiment you will find that you must keep your finger on the body, because it is in such a state that a very small motion either way causes the forces to no longer balance. Suppose, however, you were to let the cord A be wound on a pulley whose radius is equal to the distance A O; the cord B on a pulley whose radius is equal to B O, and so on, you would have the arrangement shown in Fig. 60, which differs from Fig. 59 in that a *small* motion has no effect on the balance. Now what is the

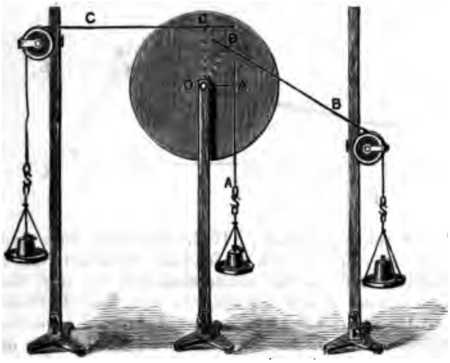


Fig. 59.

condition of balance in this case? Suppose one complete turn given to the axis, every cord shortens or lengthens by a distance equal to the circumference of the pulley on which it is wound. Let A and B lengthen, and let C shorten, then we know that the work done by A and B must be equal to the work done against C. Hence,

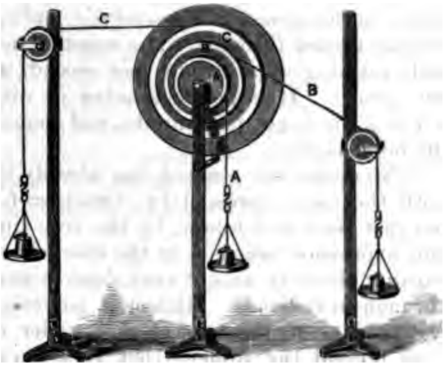


Fig. 60.

> Pull in A × circumference of A's pulley, together with pull in B × circumference of B's pulley, must be equal to pull in C × circumference of C's pulley.

We might, however, use the diameters or radii of the pulleys, and so we see that in Fig. 60 there is balance if

> Pull in A × A O, together with pull in B × B O, equals pull in C × C O.

The pull in A × A O is really the tendency of A to turn the body about the axis, and in books on mechanics it is called the **moment of the force** in A about the axis O. **The law is** then, if a number of forces try to turn a body and are just able to balance one another, the sum of the moments of the forces tending to turn the body *against* the hands of a watch must be equal to the sum of the moments of the forces tending to turn the body *with* the hands of a watch. We sometimes say:—"The *algebraic sum* of the moments of all the forces is *zero*." That is, we regard one kind of moment as *positive* and the other as *negative*. Another way of putting the proof is this:—

If a much magnified drawing be made showing a very small motion through the angle θ, it will be seen that the **work** done by C being, either C × motion of point of application resolved in direction of C, or motion of point × resolved part of C in direction of motion; in either case this work is equal to C × O C × θ, that is, moment of C × θ. Hence, total work =

algebraic sum of moments × θ, and if total work is 0, then the algebraic sum of the moments is 0. When work is done upon a body in turning it, observe that the work in *foot-pounds* is equal to the moment in *pound-feet* multiplied by the angle in radians turned through. If a constant moment M acts upon a body rotating at a radians per second, Ma **is the work done** per second. If n is the number of revolutions per minute, $2\,\pi\,n$ is the angle per minute, and hence $M\,2\,\pi\,n \div 33{,}000$ is the horse-power.

[No doubt the student has already become quite familiar with the idea expressed by "work = force × distance." It has just been shown how, by the very simplest transformation, this expression becomes, in the case where a force is producing turning about an axis, "work done = moment × angle turned through in radians." Although not required for the study of the portion of the subject now under consideration, yet we may remind the student that at a later stage it will be an advantage to him if he has accustomed himself to calculating work done when the turning moment (called the *torque*) and angle turned through are given.]

93. The Lever.—Thus, for example, a lever is a body such as I have spoken about, capable of turning about an axis. You will find that our general **rule of work** and this **rule of moments** will give the same result. *If two forces act on a lever, they will balance when their moments about the axis are equal;* that is, when P, multiplied by the shortest distance from the fulcrum or axis to the line in which P acts, is equal to W multiplied by the distance of the fulcrum from the line in which W acts.

If a number of forces balance when acting on a lever, the sum of the moments tending to turn the lever against the hands of a watch must be equal to the sum of the moments tending to turn the lever with the hands of a watch.

It must be remembered that if the body acted upon has its centre of gravity somewhere else than in its axis, then we must consider that the weight of the body is a force acting vertically through its centre of gravity.

Example.—The **safety valve**, Fig. 61, must open when the pressure on the valve is just 100 lbs. per square inch. The mean area of the valve A, on which we assume that the pressure acts, is 3 square inches; C D is 2 inches, E is 50 lbs., the *weight of the* lever is 6 lbs., and its centre of gravity is 6 inches from D; where must E be placed? All distances are measured horizontally. Here, the upward force is 100 × 3, or 300 lbs., and its moment about D is 300 × 2, or 600. The moment of the weight of the lever is 6 × 6, or 36. The moment of the weight E is 50 × the required distance from D. Hence, 600—36, or 564 divided by 50, is the answer; 11·28 inches from D. Thus we find where the mark 100 ought to be placed.

Let the student repeat this for pressures of 90, 80, and 70 lbs. per square inch, stating in each case the distance of the weight from D. What are the distances apart of the marks?

This is not the place to consider the various forms of safety valve employed by engineers. The force which fluid at rest can exert against a valve is not the same as when the valve is open and the fluid is moving; but it is very important that a valve should stay open, allowing the fluid to escape. This is effected partly by using a peculiar shape of valve, but more usually by letting E come closer to the fulcrum when the valve opens. In the above figure the valve seat is conical; in practice, a flat seat is now much more commonly used, the breadth of the seat being very small.

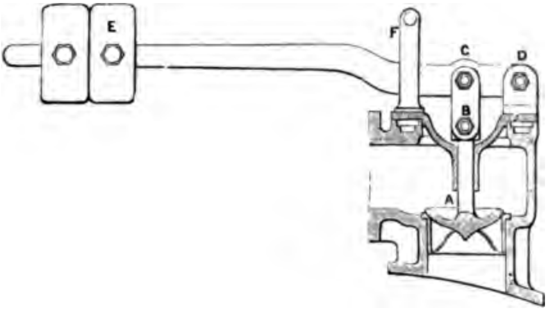


Fig. 61.

Exercise.—A **weighbridge** consists of three levers whose mechanical advantages help each other; I mean, the short arm of each supports the long arm of the next. Suppose that the weights of all parts are arranged so as just to be balanced when no load is on the bridge, and that the mechanical advantages of the three levers are 8, 10 and 12, what load will be balanced by a weight of 15 lbs.? *Answer*—14,400 lbs. Suppose that it is the *first* of these levers that is alterable (that is, there is a sliding weight), what is its mechanical advantage altered to when the load is 16,000 lbs.? *Answer*—It was 8; it now becomes increased in the proportion of 16,000 to 14,400, so that it becomes 8·8889.

Show that the **graduation** of the lever with the sliding weight is in equal divisions for equal alterations of the load. Do this by finding the position of the sliding weight for various loads.

Friction is greatly got rid of in weighing-machines by using steel **knife edges** as fulcrums of levers resting on hard steel or agate plates. Students must examine actual specimens. The common **chemical balance** must be examined. There is one thing about it which may trouble the student. "Why are short-armed balances now preferred to the older forms?" The short-armed balance has less moment of inertia, and this causes it to be quicker in its motions, so that time is saved. (See Art. 453.) But there is much more to be said about it than this. Indeed, in this as in all other cases, to thoroughly understand one machine requires a knowledge of the whole of applied mechanics and applied physics. I am not now discussing any one machine exhaustively. I was strongly tempted to take up the thorough consideration of one machine and call this the study of applied mechanics; and if I had a student with a particular interest in one machine this would be the very best way to put before him the study of applied mechanics. The method I have adopted in this book is to illustrate each principle by means of a machine in which that principle happens to appear most important. The defect of the method arises from its causing a student to think that he knows all about a machine when he only knows the most important principle of applied mechanics which is illustrated by it. The cure for this **academic** training defect comes when a student is compelled to take a special interest in some one machine, and it is then, in practical work, that he really is learning applied mechanics. We can only partially correct the defect by our **numerical exercise** and **laboratory** work and experience in the **workshop**.

Students looking at weighing-machines ought particularly to notice that when objects to be weighed are placed not on swinging scale-pans, but on fairly firm **platforms**, the construction of the balance must be such as to make the measurement to ***be independent of the*** **position** of the object upon the platform.

EXERCISES.

1. A uniform straight bar, 2 feet long, weighs 5 lbs.: it is used as a lever, and an 8 lbs. weight is suspended at one end: find the position of the fulcrum where there is equilibrium. *Ans.*, 4·5 inches.

2. A lever safety-valve has the following dimensions:—Mean diameter of valve, 3 inches; weight of valve, 8 lbs.: distance of centre of valve from fulcrum, 2½ inches; weight of lever, 16 lbs.: distance of its centre of gravity from fulcrum, 13 inches. Find where a weight of 35 lbs. must be hung from the lever, so that the steam may blow off at 85 lbs. per square inch. *Ans.*, 36·4 inches from fulcrum.

3. A B C D is a rectangle, and A C a diagonal. In A C take a point O, such that A O is a third of A C. Forces of 30, 10, and 5 act from A to B, C to B, and D to C respectively. If A B = 3 inches, and B C = 4 inches, write down the moment about O of each force, with its proper sign, and find their algebraic sum. *Ans.*, -40; $+20$; $+\frac{40}{3}$; $-\frac{20}{3}$.

4. The diameter of the safety-valve of a steam-boiler is 4 inches: the weight on the lever is 90 lbs., and distance from centre of the valve to the fulcrum is 2·5 inches; what must be the distance of the point of suspension of the weight from the fulcrum in order that the valve will just lift when the pressure of steam in the boiler is 80 lbs. per square inch? Length of lever, 30 inches; weight, 12 lbs.: distance of centre of gravity from fulcrum, 14 inches. *Ans.*, 26·059 inches.

5. A ball weighing 4 lbs. is fixed to one end of a bar hanging vertical, the other end of which can turn about a fixed axis. If the ball be pulled out by a force which acts horizontally through the centre of the ball, what will be the amount of this force necessary to keep the bar at rest in a position such that the bar makes 30° with the vertical? If the length of the bar be 30 inches, what would be the force if applied at a point on the bar 10 inches from the centre of the ball? *Ans.*, $\frac{4}{3}\sqrt{3}$; $2\sqrt{3}$.

CHAPTER VII.

ELEMENTARY ANALYTICAL AND GRAPHICAL METHODS.

94. IN Fig. 62 we have another example of the fact that when there is a displacement of a point, O, in the direction, O T, and a force, P, acts in the direction O A, the angle A O Q being called α, the **work done** is the displacement multiplied by the component of P in the direction O T. This component is $P \cos. \alpha$, or it may be defined as the **projection** of O A, the line which represents the force, upon O T.

The student must note what we mean by the *projection* of a line. The projection of O A upon the line O T is O Q. If the arrow-head is not put upon O Q, the order of the letters will indicate the sense of the action. Thus I shall use Q O to mean a sense opposite to O Q. Now note that if we have lines, O A, A B, B C, C D, and D E (Fig. 63), what I call the sum of their projections upon any line is really the projection of O E. If I have a **closed polygon,** the sum of the projections of its sides upon any line is zero, if the sense of every side is circuital round the polygon with the sense of all the rest. If the sides are parallel to, and proportional to forces, this means that the sum of all the components of all the forces in any direction whatsoever is zero.

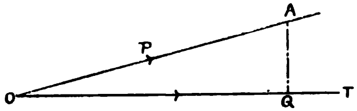


Fig. 62.

Now suppose we have a body acted upon by any number of forces in all sorts of directions (we can only illustrate this by strings and weights); and if the weight of every portion of the body itself be considered also acting; and if the body is in *equilibrium,* the **principle of work** tells us that if the body receives *a small translational* displacement, s, in any direction

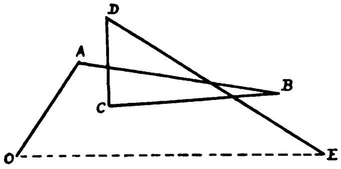


Fig. 63.

whatsoever (**translation** means that every point moves parallel to the motion of every other point, and through the same distance), then the total work done by all the forces is zero. But the work done by any force is its component in the direction of the motion multiplied by the displacement; hence, we see that the sum of the components in any direction whatsoever must be zero. This is a condition which must hold when any body is in **equilibrium.** But it is evident that we have the same rule when we say that the forces are proportional to the sides of a closed polygon, the senses of the forces being circuital round the polygon. If the forces are not in one plane, the polygon will be what is called a **gauche** polygon, but if the amounts and directions of the forces are given in **plan** and **elevation,** the polygon may be drawn in plan and elevation. To give the **analytical rule** more simply:—Take three lines mutually at right angles to one another, O X, O Y, O Z; project all the forces upon O X, they balance (that is, the algebraic sum of their projections is zero); project all the forces upon O Y, they balance; project all the forces upon O Z, they balance. These three analytical conditions are the same as the graphical rule, "the force polygon is closed."

Perhaps, however, we had better confine our attention to forces all in **one plane.** The **analytical rule** is:—The algebraic sum of all the horizontal components is zero, *or* the horizontal components balance; the algebraic sum of all the vertical components is zero, *or* the vertical components balance. These two conditions are the same as "the force polygon is closed."

EXERCISE.

Given the following forces, find their equilibrant—Force O X of 50 lbs., force O A of 20 lbs., the angle, X O A, being 33°; force M O of 56 lbs., the angle X O M being 150°; force O G of 100 lbs., the angle X O G (all angles are measured counter-clockwise) being 217°; force H O of 70 lbs., the angle X O H being 315°. Now let a student draw these on paper. I have assumed them all to be in lines through a point O, but this was for ease in setting the question. The forces need not act through one point. Let him, by drawing, or by using a table of sines and cosines, find the components of all in the direction O X and O Y, if O Y is perpendicular to O X (Art. 31). I make these projections to be—

In the direction O X.	In the direction O Y
50 cos. 0 = 50	50 sin. 0 = 0
20 cos. 33° = 16·773	20 sin. 33° = 10·893
+ 56 cos. 30° = 48·497	− 56 sin. 30° = − 28
− 100 cos. 37° = − 79·863	− 100 sin. 37° = − 60·181
− 70 cos. 45° = − 49·497	+ 70 sin. 45° = 49·497
Algebraic sum = Σ X = − 14·09	Algebraic sum = Σ Y = − 27·791

The Greek letter Σ is used to mean "the sum of all such terms as." I use X to mean any P cos. a, and Y to mean any P sin. a. We have found ΣX, the sum of all the projections on O X, and ΣY, the sum of all the projections on O Y; ΣX and ΣY are the projections of the **resultant** of all the forces. Call them $\bar{X}$ and $\bar{Y}$; change their signs and they are the projections of the **equilibrant**. Given the two projections of a force, it is very easy graphically to find the force itself. Calling it O R or R O, it is evident that its amount is $\sqrt{\bar{X}^2 + \bar{Y}^2}$, and its direction is such that $\frac{Y}{X}$ = tan. X O R.

In the above case the *resultant* is R O = 31·158, and the angle X O R is 243° 7′. The *equilibrant* is O R.

The student must get accustomed to symbols which so greatly shorten our use of language. **Our rule is**:—If P_1, P_2, etc., are forces in a plane, and if they make angles a_1, a_2, etc., with any line, say a horizontal line, then P_1 cos. $a_1 = X_1$, P_2 cos. $a_2 = X_2$, etc., are the horizontal components of the forces. The *directions* of arrow-heads must be noted, and the *algebraic* sum of the horizontal components is denoted by $X = \Sigma P$ cos. a, or ΣX. In the same way $Y = \Sigma P$ sin. a, or ΣY, denotes the algebraic sum of the vertical components $Y_1 = P_1$ sin. a_1, $Y_2 = P_2$ sin. a_2, etc. Then the resultant, R, of all the forces is the resultant of $\bar{X}$ horizontally and Y vertically, and $R^2 = \bar{X}^2 + \bar{Y}^2$, and if R makes the angle a, with the horizontal, then tan. $a = \bar{Y} \div \bar{X}$. We have only found the amount, **clinure** and sense of R. We do not yet know where it acts. (The word *clinure* was, I believe, invented by the late Professor James Thomson; the word *direction* implies more than what we *try to express* by clinure.)

Similarly, if the forces are not in the same plane, and if each P makes the angles α, β, and γ, with three standard lines at right angles to one another (usually called the axes of x, y, and z), then if P cos. α is called X and ΣP cos. $\alpha = \bar{X}$, if P cos. β is called Y and ΣP cos. $\beta = \bar{Y}$, and if P cos. γ = Z and ΣP cos. $\gamma = \bar{Z}$, then $R^2 = \bar{X}^2 + \bar{X}^2 + Y^2$ and the cosines of the angles which R makes with the axes are $\bar{X}/_R$, $\bar{Y}/_R$, and $\bar{Z}/_R$.

95. Any physical quantity which is *directional* is called a **vector** (such as a displacement, a velocity, an acceleration, a force, a stress, the flow of a fluid, etc.), and may be represented by a straight line. The *length* of the line represents the quantity to some scale of measurement; the line's *clinure* represents the clinure of the vector, and the barb of an arrow represents its *sense*. It is easy by actually drawing lines and measuring their lengths to solve problems which would otherwise require a good deal of mathematical knowledge. This sort of graphical calculation having proved useful, it has attracted the attention of men who have leisure enough to make an elaborate study of its methods. It has, unfortunately, become a complicated weapon with which these men can attack all sorts of problems which are much more easily solved in other ways.

We shall only use graphical methods where they happen to be the best methods. Now a force is a vector; it has magnitude, clinure, and sense, but it is more than a vector; it has a fourth quality not possessed by ordinary vectors—namely, **actual position** in space. Settling any one point in its line of application settles its position when its clinure is known. But if we are told that forces all act at a point, they are added exactly as all mere vector quantities are added. When, then, I speak of finding the resultant or equilibrant of forces at a point, I may be said to speak of any vectors.

96. Forces acting at a Point.—The line A B (Fig. 64) represents a force in *clinure* by its own clinure, in *amount* by its length to any scale we please, and in *sense* by its arrow-head, which shows that the action of the force is from A to B.

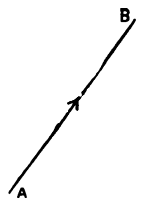


Fig. 64.

It would not be correct to call this the force B A, because this is opposed to the sense of the arrow-head. The forces A O, O B, O C, O D (Fig. 65), all act upon a small body, O, or their lines of action when produced all pass through a point, O, in a large **rigid body**. The amount of each force is shown by the length of the line, representing it to some scale. Now to *add* these forces together in the most perfect manner—that is, to find a force called their *resultant*, which shall be exactly equivalent in its effects to all the above forces acting together—we draw a polygon (Fig. 66). Each side of this polygon is parallel to, and proportional to a force in Fig. 65; thus, the side A corresponds to the force A O, and the arrow-heads agree, and lastly the action indicated by the arrow-heads is circuital. Fig. 66 is always called **the force polygon.** When it is *unclosed*, as it is in the present case, we know that the forces A O, etc. (Fig. 65), are *not* in equilibrium. To keep A O, O B, etc., in equilibrium, a new force, called the *equilibrant*, must be introduced corresponding to the side E (shown dotted), which will close the polygon, its arrow-head being circuital with the others. Now if we want the resultant of A O, O B, O C, O D, it evidently acts through O and corresponds to E, Fig. 66, but with arrow-head reversed. *The resultant of a number of forces is equal and opposite to their equilibrant.*

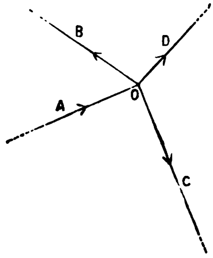


Fig. 65.

Prove now the following statements by actual drawing :—

1st. The resultant of any number of forces does not depend on the **order** in which they are drawn as sides of the polygon.

2nd. Any lines or forces whatever which form a closed polygon in any given order will form a closed polygon if drawn in any other order.

3rd. In adding forces we may first find the resultant of some of the forces, and then add together this resultant and all the rest of the forces. The answer will **always be the same**, however we may group the forces before adding them.

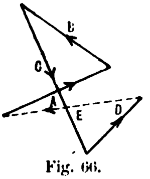


Fig. 66.

When the forces are not all in one plane the polygon must be drawn by **descriptive geometry**, and to draw it, and so find the resultant or equilibrant as the closing side, is an excellent *graphical* exercise.

In working exercises we recollect the fact that the resultant of all the forces due to gravity is called the weight of the body and acts through its centre of gravity (see page 136). This one force replaces the millions which are due to gravity. When we observe that we have only *three* forces acting upon a body at rest, we know that they **must be in a plane** and **act through a point** unless they are parallel. In the ordinary books on mechanics there are many problems which are **easily solved** if we remember this fact. *Four* forces in equilibrium, if not all in one plane, must meet in one point. When a body touches a *smooth* surface we know that the force which there acts upon the body is normal to the smooth surface. When a body touches a *rough* surface we know that the force which there acts upon the body makes an angle with the normal, and the limiting value of this angle when sliding is about to occur is what is called the **angle of repose**, or the angle whose tangent is μ, the co-efficient of friction there. It is astonishing what a number of exercises are easily worked out if one will only recollect these few **general principles.**

The student will at this place work again the exercises of page 111. It ought to be getting clear to him that the most difficult analytical work has really nothing in it more complicated than these exercises have. The exercises are usually called easy, certainly students get easily into the way of working them, and I am always sorry to notice too great an ease of this kind. It often indicates shallowness of comprehension.

EXERCISES.

1. Forces OA = 30 lbs, OB = 50 lbs., CO = 15 lbs., DO = 80 lbs., OE = 150 lbs.; the angles are BOA = 45°, COA = 90°, DOA = 135°, EOA = 270°. Find the resultant analytically and graphically.
Ans., 223 lbs.

2. Sheer legs each 50 feet long, 30 feet apart on horizontal ground, meet at point C, which is 45 feet vertically above the ground; stay from C is inclined at 40° to the horizontal; load of 10 tons hanging from C. Find the force in each leg and in stay. *Ans.*, 7·8 tons; 6·4 tons.

97. In many engineering problems, when forces A, B, C, D, etc., are given, it is sometimes important to be able to show graphically the resultant of A and B, the resultant of A, B, and C, the resultant of A, B, C, D, and so on. Thus (Fig. 67) A, B, C, D, etc., are given forces.

Draw the unclosed force polygon (Fig. 68). Join the point O with each corner of the force polygon. From the point B′

re A and B meet (Fig. 67) draw a line B′ C′ parallel to the OBC (Fig. 68) (OBC is the line from O to the corner ere B and C meet); from C′ draw C′ D′ parallel to OCD, d so on. Then B′ C′ represents the position and direction,

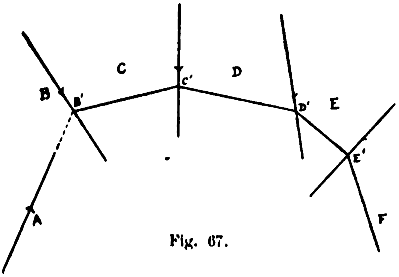


Fig. 67.

and OBC represents to scale the resultant of the given forces A and B. Similarly D′ E′ represents the position and direction, and ODE represents to scale the resultant of the given forces A, B, C, and D. Note the arrow-heads of the resultants we have found. The line A B′ C′ D′ E′ F′ (Fig. 67) is usually called a **line of resistance.**

98. We have in Art. 96 confined our attention to the forces acting upon a small body, or forces which all pass through one point if they act on a large body. But in Fig. 67 and in our description we assumed a large body, and our forces were any forces whatsoever. We gave it a small motion of translation, and obtained an important result from the consideration that, on the whole, no work was done upon the body. Now, let us assume that any point O in the body is fixed, or, rather, that an *axis* O is fixed, the axis being at right angles to the plane in which all the forces act; about this axis we assume that the body may *rotate*. Consider the wor done by all the forces during any sma rotation θ; it is zero. But the work do by any force is as we have already se Art. 92 the moment of the force multipli by θ. Hence the sum of all the moments of all the forces ab o is zero if there is equilibrium. But any small plane mot

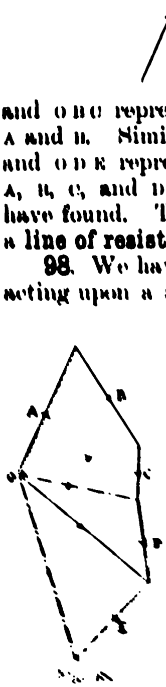


Fig. 68.

of the body whatsoever we know to be resolvable into a motion of **translation** and a **rotation** about some axis O at right angles to the plane. Hence the law of work tells us that if any system of forces is in equilibrium their components in any direction balance one another and their moments about any axis balance one another.

In the numerical exercise Art. 94 let the respective forces (only given in amount, clinure, and sense as yet) be at the following distances from a certain point, which I shall call S. The sign + means that a force tends to turn the body against the hands of a watch. O X is at the distance + 5 feet, O A is at − 2 feet, M O is at − 7 feet, O G is at + 3 feet, H O is at − 4 feet. Let the student now draw these forces in their proper positions relatively to S. The sum of their **moments** is $50 \times 5 - 20 \times 2 - 56 \times 7 + 100 \times 3 - 70 \times 4$, or -162 pound-feet. This is the moment of their resultant, which is 31·158 lbs.; so that its distance from S is − 5·2 feet.

99. If the clinure and sense of a force P be given, it is also necessary to give some point through which its direction passes. Thus, let all the forces be in one plane; let P make an angle a with the horizontal; let the co-ordinates of the given point be x and y referred to the axes of x and y.

If P cos. a is X and P sin. a is Y, then as the moment of P about any axis is equal to the sum of the moments of X and Y, taking moments about the origin, the moment of P is $\mathrm{Y}x - \mathrm{X}y$.

If R is the resultant, its components are $\Sigma\,\mathrm{X}$ and $\Sigma\,\mathrm{Y}$: call them $\overline{\mathrm{X}}$ and $\overline{\mathrm{Y}}$. Also, if $\bar{x}$ and $\bar{y}$ are the co-ordinates of a point in R, $\bar{x}\overline{\mathrm{Y}} - \bar{y}\overline{\mathrm{X}} = \Sigma\,(\mathrm{Y}x - \mathrm{X}y)$.

For equilibrium we must have $\Sigma\,\mathrm{X} = 0$ . . (1), $\Sigma\,\mathrm{Y} = 0$ . . (2), $\Sigma\,(\mathrm{Y}x - \mathrm{X}y) = 0$ . . (3).

Notice that we may have (1) and (2) true without (3) being true. In this case the system of forces reduces to a mere **couple** whose moment about any point is $\Sigma\,(\mathrm{Y}x - \mathrm{X}y)$; such a system is called a **torque.** If we choose any point in the plane, we can replace any system of forces by a single force through this point, together with a couple whose moment is the sum of the moments of the system of forces about this point. This is often an exceedingly **important** fact to remember. (See Art. 100.) The student ought to work many numerical exercises graphically and analytically.

Example 1.—A **beam**, A B C D E, is supported at A and E by forces x and y. The load at B is 3 tons, and A B = 4 feet; load at C is 2 tons, and A C is 7 feet; load at D is $2\frac{1}{2}$ tons, and A D is 9 feet; A E is 12 feet.

Here $x + y = 3 + 2 + 2\frac{1}{2} = 7\frac{1}{2}$ tons. Taking moments about A, the moments *with* the hands of a watch are

$$3 \times 4 + 2 \times 7 + 2\tfrac{1}{2} \times 9 = 48\tfrac{1}{2} \text{ ton-feet.}$$

The moment against the hands of a watch is $y \times 12$, and x has no moment because it acts at A. Hence our second equation is $12\,y = 48\frac{1}{2}$, $y = 4·0417$ tons, and therefore x is 3·4583 tons.

Example 2.—We neglected the weight of the beam itself in Example 1. If its centre of gravity is 4 feet from A, and if the weight of the beam is half a ton, and if x' and y' are the *additional* supporting forces,

$$x' + y' = \tfrac{1}{2}. \qquad \tfrac{1}{2} \times 4 = y' \times 12.$$

Hence, $\qquad y' = \frac{1}{6}$ ton. $\qquad x' = \frac{1}{3}$ ton.

Example 3.—In Fig. 69 O A is part of a beam. Considering only the portion of beam to the right of the section at O, let the loads downward, P_1, P_2, and P_3, and the supporting force upward, Q, be given. Let the perpendicular distances from O be a_1, a_2, a_3, and q. Find a force through O, and a couple to balance the given forces. Call the force S. If it acts downward at O, its amount must be $S = Q - P_1 - P_2 - P_3$. If the couple is called M and it acts tending to turn the body O A round O, with the hands of a watch

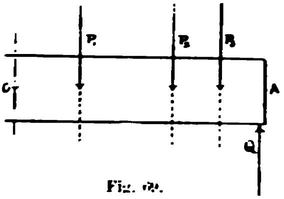


Fig. 69.

$$M = Q\,q - P_1\,a_1 - P_2\,a_2 - P_3\,a_3.$$

When we come to consider beams we shall call S the **shearing force** and M the **bending moment** at the section O.

The student will at this point work the Exercises 1, 2, 3, of pages 134–5, as well as the following:—

1. A uniform beam, 20 feet long and supported at its ends, has weights of 1, 3, 2, and 4 cwts. placed at distances of 2, 8, 12, and 15 feet respectively from one end. Taking the weight of the beam to be 5 cwts., find the reactions at each of the supports. *Ans.*, 7 cwts.; 8 cwts.

2. Draw any line O X, and lines O P, O Q, O R, O S, O T, making angles of 28°, 62°, 118°, 220°, and 305° respectively with O X. Consider that forces act along these lines, their amounts being 25, 34, 14, 42, and 18 lbs. Find the amount and direction of the force which balances these. In doing this first determine the components of each force in the directions O X and O Y, and arrange these in columns as shown in Art. 94. *Ans.*, 15·67 lbs.; 232°·2 with O X.

3. A trap-door of uniform thickness, 5 feet long and 3 feet wide, and weighing 5 cwts., is held open at angle of 35° with the horizontal by means of a chain. One end of the chain is fixed to a hook placed 4 feet vertically over the middle point of the edge on which the hinges are, the other end being fixed to the middle point of the opposite edge. Determine the force in the chain and the force at each hinge. *Ans.*, 2·65 cwts.; 2·5 cwts.

4. A uniform beam, weighing 2 cwts., is suspended by means of two chains fastened one at each end of the beam. When the beam is at rest it is found that the chains make angles of 100° and 115° with the beam; *find the tensions* in the chains. *Ans.*, 1 cwt.; 1·1 cwt.

5. A B is a horizontal uniform bar 1½ feet long, and P a point in it 10 inches from A. Suppose that A B is a lever that turns on a fulcrum under P, and carries a weight of 40 lbs. at B; weight of lever, 4 lbs. If it is kept horizontal by a fixed pin above the rod, 7 inches from P and 3 inches from A, find the pressure on the fulcrum and on the fixed pin.

Ans., 89·143 lbs.; 45·143 lbs.

100. When the forces are not in one plane, let the force P make angles α, β, and γ with the three axes of co-ordinates. Let a point in the direction of P be x, y, z. If $P \cos. \alpha$, or X, $P \cos. \beta$, or Y, and $P \cos. \gamma$, or Z be the three components of P, we can use X, Y, and Z instead of P for all purposes. Positive directions are the directions of increasing x, y or z. Thus the moment of P about any axis is equal to the sum of the moments of X, Y, and Z about the axis. Attention must be paid to the *sense* of each force, and whether it tends to turn the body *against* or *with* the hands of a clock. The student ought to spend time in fixing clearly in his mind the truth of the following **statements**:—The moment of P about the axis of x is $Zy - Yz$; the moment of P about the axis of y is $Xz - Zx$; the moment of P about the axis of z is $Yx - Xy$. Hence we see that if R is the resultant, its components are ΣX, ΣY, ΣZ, and the sum of their squares is R^2, which is therefore easily calculated; also each of them, divided by R, is a direction cosine of R. Again, if x, y, and z be the co-ordinates of a point in R, then

$$\begin{aligned} \bar{y}\,\Sigma Z - \bar{z}\,\Sigma Y &= \Sigma(Zy - Yz), \\ \bar{z}\,\Sigma X - \bar{x}\,\Sigma Z &= \Sigma(Xz - Zx), \\ \bar{x}\,\Sigma Y - \bar{y}\,\Sigma X &= \Sigma(Yx - Xy). \end{aligned}$$

The value of R and its clinure (the angles which it makes with the axes) having already been found, these equations enable the position of a point in the resultant to be found; so that R is completely determined.

For equilibrium we must have $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$, $\Sigma(Zy - Yz) = 0$, $\Sigma(Xz - Zx) = 0$, $\Sigma(Yx - Xy) = 0$.

Given a set of forces, it is evident that we can always sum them into a **resultant force** acting at any point we please to choose, together with a **couple** about some axis. If we are not given the point, it is always possible to reduce any system of forces to a resultant, and a couple whose axis is the resultant force.

101. The Link Polygon.—We shall now consider graphical methods of dealing with forces which do not necessarily act through one point. Take, for example, the forces of 1, 2, 3, 4 (Fig. 70). Draw the **unclosed force polygon** 1′, 2′, 3′, 4′ (Fig. 71), with its sides parallel to and proportional to the forces, and the arrow-heads circuital. Now the dotted line $b\,a$, with its arrow non-circuital with the rest, is parallel to and proportional to the **resultant** of all the given forces. But this does not tell us *where* the resultant force is situated, although it tells us its direction and amount.

From *any* point, o (Fig. 71), draw a line to the junction of 1′ and 2′ (it is easier to say draw the line o 1′ 2′), o 2′ 3′, etc., to all the angles of the force polygon. Now construct a **new** unclosed polygon, with its corners on 1, 2, 3, 4 (Fig. 70), and its sides parallel to o 1′ 2′, o 2′ 3′ etc. (Fig. 71), its last side being parallel to o*a*, and its first parallel to o*b*. We have now found the point, 5 (Fig. 70), where the **first and last sides of the link polygon meet.** The **resultant** of the forces 1, 2, 3, 4, passes through this point, 5, and corresponds to the closing side, *ba*, in direction and magnitude. The new polygon is called **the link polygon** of the forces relative to the **pole,** o. The position of A, the point at which we start to draw the link polygon, may be chosen anywhere on 1, and hence there may be **any number** we please of link polygons for a given position of the pole, o. Again, there are **any number** we please of link polygons corresponding to any other position of o, and we can choose o where we please. Any student who **studies** this in the light of what he did in Art. 96, will see that the link polygon really consists of a **system of links** which would be in **equilibrium** under the given set of forces and the force we have found; and since the mere links only introduce forces which are of themselves in balance, being equal and opposite in each link, the system of forces acting at the joints must balance.

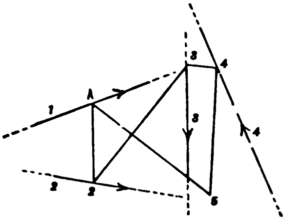


Fig. 70.

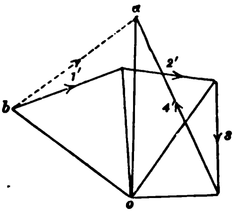


Fig. 71.

Suppose we find that when we are given the forces 1, 2, 3, and 4 (Fig. 72), and we draw the force polygon (Fig. 73), and any link polygon (Fig. 72), that these are both closed, let us **prove that the forces are in equilibrium.**

A system of forces acting on a rigid body is not affected by introducing any number of forces which separately balance one another. Now let a force represented by the length of the line o 1′ 2′ act at the point A in the direction B A, its sense being shown by the arrow-head near A, and let an equal force act at B in the direction A B, its sense being opposite to that of the force at A. These two forces are in equilibrium with one another, and they cannot therefore affect the original system of forces in any way.

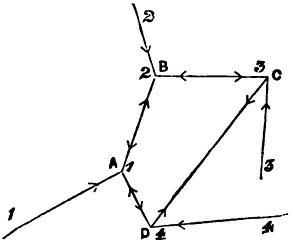


Fig. 72.

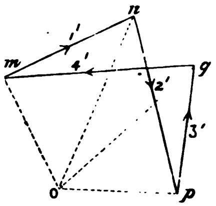


Fig. 73.

Similarly, the forces shown by the arrow-heads in B C, C D, D A are introduced, every pair balancing one another.

Now we see that the three forces at the point A are in equilibrium with one another, because they are **parallel** to and **proportional** in amount to the sides of the triangle *o m n* (Fig. 73), and corresponding arrow-heads would run right round the triangle. Similarly, there is equilibrium at every other corner of the link polygon A B C D; hence all the forces are in equilibrium, and hence the forces, 1, 2, 3, 4, taken by themselves, must be in equilibrium.

The theorems which we wish students to prove by construction can be proved to be **generally true**, reasoning from the fact that a number of forces acting at a point can only have one resultant.

102. We see, then, that the **force** polygon alone is sufficient to find the resultant of any number of forces if the forces meet at a point, but we need also the **link** polygon if the forces do *not* meet at a point.

The *link* polygon really shows that the sum of the turning moments of the forces 1, 2, 3, 4 (Fig. 70) about any point is equal to the moment of the resultant about the same point. The *force* polygon pays no regard to turning moments of forces; it merely tells us about the resultant of the forces, supposing that they all passed through the same point.

103. You ought **by actual drawing** to **illustrate** the truth of the following four ways of putting one statement. If you use coloured inks your drawings will be more instructive.

1st. The resultant of any number of forces is independent of the **order** in which we draw them in the force polygon, and draw between them the sides of the link polygon.

2nd. In adding forces we may first find the resultant of some of the forces, and then add together this resultant and all the other forces. The answer will always be the same, however we may group the forces before adding them.

3rd. If the force polygon of a number of forces is closed, and if we can draw a closed link polygon, then all the link polygons we may draw will also be closed.

4th. If any other pole be taken in Fig. 71, and another link polygon be drawn and a new point 5 (Fig. 70) is found, both of the points so found lie in a straight line parallel to *b a* of Fig. 71.

You will also find, and it is easy to prove, that the locus of the point in which any two sides of the link polygon meet is parallel to the line which closes the corresponding portion of the force polygon. Again, if *b* is taken as pole instead of o, the last side of the link polygon is found to be in the direction of the resultant of the forces 1, 2, 3, 4; and, generally, any side of the link polygon is in the direction of the resultant of the corresponding number of the given forces. Thus, if *b* is taken as pole, 4, 5 becomes the resultant of the forces 1, 2, 3, 4, and 3 4 becomes the resultant of the forces 1, 2, 3. It is evident from this that the direction of the resultant of any two forces, or of any number of forces, which meet at a point passes through their point of intersection.

A system of forces may not reduce to a resultant force, but be equivalent to a *couple*. When this is the case the force polygon is closed, and the first and last sides of any link polygon that may be drawn are parallel to one another. You may also find it worth your while to **prove** by construction this statement: if two link polygons are drawn for two positions of the pole o, the corresponding sides of the two polygons meet in points which lie in a straight line parallel to the line joining the two positions of the pole o.

If you have been able to make a few drawings such as I have been speaking about, and so take an interest in this easy and instructive method of working mechanical exercises, you ought to work by means of it some such exercises as the following:—

104. 1. *Exercises.*—In Exercise 2 of page 128, draw the force polygon, taking the forces in the order O P, O R, O Q, O T, O S, and observe that the resultant and equilibrant are the same as before. Obtain also the resultant of O P, O R, O T, and the resultant of O Q, O S, and show that these have a resultant equal to the resultant of the five forces.

2. Draw a line H K 4·7 inches long. On H K take points A, B, C, distant from H 0·4 inch, 1·5 inch, and 3·3 inches respectively. Now draw A P, B Q, C R inclined at angles of 72°, 57°, and 37° with H K. Suppose that a force of 214 lbs. acts from K to H, one of 576 lbs. from R to C, one of 132 lbs. from Q to B, and one of 237 lbs. from P to A. (1) Draw a force polygon to determine the amount, clinure, and sense of the resultant. (2) Take any pole O and draw a link polygon to determine the lateral position of the resultant. In giving your answer say what are the perpendicular distances of H and K from the resultant. (3) Draw a new force polygon, taking the forces in a different order, and observe that the resultant is the same as before as regards amount, clinure, and sense. With respect to a new pole O, now draw a second link polygon, and observe that the lateral position of the resultant agrees with that obtained before. (4) Take the first force polygon and choose a new pole, and with respect to it draw a

new link polygon. Observe that this gives rise to the same resultant. That is, the closing sides of each link polygon meet at points, all of which should lie on the same straight line. (5) Calculate the above answers according to the rules of Art. 99.

Ans., (1) 1070 lbs.; 40° with H K; sense same as force R C; (2) 1·3 inches; 1·7 inches.

105. *Example.*—Given a set of forces, find two forces, one given in position, and clinure the other to pass through a given point P, such that these two forces will balance the given set. The ingenious student will find out how to do this himself; other students will benefit by the following instructions.

Draw the force polygon with one corner unknown; choose a pole and begin drawing a link polygon from the point P; the side closing the link polygon enables the missing corner of the force polygon to be found. This problem is one of the very commonest to come before the engineering student. Thus, let any structure (a roof principal or a railway girder, for example) have given loads. Let it be supported at a hinge joint at P, and upon rollers at Q, to allow for expansion. The direction of the supporting force at Q is known, and one point, P, in the other supporting force is known.

106. A student who sees the essential ideas underlying our methods of working will have pleasure in working curious problems, such as the following exercise:—Given a set of forces and given three points, A, B, and C. Draw a link polygon with three of its sides passing respectively through A, B and C.

107. *Exercise.*—Draw a rectangle A B C D, A B = 5 inches, B C = 1·8 inch. From A, along A B, measure off lengths, A E, A F, A G = 1·75, 2·8, 5·7 inches respectively. From D along D C measure off D H, D K, D L = 1·4, 2·4, 2·85 inches respectively. Suppose that forces of the following amounts act on a rigid body—viz., 1,460 lbs. from A to H, 1,085 lbs., from E to K, 808 lbs. from F to L. These are balanced by two others, one of which acts through X, a point in D A distance 0·6 inch from D, and the other has for its line of action Y Y, the line passing through C and G. Find the magnitudes of the balancing forces and the angle between them. (The rectangle is introduced merely as a convenient way of settling, without a figure, the given forces.) Scale $\frac{1}{4}$ inch to 100 lbs. *Ans.*, 2,850 lbs.; 1,140 lbs.

Hint.—You must first find the resultant of the three given forces, and observe where its line of action meets Y Y; this point of intersection, I, joined to X, gives the line of action of the balancing force through X. Three forces now act at I. Their lines of action are known, and the magnitude of one force; hence the amounts and senses of the other two can be found.

108. *Example.*—Given a set of forces, A, B, C, D; given also three lines, X, Y, Z, as the positions of three forces which are to balance our given set. *Find these three.* The method here is not so obvious as the method of the last example. Draw A′, B′, C′, D′, known sides of the force polygon. Draw, also, after D′, X′ parallel to X unlimited in length. In it choose O the pole and join to all known corners of the force polygon. Note that O X′ is itself two radiating lines from O, because there are two corners of the force polygon in it. Now let the point where D and X meet be called a *side* of the link polygon; the intercept on X till it meets Y is another side, and it will now be found that we have sufficient information for the completion of both force and link polygons. As an example, Fig. 74 represents a ladder whose centre of gravity is at G, and weight 300 lbs. A string fastened to it at C in the direction C O keeps it in equilibrium, its end A resting on the smooth wall O A, and its end B on the smooth floor O B. Find the pull in the string and the reactions at A and B.

The forces acting on the ladder are shown by the arrows. Draw Y X (Fig. 75) vertically to represent the weight of the ladder. Draw W Y horizontally of unknown length. Draw X O parallel to O C, and take O anywhere in this line. Use O as pole of the force polygon. Join O Y. Now the link polygon is M N P M, and drawing O W parallel to P M, and W Z parallel to B M, we find that Y X Z W is the force polygon. The lengths of X Z, Z W, W Y represent the forces at C, at B, and at A.

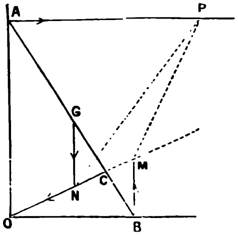


Fig. 74.

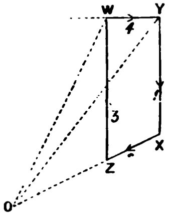


Fig. 75.

The student will find it very instructive now to introduce friction into this problem, and thus create two new problems. (1) If the pull in the cord is just sufficient to *prevent* the ladder from moving. Here the angle O B M ought to be $90° - \phi_1$ where tan. ϕ_1 is the coefficient of friction at B, and O A P ought to be $90° + \phi_2$ if tan. ϕ_2 is the coefficient of friction at A. (2) If the pull in the cord is just sufficient to *produce motion*. *In this case*, O B M $= 90° + \phi_1$ O A P $= 90° - \phi_2$.

Students ought at this stage to work a great number of exercises.

EXERCISES.

(1) In Fig. 74, A B = 20 feet, O B = 10 feet, B C = 2 feet. The weight of the ladder is 300 lbs. Find the pull in C O and the reactions at B and A. If $\mu = 0·1$, find the pull in the rope; first, if B is made to approach O; second, if B is about to recede from O.

(2) A horizontal beam A F has loads, at B of 1,000 lbs., at C of 250 lbs., at D of 1,200 lbs., at E of 600 lbs. If A B = 5 feet, A C = 9 feet, A D = 15 feet, A E = 18 feet, A F = 24 feet, find the supporting forces analytically and graphically. *Ans.*, 1,548 lbs.; 1,502 lbs. nearly.

(3) Draw a rectangle A B C D, A B = 5·2 inches, and B C = 2·2 inches. On A B take points E, F, G, H, K, L distant from A, 0·9, 1·55, 2·7, 3·25, 4·25, and 4·75 inches respectively. On B C take B M = 1·25 inch. On C D take N, O, P, Q, distant from C, 1·5, 2·85, 3·7, and 4 inches respectively. On D A take D R = 1 inch. Now, three forces act on a rigid body—one of 0·615 ton from F to O, one of 0·536 ton from H to P, one of 0·423 ton from L to N. These three forces are balanced by three others, whose lines of action are, X X drawn through R and G, Y Y drawn through E and Q, and Z Z drawn through M and K. Find the magnitudes of these balancing forces. Scale ½ inch to 0·1 ton. *Ans.*, 0·930 ton; 0·375 ton; 0·815 ton.

109. The distance x of the **centre of mass** (usually called centre of gravity, but only few bodies have true centres of gravity) of a body or system of bodies from a plane is obtained by multiplying each little portion of mass m by the distance x of its centre from the plane, adding together, and dividing by the whole mass. The symbol for this is $\bar{x} \,.\, M = \Sigma\, m \,.\, x$ where M stands for $\Sigma\, m$. Practically the engineer often finds the centre of gravity by an **experimental method.**

The distance $\bar{x}$ of the centre of an area (usually called the centre of gravity of the area) from a plane (or more usually of a plane area from a line in its plane) is obtained by multiplying each little portion of the area a by the distance x of its centre from the plane (or line), adding together, and dividing by the whole area. The symbolic way of representing this is $\bar{x} \,.\, A = \Sigma\, a\, x$, where A stands for the whole area.

If the centres of the masses m_1, m_2, m_3, etc., are at the distances x_1, x_2, x_3, etc., from any plane, let the sum of the masses $m_1 + m_2 +$ etc. be called M, and let $\bar{x} = (m_1\, x_1 + m_2\, x_2 + \text{etc.}) \div M$, or, as we prefer to write it, $\bar{x} = \Sigma\, (m\, x) \div M$. Similarly, taking distances from two other planes at right angles to the first, let $\bar{y} = \Sigma\, (m\, y) \div M$, $\bar{z} = \Sigma\, (m\, z) \div M$. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of a point. If u_1, u_2, u_3, etc., be the distances of the masses from any other plane, at a distance a from the origin, perpendicular to the direction (l, m, n), it is easy to see that

$$u_1 = l\, x_1 + m\, y_1 + n\, z_1 - a;$$

and if $\bar{u}$ is the distance of the point already found from the new plane, $\bar{u} = l\bar{x} + m\bar{y} + n\bar{z} - a$; and using the above values for $\bar{x}$, $\bar{y}$, $\bar{z}$, and re-arranging terms, we find that $\bar{u} = \Sigma\,(m\,u) \div M$. We see, then, that the above-mentioned point will be in the same position whatever be the planes of reference. We call it the **centre of mass** or **centre of inertia**; and when we are dealing with a system so small that the forces of attraction upon it due to gravity may be regarded as parallel to one another, we may call it the **centre of gravity**. In the same way, if the centres of the areas a_1, a_2, etc., all in the same plane, are at the distances x_1, x_2, etc., from a line in the plane, and if A is $a_1 + a_2 +$ etc., the whole area, and if $\bar{x}$ is the distance of the common **centre of area** (often called the centre of gravity of the area) from the line, then

$$\bar{x} = (a_1\,x_1 + a_2\,x_2 + \text{etc.}) \div A.$$

110. We can use a **graphical statics** method of finding the centre of mass, or area G. Thus, let there be masses or areas, m_1, m_2, etc., whose centres are at the points 1, 2, etc. (Fig. 76). Draw parallel forces 11, 22, etc., in any direction proportional to m_1, m_2, etc., and find the resultant by the above method. Suppose M N to be the direction of the resultant. Now repeat the process, taking the parallel forces of the same magnitudes as before, but in a different direction, and let MP be the direction of the resultant. Evidently M, where these lines meet, is the **centre of gravity** of the masses or areas. This method may often prove useful, for areas especially. Thus, to find the centre of gravity of any given area, divide it into any suitable number of parts, so that the centre of gravity and area of each part may be found easily. If we divide the area by parallel lines, these lines may be drawn equidistant, and the area of each part is approximately given by the length of the line which separates it from either of its neighbours. A repetition of the process has been employed to determine the **moment of inertia** of the area about any given line.

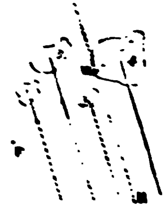


Fig. 76.

111. I do not advise students to adopt this link polygon method of finding centres of gravity or of calculating moment of inertia. A practical engineer will always apply the ordinary formula to find the centre of gravity of an area. Thus, if we want the centre of gravity of the figure M N O P (Fig. 77), draw two parallel lines, G H, K Q, touching the figure at two opposite sides. Draw a line, K G, at right angles to G H, and divide it into any number of parts, each equal to d. Draw the lines A B, C D, etc., parallel to G H, so that they are at the distance d apart, the distance from A B to G H, or from Y Z to K O being $\frac{1}{2}d$. It is evident that if N X P is a line parallel to G H through the centre of gravity, then approximately

$$GX = \tfrac{1}{2}d \cdot \frac{AB + 3\,CD + 5\,EF + \text{etc.}}{AB + CD + EF + \text{etc.}}$$

We have thus obtained one line through the centre of gravity, and in a similar way we may obtain another such line, and their point of intersection is the centre of gravity required.

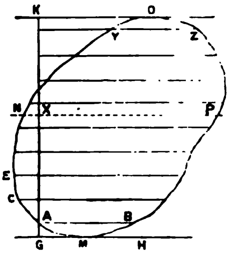


Fig. 77.

Sometimes we choose as our line of reference, a line, G H, which cuts through the area; in this case distances on one side of the line are to be considered negative; and if G H happens to pass through the centre of gravity, of course the sum $\sum a\,x$ is 0.

We often cut the shape of an irregular area from sheet zinc and balance it in two positions on a straight edge to find the centre of gravity of the area.

Exercises.—1. Masses whose centres are in a straight line at A, B, C, D are 4 lbs., 8 lbs., 7 lbs., 6 lbs.; where is the common centre of mass if A B = 0·5 feet, A C = 2 feet, and A D = 2½ feet? *Ans.*, A G = 1·32 ft.

2. A disc 8 inches diameter, 2 inches thick, with a hole 4 inches diameter; centre of hole, 1 inch from centre of disc; where is the centre of mass? *Ans.*, ⅓ inch from centre.

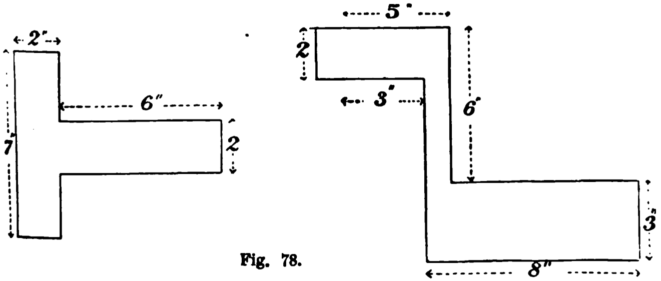


Fig. 78.

3. A B C is an equilateral triangle, each side being 3 inches long; particles whose masses are 1, 2, 3 are placed at A, B, C respectively; find their centre of gravity by construction, and note its distance from A.

Ans., 2·18 inches.

4. ABCD is a square lamina of uniform density; E, F are the middle points of AB and BC. If the corner of the square is turned down along the line EF, so that B comes on to the diagonal AC, find the centre of gravity of the lamina under the new circumstances.

Ans., $\frac{1}{48}$ of the diagonal from the centre.

5. A circular disc, 8 inches in diameter, has a hole 2 inches in diameter punched out of it, the centre of the hole being 3 inches from the circumference of the disc. Find the centre of gravity of the remaining portion. *Ans.*, 0·0667 inches from centre of larger circle.

6. A thin plate of metal is in the shape of a square and equilateral triangle, having one side common; the side of the square is 12 inches long. Find the centre of gravity of the plate.

Ans., 2·86 inches from centre of square.

7. Find the centres of area of the areas in Fig. 78.

112. To find the **moment of inertia,** I, of a great number of little masses about an axis, multiply each mass by the square of its distance from the axis, and add up. If the whole mass is M, we often write $\text{M}k^2 = \text{I}$: and when I and M are known we can calculate k, which we call the **radius of gyration** of the **mass** about the axis. To find what is called the moment of inertia, I, of a great number of little areas about a line in their plane, multiply each by the square of its distance from the line, and add up. If A is the whole area, and $\text{A}k^2 = \text{I}$, we call k the radius of gyration of the **area** about the line.

113. Just as we found centres of gravity, so we may obtain **the moment of inertia,** I, of any area about any line; or, as is often the case, suppose we wish to find the moment of inertia of MNOP (Fig. 77) about NXP, a line which passes through the centre of gravity. Evidently the moment of inertia about GH is approximately $\text{I} = d^3\,(\text{AB} + 9\,\text{CD} + 25\,\text{EF} + \text{etc.})/_4$.

Students ought to practise this method first upon a rectangle and a circle whose moments of inertia have been worked out for them by the calculus.

Now, it is well known that *the moment of inertia of an area about any line is equal to its moment of inertia about a parallel line through its centre of gravity, together with the product of the area into the square of the distance between the two lines.* Hence, the moment of inertia about NXP is $\text{I}_0 = \text{I} - \text{GX}^2 . d\,(\text{AB} + \text{CD} + \text{EF} + \text{etc.})$.

It will be found in practice that this easy way of carrying out simple ideas is better than the complicated use of the link polygon method for finding moment of inertia. If the area may be divided into rectangles whose sides are parallel to and perpendicular to the axis, we need not subdivide these rectangles. It must be remembered that the moment of inertia of any given area such as a rectangle about any axis is equal to the area multiplied by the square of the distance of its centre of gravity from the

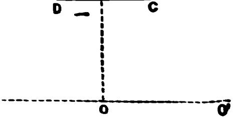


Fig. 79.

axis, plus the moment of inertia of the area about a parallel line through its centre of gravity. Thus the rectangle A B C D (Fig. 79) is known to have, about N M N, the moment of inertia $\frac{AB \cdot BC^3}{12}$, so that the moment of inertia of the rectangle about O′ O O′ is

$$\frac{AB \cdot BC^3}{12} + AB \cdot BC \cdot MO^2, \text{ or } AB \cdot BC \left(\frac{BC^2}{12} + MO^2\right).$$

It is mainly by the use of this rule that we have found the moments of inertia of the various **sections** shown in Table VI., and all these ought to be worked out as exercises by students.

The student will find it good exercise to take a few sections of angle-iron, T-iron, rails, and other specimens of rolled iron, and find by the above graphical method the position of centre of gravity of each section, and the moment of inertia of each area about any line through the centre of gravity. The exact forms ought to be taken from real specimens. If the area is *symmetrical*, one line through the centre of gravity can always be found by mere inspection.

In using this or any other graphical method, it is well to know what is the error involved in having each strip of width d, instead of being infinitely narrower. We ought to add to the sum in (2) the moment of every strip about its own central line—that is, d^3 (A B + C D + E F, etc.)$/_{12}$ or $A d^2/_{12}$, if A represents the total area. It is easy to see that in a rectangle of depth D, if we divide into strips of breadth d, the fractional error is d^2/D^2.

114. When the moments of inertia of an area about any three axes through a point are known, the moment of inertia about any other axis through the same point may be found; because if a distance be measured from the point along an axis which is equal to the reciprocal of the radius of gyration of the area about the axis, the extremities of all such measured distances lie in an ellipse. The *principal axes* of the area are in the directions of the major and minor axes of this ellipse. Thus, if for any area, M N P R (Fig. 80), the least moment of inertia is about an axis, O A, and is $\frac{c}{OA^2}$, and if the greatest moment of inertia is about O B, and is $\frac{c}{OB^2}$, then A B A′ B′ being an ellipse whose major and minor axes are A A′ and B B′, the moment of inertia about an axis, O C, is $\frac{c}{OC^2}$.

115. To know the **principal moments of inertia** of an area is important for many purposes. It is specially important in regard to struts. A strut will bend in such a way, that the axis through the centre of gravity of a section, at right angles to the plane of bending, is the axis about which there is least moment of inertia of the section. The ellipse lets us see the moments about all axes. To draw it for any particular section, if the section is symmetrical, as in sections of Fig. 80, we know that the axis of symmetry and the axis at right angles to it are the principal axes of inertia. If the section is not

symmetrical, as in the case of an angle-iron, we find the mo
of inertia about three axes as O P, O Q, O R of Fig. 81, and w

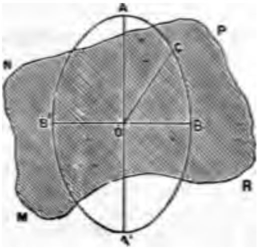


Fig. 80.

off the distances O P, O Q, O
represent the reciprocals of the
of gyration. We now have
graphical problem: given
points P Q R of an ellipse, a
centre O to draw the ellipse.
Harrison thinks the following
tion better than any other.

With centre O, radius O Q
scribe a circle, and throug
where P R cuts O Q, draw the
pr such that *p*H : H*r* = P H
Draw the radius O*s* perpendi
to O Q; through *s* draw *s*M
parallel respectively to *p* O, O *r*; and, through N and M draw lines pa
to R O, O P, intersecting in S; then O S is conjugate to O Q.

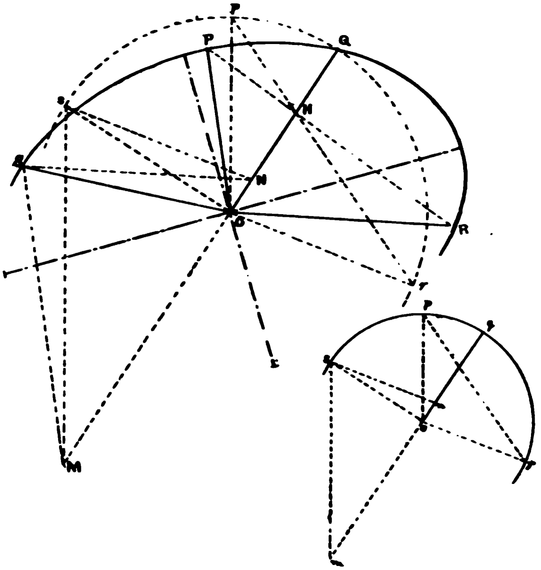


Fig. 81.

For proof, suppose the ellipse orthogonally projected i
circle, and let the smaller figure be *similar* to this proje
This figure can be drawn, remembering that parallel lines p
into parallel lines the mutual ratios of whose lengths r

unaltered; also that conjugate diameters project into perpendicular diameters. The solution given consists in drawing this figure with *oq* coinciding with O Q, and then locating S.

To determine the principal axes, through S draw a line (not shown) perpendicular to O Q, and on it take two points D, E, opposite ways from S, such that S D = S E = O Q. Then the axes of the ellipse are respectively equal to the sum and difference of O D and O E, and the major axis bisects the angle D O E.

MOMENT OF INERTIA OF A RECTANGLE.

116. The **moment of inertia** of a **rectangle** about the line O O through its centre, parallel to one side.—Let A B $= b$, B C $= d$. Consider the strip of area between O P $= y$ and O Q $= y + \delta y$. Its area is $b\,.\,\delta y$, and its moment of inertia about O O is $b\,.\,y^2\,.\,\delta y$; so that the moment of inertia of the whole rectangle is

$$b\int_{-\frac{1}{2}d}^{\frac{1}{2}d} y^2\,.\,dy \text{ or } b\Big[\tfrac{1}{3}y^3\Big]_{-\frac{1}{2}d}^{\frac{1}{2}d} \text{ or } \frac{bd^3}{12}.$$

The moment of inertia of the area A B C about an axis O at *right angles* to the area is equal to the *sum* of the moments of inertia I_x and Iy, about O X and O Y, axes at right angles to one another in the area. For if a is a portion of area at P, Ix is the sum of all such terms as $a\,.\,PR^2$, Iy is the sum of all such terms as $a\,.\,PQ^2$; and the sum of each such pair of terms is a term $a\,.\,OP^2$.

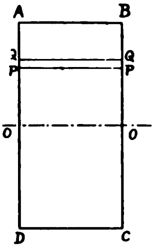


Fig. 82.

117. Moment of inertia of a **circle** about its centre.—Consider the ring of area between the radii r and $r + \delta r$. Its area is $2\pi r\,.\,\delta r$ more and more nearly as δr is made smaller and smaller, and its moment of inertia is $2\pi r^3\,.\delta r$. The integral of this is $\frac{1}{2}\pi R^4$ for a circle of radius R. The square of the radius of gyration is $\frac{1}{2}R^2$. Now, in this case $Ix = Iy$, each being half of $\frac{1}{2}\pi R^4$; so that the **moment of inertia** of a **circle about its diameter** is $\frac{1}{4}\pi R^4$, or $\frac{1}{64}\pi D^4$ if D is the diameter.

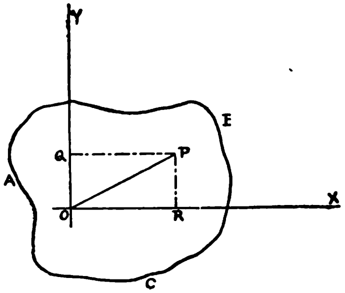


Fig. 83.

118. The student ought to be able to prove the propositions referred to in Art. 112 :—

1. **As to mass or inertia.**—To prove that the moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity together with the whole mass multiplied by the square of the distance between the two axes. Thus, let the plane of the paper be at right angles to the axes.

Let there be a little mass m at P in the plane of the paper. Let O be the axis through the centre of gravity of the whole mass, and O′ be the other axis. We want the sum of all such terms as $m\,.\,(O'P)^2$.

Now, $(O'P)^2 = (O'O)^2 + OP^2 + 2\,.\,OO'\,.\,OQ$, where Q is the foot of a perpendicular from P upon OO′, the plane containing the two axes. Then, calling $\Sigma m\,.\,(O'P)^2$ by the name I, calling $\Sigma m\,.\,OP^2$ by the name I_0, the moment of inertia about the axis O through the centre of gravity of the whole mass, then

$$I = (O'O)^2\,\Sigma m + I_0 + 2\,.\,OO'\,.\,\Sigma m\,.\,OQ.$$

But $\Sigma m\,.\,OQ$ means that each portion of mass m is multiplied by its distance from a plane at right angles to the paper through the centre of gravity, and this is zero. So that the proposition is proved. Or, letting Σm be called M, the whole mass,

$$I = I_0 + M\,.\,(O'O)^2.$$

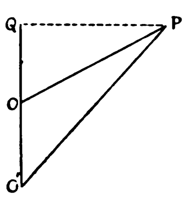


Fig. 84.

2. **To prove the proposition about areas.** Let OO be the axis through the centre of gravity, and O′O′ a parallel axis. We want I, the sum of all such terms as $a\,(O'P)^2$, and this is the same as $\Sigma a\,.\,OP^2 + \Sigma 2a\,.\,OP\,.\,OO' + \Sigma a\,.\,OO'^2$. But

$$\Sigma 2a\,.\,OP\,.\,OO' = 2\,.\,OO'\,\Sigma a\,.\,OP,$$

and this is 0 from our definition of centre of gravity; so that

$$I = I_0 + OO'\,\Sigma a.$$

EXERCISES.

1. A fly-wheel has a rim of rectangular section, the outside and inside radii being 8 feet and 7 feet. What is the error in assuming the radius of gyration to be 7·5 feet?

Ans., The true radius of gyration is 7·516 feet, and hence the assumed radius of gyration is ·21 per cent. wrong. It would give a moment of inertia ·4 per cent. wrong.

2. Find the radii of gyration of the sphere of Table II., p. 251, about a line touching its surface; the solid cylinder about a line touching its outside, parallel to the axis; the rod about a line at right angles to it at one end.

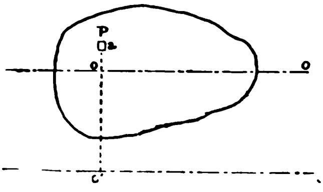


Fig. 85.

3. What error is introduced in Table I., by neglecting the size of the section of the prismatic rod?

4. Two homogeneous spheres of weights 12 and 20 lbs., radii 0·2 and 0·3 feet, their centres 5 feet apart: find the distance of G, the centre of gravity, from O, the centre of the smaller: find I_0, the moment of inertia about an axis through G at right angles to OG: find the moment of inertia about OG itself; find the moment of inertia about any line GA if the angle OGA is a. *Ans.*, 37·5 inches; 27131; 131·3; $131{\cdot}328 + 27000 \sin.^2 a$.

119. Given the moments of inertia of a lamina about a pair of axes at right angles in the plane of the lamina, to find the moment of inertia about any other axis through the intersection and lying in the plane. Let the moment of inertia about O X be I_x and the moment of inertia about O Y be I_y; O P is any other axis, the angle X O P being a. Let the distance of a small portion of area at Q from the three axes be called y', x', and γ; then

$$\gamma = y' \cos.\, a - x' \sin.\, a,$$

and $$\gamma^2 = y'^2 \cos.^2 a + x'^2 \sin.^2 a - 2\, x'\, y' \sin.\, a \cos.\, a.$$

Hence, if I is the moment of inertia of the whole area about the new axis, $I = I_x \cos.^2 a + I_y \sin.^2 a - 2\, I_{xy} \sin.\, a \cos.\, a \; .\,.\,.\,.$ (1), where I_{xy} is written to mean the product of inertia about the axes x and y, or the sum of all such terms as "portion of area × xy."

Now, let $I = \frac{A}{r^2}$, $I_x = \frac{A}{r_2^2}$, $I_y = \frac{A}{r_1^2}$, $I_{xy} = \frac{A}{s^2}$.

When A is the area, r, r_2, and r_1 are the reciprocals of radii of gyration; but s has no name.

If we take a point P in the line O P, such that O P $= r$, and let the co-ordinates of this point relatively to the original axes be x and y, then (1) becomes

$$\frac{x^2}{r_2^2} + \frac{y^2}{r_1^2} - \frac{2\, xy}{s^2} = 1,$$

which is the equation to an ellipse. We see, then, that if a distance proportional to the reciprocal of the radius of gyration about an axis be measured along that axis from O, the points so found lie in an ellipse. When the point O is the centre of gravity of the area this ellipse is called the **momental ellipse** of the area, and its principal axes are the principal axes of inertia. If they had been chosen as the axes of reference, evidently I_{xy}, the product of inertia relatively to them, would have been zero.

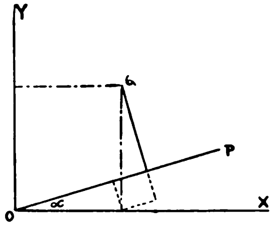


Fig. 86.

APPENDIX TO CHAPTER VII.

Mr. J. Harrison, of the Royal College of Science, has been kind enough to prepare the following short account of the general principles involved in the graphical study of forces when they do not act in one plane, and readers are referred to "Graphics," by Prof. R. H. Smith, for more detailed information.

Forces in space and framed structures in three dimensions.

Problem 1.—*To find the resultant of a given system of forces in space, whose lines of action all pass through a point.*

A force in space is conveniently defined by *two orthogonal projections* of the line which represents it. These projections represent the components of the force respectively parallel to the planes of projection.

To compound the given system, add the forces as vectors. Two projections of the **gauche polygon** representing the vector addition are required.

Let P Q R, P′ Q′ R′ (Fig. 87), be the given plans and elevations of the lines of action of the forces.

The plan *p q r*, and the elevation *p′ q′ r′* of the vector polygon can be at once drawn, since the lengths of the sides are supposed given as part of the data. Then a line, S, S′,

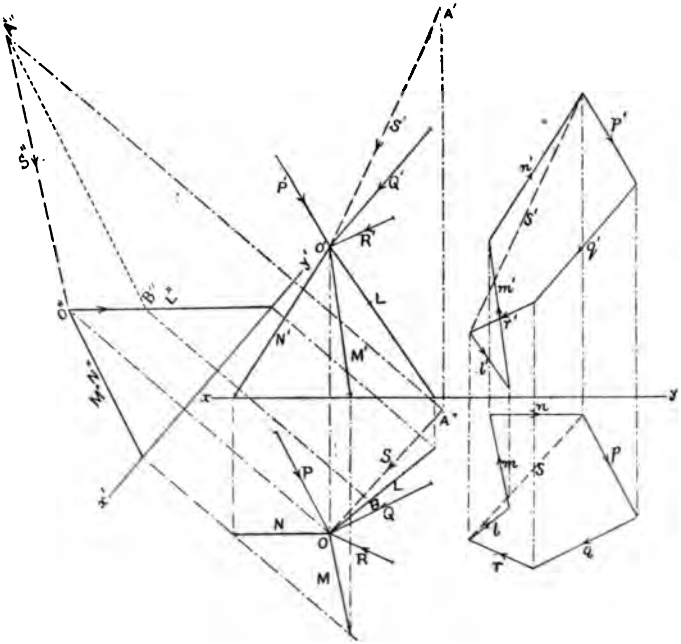


Fig. 87.

through the given point O, O′, parallel and equal to the closing side *s*, *s′* of the vector polygon, represents the required resultant.

Problem 2.—*The lines of action of a concurrent system of forces in equilibrium in space being given, and the magnitudes of all the forces except three, to find the three magnitudes.*

In Fig. 87 the forces P P′, Q Q′, R R′, are those given completely; the magnitudes of L L′, M M′, N N′, are required.

Compound the known forces into a single force S S′, by means of the vector polygon shown in plan and elevation.

To find L L′, observe that its magnitude must be such that the component perpendicular to the plane of M N shall be equal to the component of S S′, perpendicular to the same plane.

Draw a new elevation on a plane perpendicular to the plane of M N. Let S″ or O″ A″ be the new elevation of S. Draw A″ B″ parallel to M″ N″, to meet L″ in B″; then O″ B″ is the magnitude of L″, the elevation of L on *x′ y′*. Project B″ to B. Then O B is the length of the plan L.

Draw *l* parallel and equal to O B, and close the vector polygon in plan by drawing the lines *m*, *n* respectively parallel to M and N. The elevation of the polygon can now be drawn. The *true lengths* (not shown) of the lines *l l′*, *m m′*, *n n′* would give the actual magnitudes of the three forces, L, M, N.

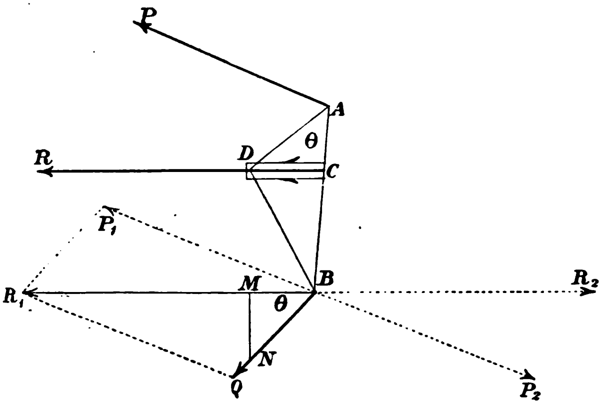
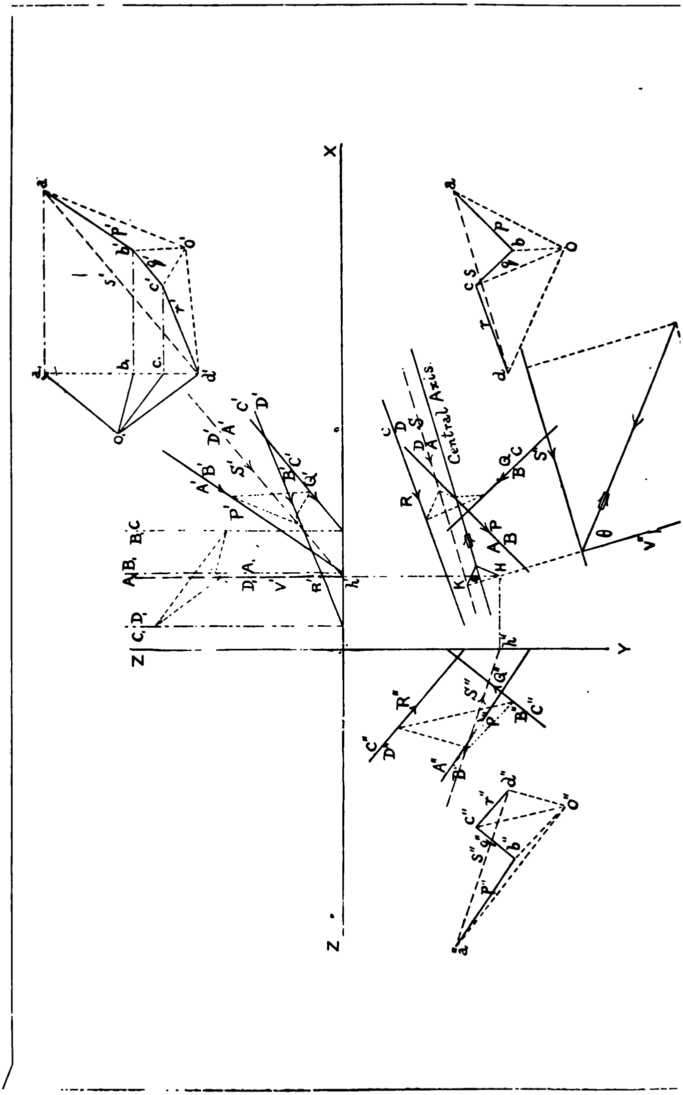


Fig. 88.

Problem 3.—*To reduce two given forces, acting in directions at right angles to each other, to a single force and a couple, the plane of the couple being perpendicular to the line of action of the single force.*

Let P and Q (Fig. 88) be the given forces, and let A B $(= a)$ be the line meeting both P and Q at right angles.

At B introduce the forces P_1 and P_2 as shown, each equal and parallel to P.


X
Y
Z
Central Axis
a
b
c
d
o
p
q
r
s
A B C D
P Q R S
H
K
h
θ
V

Compound P_1 and Q into R_1. In the plane A B R_1 draw A D, making, with A B, the angle D A B = Q B R_1 = θ; and draw B D perpendicular to A D. Introduce the forces R and R_2, each equal and parallel to R_1 as shown.

Then the required single force equivalent to P and Q is R, and the required couple R × C D.

For let the couple P P_2 be represented by the axis $B\,N = P \times a$; resolve this into B M, M N, where M N is perpendicular to B M.

Then $B\,M = B\,N \times \cos.\ \theta = P\,a \cos.\ \theta = R\,a \sin.\ \theta \cos.\ \theta = R \times C\,D$.

And $M\,N = B\,N \times \sin.\ \theta = P\,a \sin.\ \theta = R\,a \sin.^2 \theta = R \times B\,C$.

So R × C D represents the component couple B M, and R, R_2 may be taken as the couple represented by M N. Now R_2 cancels R_1, and there remain R and R × C D.

The two outer lines at C D may be taken to represent the couple. The convention adopted as to *sign* is, that when the arrow-heads on the couple axis, and on the force line point the same way, as in the figure, the tendency of the forces is to produce a right-handed screw motion, and conversely.

Problem 4.—*To compound a given general system of forces in space.*

Let the lines of action of the forces be supposed cut by any plane, and at the points of intersection resolve the forces respectively along and perpendicular to the plane. Compound each of these two sets. The system is thus reduced to *two non-intersecting forces at right angles.* If desired, these two forces may be compounded, as in the last problem.

The given system illustrated in Fig. 89 consists of three forces, P, Q, R, the projections of the lines of action of which on three planes mutually perpendicular are shown, as are also the three projections of the vector polygon, drawn as if the given system were concurrent.

The system may be reduced to two forces—one in the horizontal plane X Y, the other vertical.

To find the former, draw a link polygon in plan with respect to any pole O. Thus, S = *s* is the horizontal force.

To find the vertical force, two methods are available.

(1) Compound the vertical components. In the figure, the elevations of the components on Z X are shown, and their resultant V′ is found by means of a link polygon drawn with respect to any pole $O_{\prime}$. In a similar manner (not shown), the elevation on Y Z of the *resultant* vertical force could be found,

thus completely determining the vertical force and giving the plan H of its line of action.

(2) Or thus:—On Z X by means of a link polygon drawn to any pole o′, find s′, the elevation of the line of action of the component force of the system parallel to the plane Z X. And similarly, on Y Z, find s″, the elevation of the component parallel to Y Z. Let these intersect the ground lines respectively in *h′* and *h″*, projections from which give H, the plan of the line of action of the vertical force required. Its magnitude is *a, d′*.

A further construction is shown for reducing the forces, as in the last problem. The line of action of the single force thus obtained is called the *central axis* of the system. The couple is equal to the moment of system about the central axis, and may be shown to be the *minimum couple*. There is no other axis about which the forces have a less moment.

Note.—If the given system contain couples as well as forces, treat those independently, adding their axes as vectors. Resolve this couple parallel and perpendicular to the central axis, and add to the other part of the system.

Problem 5.—*To determine the stresses in a non-redundant framed structure of three dimensions under given loads.*

The *criterion* for a non-redundant stiff frame in three dimensions, with non-rigid joints, is that the *number of bars* must be *six less than treble the number of joints.* For this relation is evidently true in the simplest example—viz., for a frame of six bars forming a pyramid; and in building up a frame which shall be stiff, each new joint requires three new bars. If any portion consist of a plane frame with redundant members (such, for example, as a plane quadrilateral with crossed diagonals), such redundant members are not to be counted in applying the criterion.

Fig. 90 shows a derrick crane carried by a **braced triangular pier.**

The stress diagram for the frame is built up in plan and elevation, applying Problem 2 in succession to the several joints. The order of taking the joints is that indicated by the Roman numerals, and the various points on the stress diagram are marked *a*, *b*, *c*, etc., in order as they are found.

If the pier had been braced as shown in Fig. 91, then after having drawn the stress diagram for the joints I and II, there would be no new joint with less than four unknowns. *In this* case the stress in the bar marked P could be found by

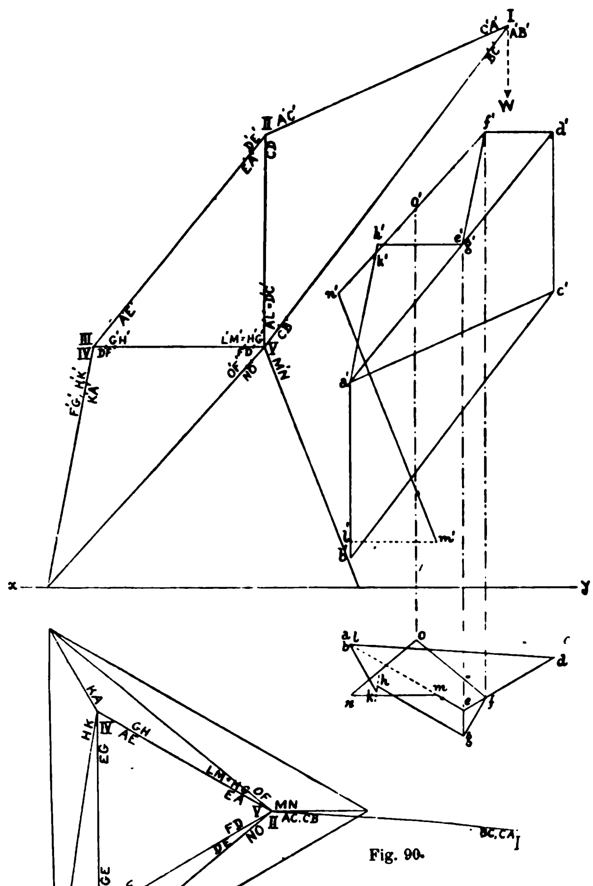

I
CA'
AB'
BC'
W
II
AC'
DE'
EA'
CB'
f'
d'
o'
h'
e'
g'
k'
n'
c'
a'
III
IV
GH'
DF'
AE'
LM'=HG'
FD'
V
AL'=DC'
CB'
OF'
NO'
MN'
FG'
HK'
KA'
l'
b'
m'
x
y
a l
b
o
c
d
h
m
n
k
e
f
g
KA
HK
IV
GH
AE
EG
LM=HG OF
EA
MN
V
II
AC.CB
FD
NO
DE
GE
III
DF
ED
FG
BC.CA
I

Fig. 90.

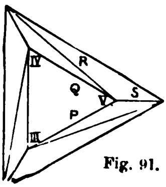

IV
R
Q
V
S
P
III

Fig. 91.

resolving the forces at the joint V perpendicular to the plane which contains the other three unknowns, Q, R, S. The building up of the diagram would then proceed in the same order as before.

EXERCISES.

1. A vertical crane post is 10 feet high, jib 30 feet long, stay 24 feet long, meeting at a point C. There are two back stays making angles of 45° with the horizontal: they are in planes due north and due west from the post. A weight of 5 tons hangs from C. Find the forces in the jib and stays—1st, when C is south-east of the post: 2nd, when C is due east; 3rd, when C is due south.

2. A tripod whose vertex is A, and whose legs are A B, A C, A D, of lengths 8, 8·5, and 9 feet respectively, sustains a load of 2 tons. The ends B, C, D form a triangle whose sides are B C = 7 feet, C D = 6 feet, B D = 8 feet, find by graphical construction the compressive forces in each leg.

Ans., 1·16 ton, 0·55 ton, 0·53 ton.

3. Three ropes, each 12 feet long, hang from the three corners of a horizontal isosceles triangle, A B C, in which A B and A C are each 20 feet, and B C is 10 feet. The ropes are joined at their ends and support a load of 1 ton. Find the pull on each rope. *Ans.*, 0·52 ton, 0·52 ton, 0·92 ton.

CHAPTER VIII.

EXAMPLES IN GRAPHICAL STATICS.

120. In any structure, such as the **principal of a roof** and many **girders of bridges** formed of many different bars, if we neglect the weights of, or upon, the bars, and we assume that the joints are **frictionless hinges,** it is easy to see that the force exerted by any bar is in a straight line between the centres of the hinges at its ends. For whatever may be the many forces between a piece and pin, at the surface of the pin, these forces must all be *normal* to the surface, since there is no friction; they must therefore all be directed in radial lines, and hence their resultant must be a radial line through the centre of the pin. In this case, then, the joints of a structure only being loaded with known forces, it is easy to **calculate the pushing** or **pulling force** exerted by each piece.

To illustrate this—let there be three pieces, whose centre-lines are A O, B O, and C O (Fig. 92), meeting on a pin whose centre is O. Suppose we know that the piece C O pulls the pin in the direction O C with a force of 2,000 lbs. We have then to find the two forces in the given directions of A O and B O to balance the known force O C.

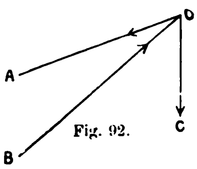


Fig. 92.

Draw the triangle (Fig. 93), whose sides, *c*, *a*, *b*, are parallel to O C, A O, and B O; and let *c* represent the force O C or 2,000 lbs. to some scale, and let the arrow-head on *c* represent the sense of O C. Then the lengths of *b* and *a* represent the other two forces to scale. Put the circuital arrow-heads on *b* and *a*, and we see that O A is a **pulling force,** and the piece O A is called a **tie-rod;** B O is a **pushing force,** and the piece B O is called a **strut.** Note that B O is attached to some other pin than O. If we study the equilibrium of the new pin we must remember that B O pushes it, and we must draw the *new* arrow-head when we study the *new* pin.

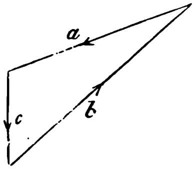


Fig. 93.

It is very important that the student should illustrate an important fact like this for himself in the laboratory. Fig. 94 shows two real pieces A O and B O hinged at O. Hang on any weight C. Adjust the screw at A′ until the spring-balance A′ A is in a line with O A, so that it indicates the pull on O A. The push in B O is recorded by the spring-balance B′ B. When things are at rest, open the parts of a two-foot rule until they just fit the angle between A O and B O, and transfer this angle to a sheet of paper. Now do the same with the angle between B O and C O, and test on the paper if the sides of a triangle parallel to A O, B O, and C O really represent, to some scale, the three forces. If they do not, speculate for yourself upon the discrepancy; how much discrepancy is due to the fact that the forces as measured are not truly the forces at O, because of the weights of the parts? How much is due to the fact that the rule presupposes a pin with absolutely *no friction?* If you are ingenious and an advanced student, you will try a pin with much friction, or perhaps a riveted joint at O, and so learn more than I can tell you.

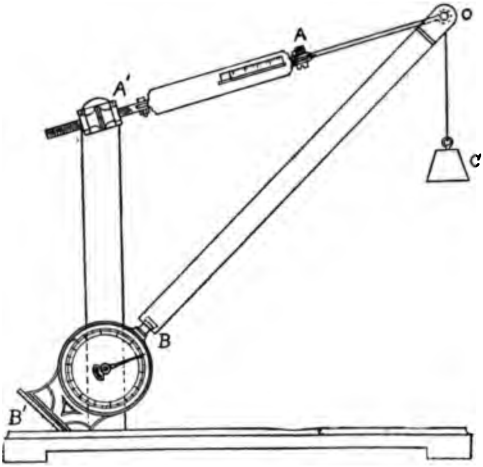


Fig. 94.

EXERCISES.

1. In Fig. 94 the jib O B makes an angle of 45° with the horizontal; angle A O B, 15°; weight at C, 5 tons; find the forces in O A and O B.

Ans., 13·66 ton, 16·73 ton.

2. A chain fastened at O (Fig. 94) goes round a snatch-block at C, and then over a pulley at O in the direction O A to the barrel, the total

weight at C being 5 tons; find the forces in O A and O B. Dimensions same as in 1. *Ans.*, 10·76 tons; 16·73 tons.

3. If a wharf crane, the post, tie-rod and jib measure 15, 20, and 30 feet respectively, what would be the stresses in each of the three members when a load of 7 tons is suspended over the pulley at the jib-head (1) when the lifting-chain passes from the pulley to the drum parallel with the jib; (2) when the drum is placed so that the chain passes from jib-head parallel to tie-rod?

Ans., 7 tons; 2·3 tons; 11·67 tons; 11·2 tons; 9·3 tons; 18·67 tons.

4. A contractor's portable hand-crane has a vertical post A B, to which the jib A C is inclined 45°, and the tension-rod B C makes with A B an angle A B C of 120°. The back-stay, from the head of the post B to the extremity D of the horizontal strut A D, is inclined at an angle of 45° to A D. Find the counterbalance weight required at D to balance a load of 10 tons suspended from the end C of the jib. Determine also the nature and amount of the force in the jib A C, and in the rods B C and B D. (The tension in the chain may be neglected.)

Ans., 23·66 tons; 33·46 tons; 27·32 tons; 33·46 tons.

121. Let us now consider the **roof-principal** shown in Fig. 95.

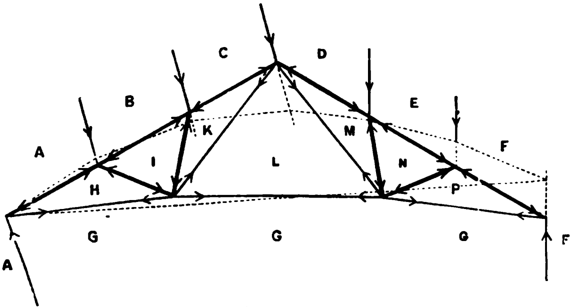


Fig. 95.

Certain loads are given acting at the joints, and we know that the structure is supported by two forces or **reactions** at its two ends. Our first step is to find these two supporting forces. They must be in equilibrium with all the external loads.

Now, it is well known that we must be given either the *direction* or some other information about one of these supporting forces, else the problem becomes indeterminate. It is usual to be told that one or other supporting force is vertical. This condition is arrived at in practice by having at one end of the structure a shoe with rollers resting on a horizontal plate of iron.

The notation which we use very materially simplifies the process of calculation. You observe that every *space* between two forces in Fig. 95 is indicated by a letter. The *line* which separates the space A from the space B is called A B, and corresponds with the line A B in Fig. 96. A *point* is indicated by the letters of the spaces which meet at that point. Thus, A G H is the end of the roof-principal.

Suppose the supporting force at the point F G P is known to be vertical. We must first find the amount of the force F G, and the direction and amount of the force A G.

Draw the force polygon, A B C D E F, Fig. 96.* We see that to close it we need two lines to join F and A. Now, one of these, F G, is vertical. Take O as pole. Join O A, O B, O C, O D, O E, O F in the usual way. Draw the link polygon, shown dotted in Fig. 95, commencing at the only known point of the force A G, namely A G H. Now, O G (Fig. 96) is parallel to the last side of it, and thus we find F G and G A, the supporting forces at the end of the principal. Having found the two supporting forces, F G and G A, we proceed as follows:—We have the closed force polygon A B C D E F G A. The arrow-heads shown on this force polygon are not to be rubbed out during the calculation, and in practice we mark them in ink. All

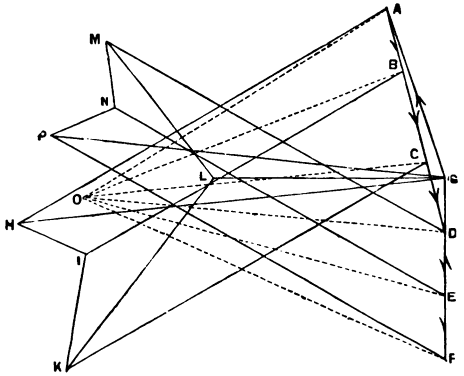


Fig. 96.

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* In an actual case there would usually be loads given at the end joints as well as the others. In this first example I have only taken the loads as *shown*.

other arrow-heads which we draw on Fig. 96 may require to be rubbed out, and ought only to be marked in pencil. We must begin our calculation at a joint where only two pieces meet, and where one force which acts there is given. Now at the joint A G H we know the force A G. In Fig. 96 draw A H and G H parallel to the pieces A H and G H of Fig. 95. Put arrows on the sides of the triangle G A H circuital with the arrow on G A. Now we see by the arrows that the piece A H *pushes* the joint with a force represented to scale by the length of the line A H (Fig. 96). We know, then, that A H is a **strut**, since it *pushes*, and we know the total pushing force in it. Similarly, H G is a **tie**, and the *total pulling force* in it is represented by the length of the line H G in Fig. 96.

We now rub out the arrows which we are supposed to have drawn in pencil on the lines A H and H G (Fig. 96), and proceed to the joint A B I H. It must be remembered that although the pieces A H and B I are in the same straight line, we regard them as two separate pieces.

We know the force A B, we also know that the force with which the piece A H *pushes* the joint (we have already found it to be a strut, therefore it pushes both joints at its ends) is represented by the length of the line A H (Fig. 96). Draw, then, H I and B I (Fig. 96) parallel to the pieces H I and B I (Fig. 95). We have thus a polygon A B I H. The force A B (Fig. 96) tells us how to pencil arrow-heads circuitally round this polygon. When we do this we find that the piece B I pushes the joint with a force represented by the line B I (Fig. 96), so that B I is a strut. Also I H is a strut, in which the stress is represented by the line I H (Fig. 96). We proceed in this way from joint to joint, always taking care to rub out our pencilled arrow-heads when we proceed from one joint to the next. The *lengths* of the lines in Fig. 96 give the *magnitudes* of the forces in the pieces of the structure. It is easy to prove that, if no mistake is made, no discrepancy will appear when the drawing is being finished.

If in Fig. 96 the points K and I were found to coincide, this evidently means that the piece K I is unnecessary in the structure. If, again, we find that we cannot close one of our little polygons in Fig 96, we ought to proceed to new joints, and, possibly, when we again consider the joint with which we had difficulty, we shall be able to close its polygon. If we still find difficulty, it must be caused by two or more joints, and the pieces connecting these are evidently unnecessary to the structure. If we find in Fig. 96 two points with the same letter, we evidently require to add a new piece to the structure, which will exert a force represented by the distance between these two points.

No explanation in writing will enable the student to master this beautiful **method of determining the forces in structures.** He must select structures, apply loads to the joints, and calculate the various forces for himself. When he has made four such calculations, he will know nearly all that can be said on the subject. (Figs. 99 and 101 show two examples.)

122. **Roofs.**—It is not my object here to describe the construction of a roof or a bridge. For such information the student must examine real structures and good drawings of roofs and bridges for himself.

Suppose, for instance, that he finds a roof, somewhat like his own, to weigh—including possible snow, etc.—40 lbs. per square foot of horizontal area covered. Suppose his principals are to be placed 8 feet apart, the span being 50 feet, then each principal has to support about

$$8 \times 50 \times 40, \text{ or } 16{,}000 \text{ lbs.}$$

Now, if Fig. 97 is the shape of his principal, as A B, B C, C D, and D E are all equal, we may suppose that, however the roof covering may be supported by the principals, the piece of rafter, A B, or any other of the divisions, supports 4,000 lbs. The joint B gets half the load on A B and half the load on B C; consequently the load at the joint B is taken to be 4,000 lbs., and similarly for C and D. The joints A and E have 2,000 lbs. each.

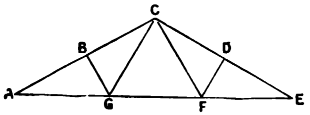

Fig. 97.

When the above vertical loads have been given to the joints, we have to consider **wind pressure** on one side of the roof. If we suppose, as we reasonably may, that 40 lbs. per square foot is the greatest pressure of wind ever likely to occur on a surface at right angles to the direction of the wind, then the normal pressure per square foot on roofs of the following inclinations may be taken from the following table, which is obtained from *experiment.*

TABLE I.

Normal Pressure of Wind against Roofs.

Angle of Roof	Normal Pressure.	Angle of Roof.	Normal Pressure.
5°	5·0	50°	38·1
10°	9·7	60°	40·0
20°	18·1	70°	40·0
30°	26·4	80°	40·0
40°	33·3	90°	40·0

Thus, if the portion of one slant side of the roof between two principals has an area of 240 square feet, and if the inclination of the roof is 30°, say, then 240 × 26·4, or 6,336 lbs., has to be supported by each bay. Transferring this to the joints, we see that at B (Fig 98), in case the wind pressure is upon the side A B C, we have the vertical load X B, or 4,000 lbs, due to weight of roof, snow, etc., and also Y B, or one half of 6,336 lbs., normal to the roof, and due to wind. Complete the parallelogram, and evidently Z B is the load at the joint B which we must use in our calculations.

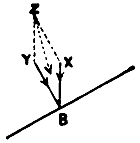

Fig. 98.

The student will find that if a roof-principal can only be supported by a vertical force at a certain end, the stresses in the structure are greatest when the other side of the roof is acted on by the wind.

123. Many joints in a real structure are usually stiff joints, so that many pieces may really be subjected to bending, as well as to direct compressive or tensile stresses. A general method of taking stiffness of joints into account is quite unknown; but when we discuss **bending** we shall see pretty clearly what is the effect of a stiff joint, and in some cases we shall be able to make calculations on the subject. It may generally be assumed that the strength of a structure is greater if the joints are stiff than if they are merely hinges. This is not always the case, and, from the indeterminateness of the problem of finding the stresses in a structure whose joints are stiff, many large bridge trusses are made with

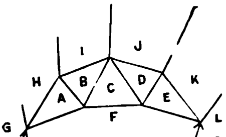

Fig. 99.

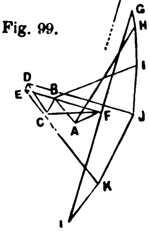

Fig. 100.

nearly all their joints hinged. In roof-principals the joints are often made stiff, rather for the purpose of **stiffening** the whole structure than for the sake of strength. We shall presently see the distinction between *stiffness* and *strength* in structures. In a roof all joints of struts are usually made stiff. What we shall now say is of more importance in bridges than in roofs.

If two or more pieces of a structure are in a straight line with one another at joints where they meet, it is usual, for strength, to make the joints between them quite **rigid**. Thus, the pieces A H and B I of Fig. 95, or A B and B C of Fig. 97 ought to form one bar. But this is only useful when the pieces in question are struts, and our reason for the continuity of the pieces is that a strut is **stronger** when its ends are **fixed** than when its ends are not fixed. Thus the piece B I (Fig. 95) will resist a greater thrust if it is continuous with A H and C K than if it were hinged with these pieces. (See Art. 372.) It is not good in all cases to fix the end of a strut by rivets, etc., instead of a hinge; because the benefit due to fixing an end may be more than counterbalanced by the evil effects of bending introduced to the strut through the joints by a tendency to change the angle which the strut makes with the piece to which it is fixed. The common-sense of the engineer will always enable him to decide as to the judiciousness of fixing the end of a strut.

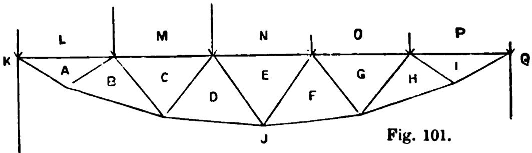

Fig. 101.

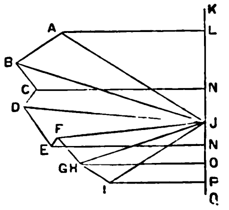

Fig. 102.

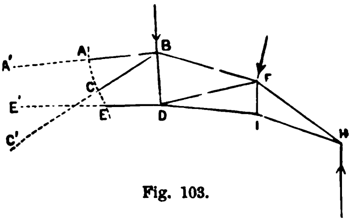

Fig. 103.

124. Sections of Structures.—It is often of considerable importance to find immediately the forces in pieces of a structure which are not near the ends. If we can draw any surface which will cut through the pieces in question, we can calculate the stresses in these pieces directly, supposing the pieces are only three in number. Thus, the section A C E (Fig. 103) cuts the pieces B A, B C, and D E. Now, not only the whole structure, but every part of it is kept in equilibrium. What forces keep the part A H E in equilibrium? They are the known forces at B, F, and H, together with three unknown forces whose directions are B A A', B C C', and D E E'. Given the directions of three forces which equilibrate a number of known forces, we know that they may be determined in magnitude by the link-polygon method (Art. 101). Sometimes the link-polygon method is more troublesome than the following:—**To find the push or pull** in D E E'. We know (Art. 98) that the moment of the force in E E' about the point B is equal to the sum of the moments about B of all the external forces (for the forces in the directions A A' and C C' have no moment about B, since they pass through it). Let the algebraic sum of the moments of the external forces be actually calculated, multiplying numerically each force by its perpendicular distance from B. This sum, divided by the perpendicular distance from B to E E', will give the force in E E'. If the algebraic sum gives a moment tending to turn the structure about B against the direction of the hands of a watch, the force in E E' is a pulling force acting from E towards E', and therefore the piece D E is a tie. The engineer ought to practise this common-sense way of applying our **fundamental principles.** He will regret it if he fetters himself to graphical statics methods of working.

It will be observed that if we wish to know the forces at any section of any loaded structure, we must consider that the parts of the structure on **any one side** of this section are in equilibrium. Thus, if A and B are the two parts of the structure, consider the equilibrium, say, of B. Now, B is kept in equilibrium by the external forces or *loads* which act on B, and by the forces which act on B at the section. Of course, it is A which causes these forces to act on B through the section; but in calculations concerning them we do not need to consider A or the loads on A.

125. Loaded Links.—Let A C, C D, D E, and E B be four links hinged together at C, D, and E, and supported somehow by hinges at A and B, and, neglecting the weights of the links themselves, let x, y, and z be forces acting at the three joints, so as to make the links take the positions shown in Fig. 104. Take any point, o (Fig. 105), and draw lines $o\,m$, $o\,n$, $o\,p$, and $o\,q$ parallel to the links, and from any point, m, in $o\,m$, draw $m\,n$ parallel to the force z, $n\,p$ parallel to the force y, and $p\,q$ parallel to the force x: then it is easy to prove that the lengths of the lines $m\,n$, $n\,p$, and $p\,q$ are proportional to the forces z, y, and x, and the tensile forces in the links are proportional to the lengths of the lines $o\,m$, $o\,n$, $o\,p$, and $o\,q$. For it is evident that the three forces at E, keeping the joint in equilibrium as they do, must be proportional to the sides of the triangle $o\,m\,n$. If you put arrow-heads on $o\,m$ and $o\,n$ circuital with the one already on $m\,n$, you will see that the bars B E and D E do not *push* the joint E; they *pull* it and are *tie-bars*. Thus, then, the lengths of the lines in Fig 105 represent, to some scale, all the forces acting at the joints, C, D, and E.

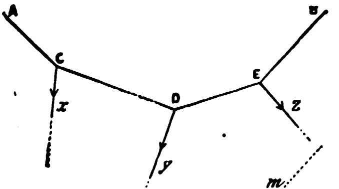

Fig. 104.

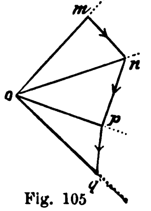

Fig. 105

All so easy as it is to prove that the above proposition is correct, it is well to illustrate its truth in the laboratory. Let A, C, D, E, B be a string fastened to a vertical board at A and B, with loads x, y, and z applied at C and D by means of strings passing over pulleys. The string, instead of being fastened at A and B, may there pass over pulleys with balancing weights. Let the vertical board be covered with paper; with a pin prick points on the paper showing the directions of the string everywhere, and transferring the paper to another board, find what is the pull in A C, C D, D E, and A B. We have no actual test of the pulls in C D and D E, unless an ingenious student can introduce very light spring balances whose own weights are negligible.

The student may compare this construction with exactly the same construction in Art. 97. He will see that in Fig. 104 the resultant of the forces E B and z is in the direction D E; the resultant of the forces E B, z and y is in the direction C D; the resultant of the forces E B, z, y and x is in the direction A C. Fig. 104 shows the *positions*, and Fig. 105 shows the *amounts* of these successive resultants. B E D C A is what we called a **line of resistance.**

126. Loaded Chain.—If we want to find the pull in every part of one chain of a **suspension bridge,** and to draw the shape of the chain, it is first necessary to know the weight of the bridge at every place. This weight is probably supported by two chains, so, as we have only one chain to deal with, we only take half the weight of the bridge. We shall suppose that there is no long girder or other support for the bridge but the chain. It is usual to suspend the supporting beams of the roadway from the chain by vertical iron rods, placed at equal horizontal distances from one another. We may imagine the roadway to be as heavy at one place as another, so that the pulls in all the rods will be the same. Suppose there are ten rods, and in each a pull of 20 tons. Draw ten equidistant vertical lines (Fig. 106) to represent the rods. We must get **another condition** before we can draw the chain. Let it be this, that the chain in the middle, where it is horizontal, shall be capable of withstanding a pull of 200 tons. Now draw O H horizontally (Fig. 107), and make its length on any scale represent 200 tons. Make H A and H B on the same scale represent each 100 tons (if your chain is to be symmetrical), and divide them up, so that each portion represents 20 tons—that is, the vertical load communicated to the chain by each tie-rod. Now join O with each *point of* division in A B. Suppose

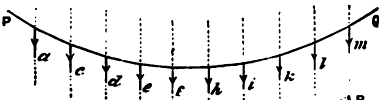

Fig. 106.

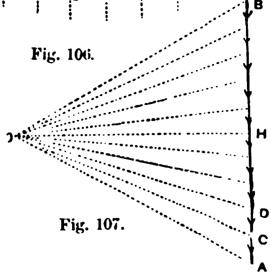

Fig. 107.

that P (Fig. 106) is one point of support of the chain, draw P *a* (Fig. 106) parallel to O A (Fig. 107), *a c* parallel to O C, *c d* parallel to O D, and so on till you reach the point Q, which I suppose to be on the same level as P. Of course, the points of support, P and Q, may be anywhere on the lines *a* P and *m* Q. It is quite evident from what you have already learnt that the pull in any part of the chain is represented by the length of the line from O, which is parallel to it in Fig. 107, and it is also evident that the chain will take this shape without any tendency to alter. Note that when all the loads on a chain are vertical, the *horizontal component* of any of the sloping forces is the same as that of any other, being O H. This is always called the **horizontal pull of the chain.**

127. We began by assuming a pull of 200 tons in the part *f h*, where the chain is horizontal. We might have assumed a pull of 300 tons in *f h*: this would have caused the chain to hang in a flatter curve. Assuming a pull of 100 tons in *f h*, we should have obtained a greater difference of level between P and *h*.

It will be found that in the present case, where the load is supposed to be *uniformly distributed* along the horizontal, the links would just circumscribe the curve called a **parabola.** With any other distribution of load they will fit some other curve than a parabola, but in any case you know now how to draw the shape of such a chain, and to determine the pull in any part of it.

128. Arched Rib.—If instead of a hanging chain you wanted to use a thin arched rib to support your roadway, then if you have numerous vertical rods by which to hang your load to the rib, and if the distribution of the load is known, you can draw the curve of the rib in exactly the same way, but it will now be convex upwards, of course. With uniform horizontal distribution of your load you will get a **parabolic rib.** The difference between the two cases is this: a slight inequality in your loads or a temporary alteration will only cause the chain to take a slightly different position for the time, and it will get back to its old shape when the old loading is returned to; whereas the arch is in a state of **unstable equilibrium**, and as it is very thin, so that it cannot resist any bending, a slight change of loading will very materially alter its shape and it will get destroyed. Such a rib or series of struts is either stayed with numerous **diagonal** pieces or else it is made very massive, so that should the line like P *h* Q (inverted), which is supposed to pass everywhere along its axis, deviate a little from this position, the rib may resist alteration of shape by refusing to bend.

129. **The load carried by an arch** may either be hung from it by means of tie-rods, or else it may rest on the top of the arch, the weight being carried from the different parts by means of struts or pillars of iron, stone, or brick, or the arch may be levelled up to the roadway by means of a solid mass of masonry, or merely by one or two pillars of masonry, the roadway being carried on little arches from one to the other; or there may be a filling-in of earth. It is rather difficult in a stone or brick bridge to say exactly what is the load on every portion of the arch, but it is guessed at, and a curve or **line of resistance,** such as P *h* Q, Fig. 106 (inverted), drawn. It is shown in Art. 368, that in a stone or brick arch it is dangerous to have the arch so thin that the line P *h* Q (inverted) passes anywhere outside **the middle third** of the arch ring. Thus, in Fig. 108 we have a section of a stone arch, the various stones or *voussoirs,* as they are called, being separated by joints of mortar or cement. Now divide each joint into three equal parts and draw two polygons, *m m m* and *n n n,* marking out the middle third of every joint. Let us suppose we know the weight which each voussoir supports, including its own weight (it is usual to consider the arch as one foot deep at right angles to the paper), and let these weights be the weights w_1, w_2, etc., shown in Fig. 108. I have taken a case in which these loads are symmetrical to right and left of the crown. Now draw the force polygon, Fig. 109; it happens to be all in one vertical line, the forces being all vertical. And now we come to the drawing of our line of resistance, but we are stopped at the outset *by not* knowing what is the thrust

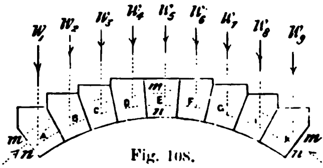

Fig. 108.

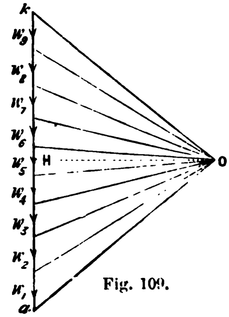

Fig. 109.

at the crown of the arch. The pull at the middle of our suspension bridge chain was quite definite, but the thrust at the crown of the arch may be what we please, and the arch will remain stable if the link polygon which we draw never passes outside the middle third of any of the joints.* Suppose we draw any symmetrical link polygon to begin with, by bisecting *a k* in H (Fig. 109), draw H O horizontal, and take O anywhere we please. O H will be the thrust in the crown of our arch, if this link polygon is the correct one. Join O *a*, O 1 2, O 2 3, etc. Start from any convenient point in w_5, Fig. 108, say E, within the space which contains the middle thirds of all the joints. Draw E D, Fig. 108,

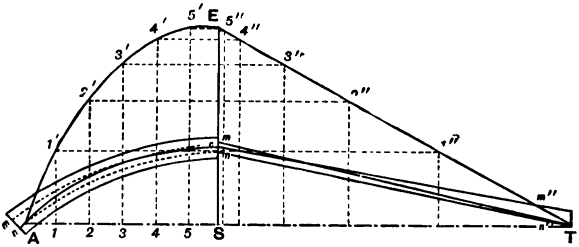

Fig. 110.

parallel to O 4 5, Fig. 109; draw D C, C B, B A, A P, in succession parallel to the corresponding lines in Fig. 109, and so also for E F, etc., to K Q. If any of the lines so drawn passes outside the space *m m*, *n n*, we must choose some other point E to begin at, and if we find that no choice of E will allow the link polygon to lie altogether within the space *m m*, *n n*, then we must choose another pole, O, in Fig. 109, until at length we find, as in the figure, a link polygon, P E Q, which cuts within the middle third of every joint. The lengths of the lines in Fig. 109 tell us the forces acting at the joints of Fig. 108. Thus O *a*, Fig. 109, is the force P A, Fig. 108, the resistance of the abutment of the bridge. Again, the length of O 4 5 is the force acting in the direction E D between the stones E and D.

130. Professor Fuller has made the work of drawing such a link polygon very easy. In case the loads are all parallel to one another, it can be shown that if a number of link polygons are drawn in Fig. 108 for different lengths, O H, Fig. 109, then the vertical distances between the points A, B, C, D, etc., are in the same proportion in all the link polygons.* Beginning with the first load w_1, draw (Fig. 110) A 1′ 2′ 3′ 4′ 5′ E, the half of any link polygon corresponding to P A B C D E in Fig. 108. Divide the horizontal A S into any number of equal parts; I choose six. Erect perpendiculars at A, 1, 2, 3, 4, 5, S. Draw horizontal lines A S T and from 1′, 2′, 3′, etc.; draw any inclined straight line of convenient length, E T; draw vertical lines from T, 1″, 2″, 3″, etc. From the points where the verticals A A′, 1 1′, 2 2′, etc., cut *m m* and *n n*, draw horizontals to cut the corresponding verticals from T 1″, 2″, 3″, etc. Join the points so found by the curves *m m″* and *n n″*; then, just as the straight line E T represents a link polygon, *m m″ n n″* represents the area bounding the middle third of all the joints, and any link polygon will be represented on the right hand side by a straight line. Now draw a straight line lying altogether within the space *m m″ n n″*. If you can draw several, then *draw that one which is steepest*, in this case C T. Project this over to the left hand side, and you will find that you have the link polygon, which supposes **the least thrust at the keystone.** The corresponding force polygon has its O H less than the O H of A E in the proportion S C to S′ E. The proof of this is easy.

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* It is obvious also that the link polygon, wherever it crosses a joint, must make an angle so near a right angle with the joint that there can be no *slipping or rupture* by shearing there.

In the figure the possible line A′ S′ T′ is so close to A S T that it is not shown. The proposition that the line of resistance must lie within the region of middle thirds will be dealt with in Art. 368. Fig. 109 shows the amounts of thrust between the **voussoirs**, and it is worth while comparing these forces with the areas of the joints. In Art. 368 the strength of such joints is more particularly studied. It is also worth while to consider whether the line of resistance may not be made to pass outside the region of middle thirds by a possible unsymmetrical load.

But we cannot give a proof of the proposition on which the whole work depends. If any line of resistance may be drawn to pass inside the region of middle thirds, the arch is *stable*, and the *steepest* of such lines is the actual line. We have a lecture model in which a number of rounded blocks of wood C D E F G H I J represent voussoirs, the fixed parts A and B being abutments. We can load these voussoirs in all sorts of ways. When the loading is changed, the voussoirs will be seen to roll on one another into new positions; and if the points of contact are joined, we have the line of resistance.

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* See Art. 349. *Proof.* Each link polygon is really a diagram of bending moment, supposing the loads were acting on a horizontal beam, and the scale of each diagram is proportional *to the length* of O H.

When the loading is *symmetrical*, loads on the haunches cause the line to rise at the haunches and get lower at the crown, and a load on the crown reverses this effect. A student may see on this model how change of loading causes such a yielding in the arch as produces pressures at the abutments just suited to equilibrium, provided only that a line of resistance cuts all the joints. But we must confess that the sudden step in the reasoning from this to the mere statement of the above proposition does not satisfy us. At the actual plane joints of an arch, changes occur when the load is changed, but these changes are very different from the changes that occur in the model: and it seems to us that the only legitimate method of study is that of assuming that a **masonry arch** behaves exactly like an **arched rib of iron** fixed at the ends, and

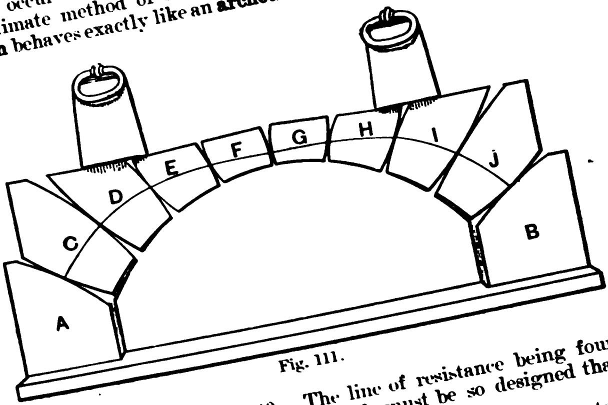

Fig. 111.

this is studied in Art. 380. The line of resistance being found as for an iron arch, the masonry arch must be so designed that this line is kept within the middle third.

In a few cases in Germany an attempt has been made to build masonry arches hinged at the ends, and, indeed, also with a hinge at the crown. This can be done by bedding the masonry at these places upon iron. A **quasi hinge** has also been employed by introducing plates of lead at the two joints at the springings, and one at the crown, the lead only extending over the middle thirds of the joints.

131. Buttresses. To find the force which acts from one stone to another in a buttress, it is necessary to know the force acting on every stone from the outside, and also the weight of the stone. Find the resultant of these two forces for eac stone, and draw the *link polygon* whose first side is the for on the top stone. In Fig. 112 F A B K P is the link polygon drawn. Each side of it shows the *resultant* of the forces acting at every joint, and the length of the corresponding line in Fig. 113 shows its *amount*. Thus the resultant of the forces acting at the joint S T is shown by the position of the line B K, and its amount is shown by the length of the line *o t* in Fig. 113.

If we see that any of the sides of the link polygon passes outside the middle third of the corresponding joint between two stones, we know that part of that joint will be subjected to tension, a condition to which we suppose that a common masonry joint ought not to be subjected.*

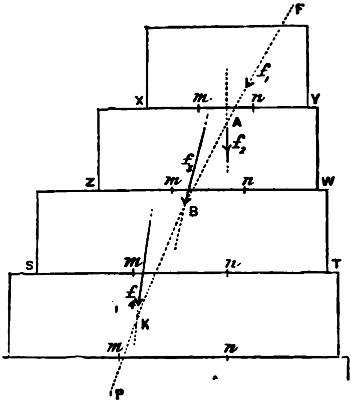

Fig. 112.

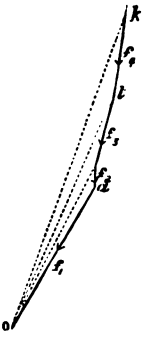

Fig. 113.

The student ought to work examples in which, besides the force F acting on the top stone, there are forces acting on the other stones, due, say, to the **pressure of water** (Art. 173), or to the **pressure of earth** (Art. 293).

132. To return to the **hanging chain.** If the total weight of a chain and all the vertical loads upon it is W, and if its ends are supported at two points, A and B, and at these points it makes angles α and β with the horizontal, the tensions there are represented by the two sides of a triangle, which are parallel to the directions there, the third side, representing W, being a vertical line. Let the student sketch examples.

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* In many cases it will be found well to magnify all the horizontal components of all the forces, magnifying the horizontal dimensions of all the stones in the same proportion. In this way the points in which each side of the link polygon cuts each joint may be found more accurately.

The tension T at A is evidently W cos. β/sin. $(\alpha + \beta)$, and the tension at B is W cos. α/sin. $(\alpha + \beta)$.

If at B the chain is horizontal, then T at A is W/sin. α, and the tension at B is W cot. α. Call this T_0.

If x is horizontal distance of any point in the chain from B (where the chain is horizontal) and y is vertical height above B; if the load on the chain between B and any point A is wx where w is the load per unit length of horizontal projection of the chain [here I assume that the load is uniformly spread *horizontally*, and this is nearly the case in any very flat chain or telegraph wire], then, as before, if α is the angle of inclination of the chain at A, we have $T_0 = wx$ cot. α; or, in the language of the calculus, $T_0 \frac{dy}{dx} = wx$;

and it follows that the shape of the chain is $y = \frac{1}{2} \frac{w}{T_0} x^2 \ldots . (1)$, a **parabola** with vertical axis whose vertex is at B, the lowest point. Given the heights of the two points of support of such a chain above its lowest point, and also the horizontal distance between them and w, it is easy to calculate the value of T_0 and everything else.

EXERCISE.

In a suspension bridge of 800 feet span and 80 feet dip, the weight of the platform, chains, and rods, etc., is $2\frac{1}{4}$ tons per foot run, what will be the horizontal tension in each of the two chains, and the tension in each at the point of support at the piers? *Ans.*, 1125 tons; 1212 tons.

133. In a **telegraph wire** of span l, the points of support being at the same level, if the **dip** (supposed to be small compared with l) is a, the weight of wire being W or wl nearly, it is easy to show that the tension, which I shall here call P, is $wl^2/8a$ or W$l/8a$ at the lowest point, and the tension elsewhere is not much greater. Now suppose s, the whole length of a wire, to be $b\ (1 + k\theta + \beta\,\text{P})$ where b is its length at the lowest **temperature**, and this we shall call the zero of temperature, under no tension, θ being the temperature Centigrade at any other time above the zero, k the coefficient of expansion, and β a constant easily determined when one knows Young's modulus for the material and the cross-section of the wire, just as w is known from the material and cross-section. If a is the dip, find the tension P, and find the length of the wire. The length of a parabolic curve is easily found by integration, but $x + 2\ y^2\ 3\ x$ gives the length from the origin to any point with sufficient accuracy for such purposes as the present. [Check this statement when you have leisure.] So that length of wire is $l + 8\ a^2/3\ l$.

If a_0 is the dip at the zero of temperature we are led to the result

$$k\,\theta = (\alpha_0 - \alpha) \left\{ \frac{\beta w l}{8\,\alpha\alpha_0} + \frac{8}{3}(\alpha + \alpha_0) \right\} \ldots . (1)$$

where $\alpha = a/l$, $\alpha_0 = a_0/l$.

If, now, the span l and w (the weight per unit length of a wire) are given, and if P_0 be taken as the greatest pull to which the wire ought to be subjected, this will be at the lowest temperature, which we here call zero. Then $\alpha_0 = wl/8\ \text{P}_0$.

Choose now some greater value of α, and from this, by (1), calculate θ, and note also that $\text{P} = wl/8\,\alpha$; so that it is easy to make out a table showing α and P for various temperatures.

This calculation may be regarded as a mere exercise for students, and yet there may be occasion for its use in practice.

The linesman who puts up telegraph wire pays very little attention to the result of such a calculation. He gives such a dip to his wire as seems good to him, making it somewhat less in winter than he would in summer. However much he may have *killed* the wire already, it will permanently stretch a little more if a very cold night comes, without real hurt to its material.

If a chain is uniform, and is loaded only with its own weight, and if we desire an exact answer instead of the above approximate answer for flat chains or wires, we let $\text{W} = ws$ where w is the weight of unit length of the chain and s is the length of chain from B, its lowest point, to any point A. Then $\text{T}_0 = ws \cot.\ \alpha$ leads, by the calculus, to the result $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$, where $wc = \text{T}_0$, and c is the vertical height of B above the origin. This curve is called the **catenary**.

As we have $\text{T} = \text{T}_0 \tan.\ \alpha$, it is easy to find T. It is also easy to find s in terms of x and y. The properties of this curve give very easy exercises in the calculus, and so they are well known.

CHAPTER IX.

HYDRAULIC MACHINES.

134. **Hydraulics.**—Hydraulic machines are very wonderful to people who observe their action for the first time. Ancient drawings show armies of slaves dragging on ropes to lift a single weight. Three hundred years ago, Fontana raised an obelisk at Rome with 40 capstans, worked by 960 men and 75 horses. A few years ago a similar obelisk was raised in London by four little 100-ton hydraulic jacks, each of which can be worked by one man. Large modern guns would be almost impossible to work with any other machinery. If you go to any large docks you will see how, by the manipulation of a few handles, a boy can remove heavy objects rapidly from ships and place them on the dock by means of an hydraulic crane. Visit any large steel works, and you will see great armour-plates and Bessemer converters and their appliances passed about nearly as readily as small objects are moved by blacksmiths and moulders. The steam-hammer, powerful as it is, is giving place to **hydraulic forging** and squeezing machinery, because the new forces are enormously greater, and their effects can be more uniformly distributed over large masses of metal, leaving them more homogeneous. Visit the Victoria Docks, and you will see vessels of 3,000 tons raised out of the water on a floating gridiron and towed off for repairs. Visit the River Weaver, in Cheshire, and you will see sections of a **canal** rising and falling 50 feet with canal boats, which are no longer delayed for hours in floating through a flight of locks. Instead of bringing great iron girders near a **riveting-machine**, we now take little riveting machines to the girder, and work them in any position through small flexible pipes from a distant steam-engine and pumps.

135. **Hydraulic Press.**—An hydraulic press is a machine which enables large weights to be lifted or great pressures exerted, but in which, instead of levers and wheels, we use water to **transmit the energy.** In Fig. 114 we have rams of large and small cross sections A and a square inches, and weights (neglecting their own weights) R and E. Because the vessel C is full of water which cannot escape past the rams on account of two **leather collars** or other packing at D and G, the velocity *ratio* is A/a. Thus if A/a is 100, and if a falls through 100

inches it displaces $a \times 100$ cubic inches of water. We assume that the water cannot escape, and that all the room needed is procured by the lifting of A; then A must lift one inch to make the necessary room.

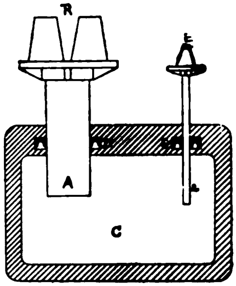

Fig. 114.

In actual specimens, instead of one large vessel C, we usually have vessels round A and a slightly larger than themselves, called a press and pump-barrel with a pipe connecting them. The **friction at a leather collar** is sometimes as little as one per cent. of the weight on the ram, and sometimes as much as 5 per cent. Notice that the amount of the axial force R which may be exerted, neglecting friction and weights of water, does not in any way depend upon the shape of the end of its ram, or where it is, or the direction of the ram's motion.

136. The internal construction of an hydraulic press is shown in Fig. 115. Three men press, each with a force, say, of 60 lbs., on the end of the lever G, whose mechanical advantage is, say 20, and hence the plunger, Ea, is pressed downwards with a force of 3,600 lbs. Just consider for an instant what is the condition of things before E moves. There is a ram, R, which carries a heavy weight, the weight to be lifted. Observe that this ram is wanting to fall, but it can only fall into the vessel D. Now the space between the vessel, or press, and the ram, and all the space in the tubes, T, is filled with water which has no means of getting away. It might get away by the little valve F, but that valve can only open upwards, and the more the water tries to escape, the more it really closes the valve, just as a closely-packed crowd in a panic keeps an inward-opening door closed, by which they might otherwise escape from a theatre. There is no escape for the water on the pump side; there is just as little on the other side, for you see that the water, if it escapes into the the space N, finds that it has still to get past the leather collar, which is, however, so placed and shaped that the *greater* the water pressure the *tighter* the leather fits the ram. Fig. 117 shows such a leather collar. There is then no escape for the water, and when this is the case, no matter what weight is placed on the top of the ram, it cannot fall. The falling of the ram would mean some escape of the water; but as there is no escape for the water, the ram will fall no more than if it were supported on some quite rigid material.

137. I have been supposing that a certain quantity of water will absolutely refuse to occupy a smaller space, but this is not quite correct. We know that if it were air that filled the space N, instead of water, and there were no escape for it, the ram would fall when a greater weight was placed on it; for although the air cannot escape, it is contented to occupy a smaller bulk. I have been supposing that this does not occur in water, and that water will refuse to go into a smaller space, whatever the pressure. This was an old notion which people deduced from the famous Florentine experiment. A hollow globe of gold was quite filled with water, and was hermetically sealed. It was then beaten to diminish its cubic contents, and the result was that drops of water made their appearance on the surface, having oozed out through the pores in the gold rather than *submit to the* lessening of the total bulk.

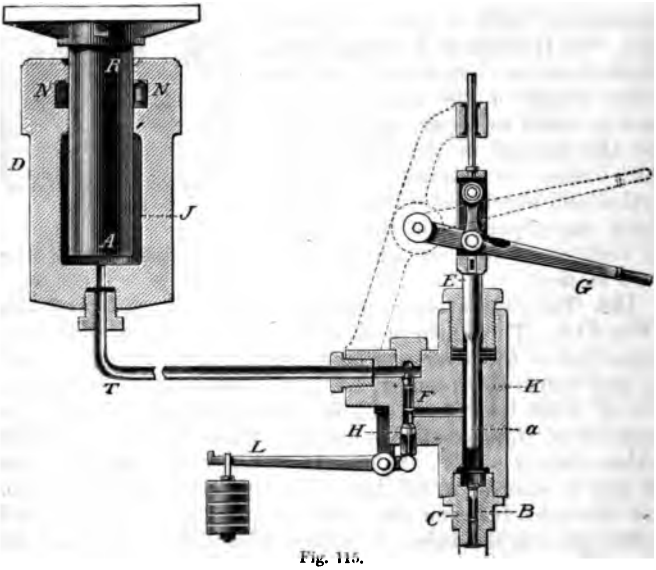

Fig. 115.

I am told that a lecturer at Chatham subjected the water inside a **cast-iron shell** to so much pressure that it came through the pores of the iron, and appeared as a fine spray or mist on the outside, and soon afterwards the shell burst, or, rather, fell gently in pieces. But with a **piezometer** it is easy to show that water and all other substances will submit to a diminution of their bulk when subjected to a pressure; and we find that this diminution for water is 1-20,000th of its total bulk for a change of pressure of one atmosphere, or 1-70th of its volume for a pressure of 2 tons per square inch, such as we find in hydraulic presses—that is, 70 cubic inches become 69. Now this diminution in bulk is far too insignificant to be of much practical importance in hydraulic machines, and we may take it for granted that whatever weight we place on the ram, it will not perceptibly fall, because the water refuses to become smaller in bulk, and because it cannot escape anywhere.

We understand that it *tries* to get away; it is *trying* to burst the thick cast-iron press; it is *trying* to burst the pipes and the pump. Before it will burst the pump, it will open the safety valve H, and escape; but we can assume for the present that the pressure never reaches the bursting pressure of the arrangement.

Now the labourer acts on the lever, forcing down the plunger E *a*. The water in the pump-barrel is just like the water in the press; it tries to escape, it tries to burst the pump-barrel, it resists the motion of the plunger. It tries to escape through the valves F and H, and it will open the valve F, and pass through, if the labourer acts sufficiently on his lever; but if the water passes through F, the ram A R must rise, however great the weight may be that is pressing it down. **The question is,** then, what force on the plunger E *a* is sufficiently great to cause motion—that is, to cause the water to pass through the valve F, and so make the ram A R rise?

Suppose that the plunger E *a* is one square inch in section, and that one inch more of its length is forced into the pump; evidently the metal takes up the place of an equal bulk of water, or one cubic inch. This cubic inch of water has found one cubic inch of room for itself somewhere else. As we suppose no greater compression of the water, and no yielding of the sides of the press, it is evident that one cubic inch of the ram must leave the press *to give the* water the space it must have.

Now, if the ram is 100 square inches in section, then 1-100th of an inch of its length contains one cubic inch in volume. If the ram lifts through the distance 1-100th of an inch, it will therefore leave one cubic inch of room behind it.

We are not concerned with the shapes of the ends of the plunger and ram; we know that if one more inch of the plunger enters the water, 1-100th of an inch of the ram must leave the water; that is, the relative speeds of plunger and ram are as one inch to 1-100th of an inch, or as 100 to 1. The plunger must move 100 times as quickly as the ram, and by the **law of work**, if there was no friction, a force of one pound on the plunger would balance a force of 100 pounds on the ram. We know the **mechanical advantage**, then, if there were no friction, in the portion of this machine from plunger to ram, if we know how many times greater is the area of the ram than the area of the plunger.

138. Different experimenters give different results as to the **loss of energy**, and therefore of mechanical advantage, in friction. Rankine states that there is 20 per cent. of loss. Mr. Hick found results less than one-tenth of this. It is known that in accumulators, the pressure which is great enough to lift a load, being, say 1,010 lbs. per square inch, if the pressure is reduced to 990, the same load will fall; thus only one per cent. of the energy seems to be wasted in friction at the leather collar, when the motion is slow. It is probable that there is always less than 20 per cent. of loss of energy altogether in large hand-worked presses. As an instance of this greater efficiency of hydraulic machines, a platform weighing 12 tons had to be lifted, and a little **hydraulic jack** was placed under one corner, a **screw jack** under the other. One man was told off to work the hydraulic jack and three men to the screw jack. The man at the hydraulic jack, with one hand in his pocket, would pump a few strokes and then quietly wait a little, whereas the three men were hard at work all the time.

My students in Japan used to employ a little hydraulic press to crush bricks, stones, and wood, and it was roughly assumed that the friction at the glands was insignificant. Of course, we only made this assumption when we wanted rapidly to get a rough idea of the relative strengths of materials to resist crushing. Still, it was a sort of thing that we should not have been able to do with any other except a lever *machine,* and I am inclined to think that there is more curacy with many lever-testing machines than there was my hydraulic press.

xample.—A three-ton hydraulic jack, like Fig. 116, not in good order, was examined in my laboratory; weights A hung at the end of a lever, and we calculated R the valent weight which the jack L was actually supporting. handle was replaced by a wooden sector, to which the ht B gave a slow motion, after one or two strokes had made by the experimenter to get things into a steady ition. The weight B, which would slowly overcome a R being noted, the experiment was repeated many

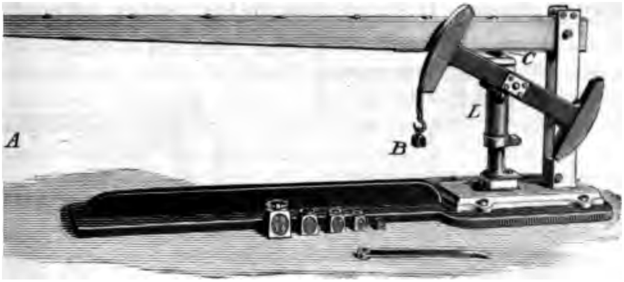

Fig. 116.

with other loads, and the values of R and B tabulated plotted on squared paper. We found the result: ·024 R + 2·2 where B is the effort, in pounds, applied e end of the handle, and R, in pounds, is the load on jack, not including the weight of the ram itself. The anical advantage of the lever was 14·75; the ram was hes in diameter and the pump plunger 1 inch, so that the velocity ratio was 14·75 × 4, or 59. If there had been iction the effort, B^1, would have been R ÷ 59; hence the ency is—

$$e = \frac{B^1}{B} = \frac{R}{59} \div (\cdot 024\,R + 2\cdot 2) = \frac{R}{1\cdot 42\,R + 130}$$

$$= \frac{1}{1\cdot 42 + \frac{130}{R}}$$

hus when R is 3 tons, or 6,720 lbs., the efficiency is 0·69. e can be no doubt *that the loss* of energy mainly occurs by solid friction, and so we are led to ask, **where does friction occur** in the hand-worked hydraulic press? It occurs at

1. The rubbing surface at the fulcrum of the lever. This is the friction of solids.
2. The rubbing surfaces of the two glands; one, that of the plunger, the other, that of the ram. Here, again, we have friction, as if between solids.
3. Everywhere in the water where there is motion. This is **fluid friction**, which is quite different from that of solids.

139. Figs. 117, 117A, show sections of the **leather collar** used in presses. It is made from the best leather, softened in hot water, and pressed between cast-iron moulds to its present shape, the pressure gradually increasing. It is then left for several days under pressure in the mould. When it is in its place in the press the water gets behind the leather, and presses it tightly against the ram. The friction seems mainly to occur at the part A, and increases as the pressure of the water increases. There does not seem to be much friction at the portion between A and B, and the efficiency of the press is but little altered by making A B greater or less. It is, however, asserted by some makers of presses that the distance A B is of importance; but I rather think that this is on account of deterioration. The part B A is constantly being in states of tension and compression, and is liable to crack. When the leather deteriorates, much time is wasted in *renewing it.*

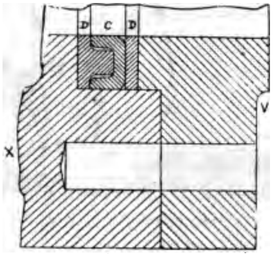

Fig. 117A.

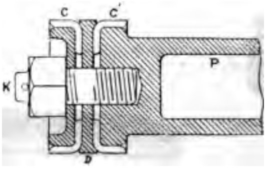

Fig. 117B.

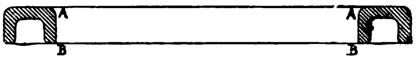

Fig. 117.

140. **Hemp-packing** is invariably used instead of the leather collar at low pressures; some manufacturers never use hemp or cotton when the water has a greater pressure than 700 lbs. per square inch, but others use hemp to 2,000 lbs. per square inch.

Our subject is too large to allow me to enter into many details as to the advantages and disadvantages of leather, india-rubber, and gutta-percha for packing purposes. There is a great divergence of opinion, and I believe that much of the evidence against one or another material is based on the bad preparation of the materials against which the evidence is given.

141. In Fig. 117B you will see the form of **india-rubber cup** used in hydraulic plungers, jacks, and bears. It is moulded in the form shown, and is fastened to the end of the ram by means of a screwed bolt and metal washer, as shown in the figures. The water pressure keeps it tight against the cylinder at the edge, and this is where the friction occurs.

The **quasi-solid friction** between lubricated leather or hemp or rubber and metal, follows the laws of solid friction, and we have now described what friction of this kind occurs in hydraulic presses. The remaining source of loss of energy is the **fluid friction.**

The flow of water even in the narrow passages past the valves may be very slow, and therefore the fluid friction may be as small as we please. So small, indeed, is the fluid velocity that when **oil** is used the loss of energy is much the same. Probably honey or tar would not give very different results; but the more viscous mixtures of tar and pitch would be unsuitable, because in these the fluid friction, at even such small velocities as we have to consider, would be considerable. Even solid pitch would, however, act as a fluid, and might give the same mechanical advantage as water if we made only one stroke of the plunger in a month.

142. In applying the **law of work** to our machines, equating the energy given to, and the energy given out by, each machine, we assume no store of energy in the machine. In the press, even if we disregard the compression of the water and the elastic yielding of the press, we must remember that the ram itself is lifted, and also that some water is lifted in level. The lifting of the ram is easy to take into account; consider it part of the weight to be lifted. In ordinary presses it is of no great

consequence. But when the dead weight of the ram and other things is greater than the weight lifted usefully, as in warehouse and hotel **hoists**, it cannot be neglected, and it is usually easily **balanced**. In very high hoists the weight of water which changes its level becomes important. (See Art. 169.)

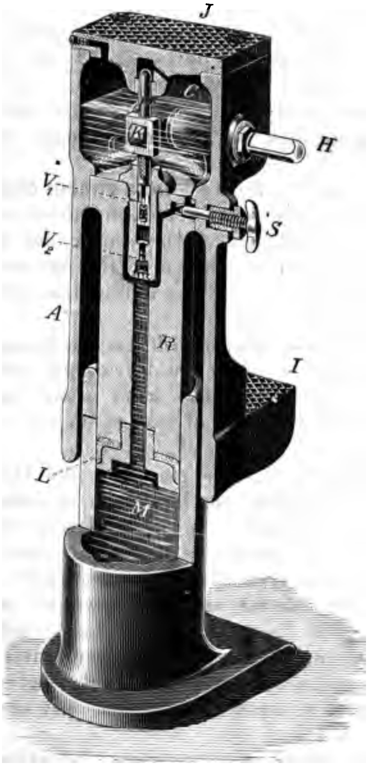

Fig. 118.

143. Fig. 118 shows a section of a **lifting jack** with its ram R lifting any weight which may rest on J or I. Notice how the ram R fits the press M, and is made water-tight by the india-rubber dish L. A handle or lever is attached at H, and when worked causes great pressure on a projection or cam K, which communicates with the plunger V_1 of the pump, and gives it a downward motion, pressing the water in the pump-chamber through the valve V_2 and the passage in R to the press M, which rests on the ground. A small pressure on the lever thus forces more and more water into the press, and necessitates the upward movement of the ram R. The upward motion of the lever causes a partial vacuum in the pump-chamber as the plunger V_1 is withdrawn, and the pressure of the air and liquid in the cistern C overcomes the resistance of the spring on the inlet valve V_1, which is really a part of the pump plunger, and thus effects a passage for the water into pump-chamber. *During the downward* stroke the previously-described operation is repeated. If we want to lower the weight we open the lowering screw S, and allow the water to return from the press M to the cistern C, the ram falling.

144. The **punching-bear,** shown in section in Fig. 119, is similar in construction. The plunger, pump, valves, etc., are much the same as in the last case. The upper part of the stout ram F terminates in an india-rubber dish, which is fastened by the washer and a bolt passing through the middle of the washer into the ram.

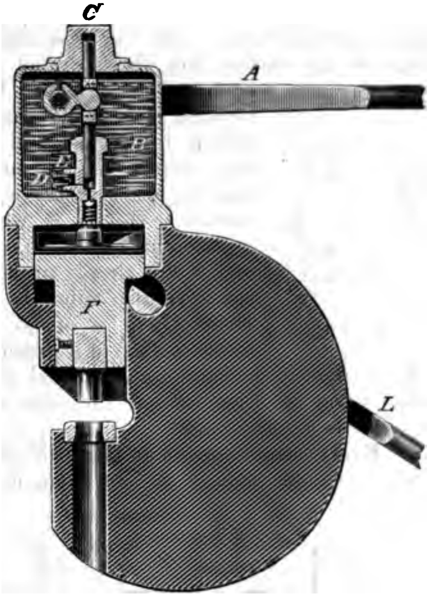

Fig. 119.

As we work the top lever A, the ram F holding the punch is pressed down by water through a valve arrangement exactly similar to that in the lifting jack.

145. We are now well aware of the way in which the water in these machines acts. It is nearly incompressible, and tries to find an outlet in every direction. Consider what occurs from particle to particle of the water. Each particle presses on all its neighbours, because they all press in upon it, and it presses equally in every direction.

Wherever the water comes in contact with a solid surface, it presses **normally** against the surface. There can be no such thing as oblique pressure in water when the water is at rest, for oblique pressure means a force partly along the surface, and this would imply some frictional resistance to sliding, which we know cannot occur in water at rest. It is evident from our discussion of the hydraulic press as a machine in which there is no store of energy, *that on every square inch of the solid* surface touched by the water there must be the same force acting, the water tending to escape everywhere; and when we consider the whole case mathematically, we find that every little interface separating any two portions of water is acted on by this same pressure per square inch

146. We shall be perfectly safe in all our notions of fluid pressure if we consider each particle of water to be a very small being, greased all over, so that it cannot possibly resist sliding past its neighbours. It can press normally against a wall or against any surface, but there cannot possibly be a tangential or frictional pressure between it and a wall, because it is well greased. All the water is trying to escape, and the total pressure on any surface is evidently proportional to the number of water particles pressing against the surface. Hence, if we have a piston A and a piston B (Fig. 120), the total pressures on A and B are simply **proportional to the areas** of the cylindric tubes in which they can move. Evidently it is of no importance whether the inner surface of a piston has projections or not. Thus there is the same total vertical pressure on piston M and on piston N, Fig. 121, if the cross-sectional areas of their cylinders are the same.

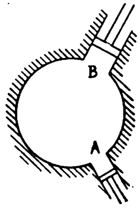

Fig. 120.

Everything depends on this: will they leave the same empty space behind them if each of them moves one inch?—and we know that in each case the empty space is simply the volume of one inch of length of the cylinder.

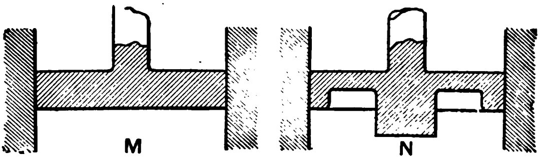

Fig. 121.

147. In the same way, although the end of the ram may be curved, as in Fig. 122, and is therefore being acted on by a series of pressures normally to the curved surface, as in the figure, it is easy to show that the resultant action of these is in *the direction* A B, and is really the same as if the ram had a flat end. Every one of these forces has a *horizontal* tendency, more or less, and when we leave out of account these horizontal actions, we get the same *vertical* result for all shapes of ends.

You will understand this better, perhaps, if we consider a vessel, A, B, C (Fig. 123), to be filled with fluid at such a great pressure that we can neglect the pressure due to the fluid's own weight. We are assuming now that we can do this in the hydraulic press and in steam boilers. The pressure on the surface everywhere is shown by the arrows.

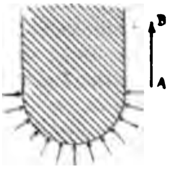

Fig. 122.

It is evident that the total horizontal force on the curved surface A C B is exactly equal and opposite to the total horizontal force on the flat surface A B, because if there were on the whole more force on one than the other, the vessel would move bodily, an idea which is absurd. Hence, when we want to find the **total horizontal force** on the curved surface, we never dream of going into the long calculation which you might think necessary, for it is simply equal to the *area in square inches of the flat surface* A B, *multiplied by the pressure per square inch.*

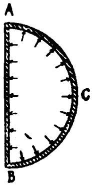

Fig. 123.

148. Suppose we want to find the horizontal **bursting tendency** of the egg-ended boiler M N, Fig. 124, that is, say, the force tending to burst it by direct pull of the iron at the section A B—we do not trouble ourselves with the **shape** of the boiler anywhere except at A B itself. The bursting force is the inside area of A B in square inches, multiplied into the pressure per square inch. The area of the iron in the section A B is exposed to this pull. These are the two **important facts** to be remembered. Consider any section whatsoever of a boiler, or ram, or pipe. Remember that the fluid pressure is calculated over the whole area. The resistance of the iron is only calculated over the actual sectional area of the metal.

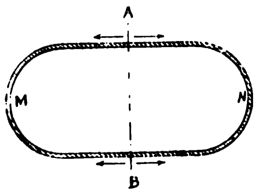

Fig. 124.

Thus, if we want to find **the tendency to burst** along such a section as A B, we take the total inside area of the section multiplied by the pressure, and this is equal to the stress in the iron all along this section, multiplied by the whole sectional area of the iron.

In a cylindric boiler or press, which is everywhere of the same thickness, it is easy to show that if we neglect the effect of the ends, the tendency to burst *laterally* is *twice* as great as the tendency to burst *endwise.*

Thus in the endwise bursting, if p is the bursting pressure in pounds per square inch, f the tensile strength of the material in pounds per square inch, r the radius of the boiler, and t the thickness of metal, the total force tending to produce bursting is the area of the circular cross section, $3{\cdot}14\ r^2$, multiplied by p, and the total force resisting fracture is the circumference of the circular cross section, $6{\cdot}28\ r$, multiplied by t and by f. Hence—

$$3{\cdot}14\ r^2 p = 6{\cdot}28\ r f t,$$

or the bursting pressure $p = 2 f t \div r$.

Again, if a boiler is l inches long, the total force tending to burst the boiler laterally is the area l times the diameter of the boiler, multiplied by p, or $2\ r\ l\ p$, and the total force resisting fracture, if we neglect the ends, is the area of the iron $2\ l\ t$, multiplied by f. Hence—

$$2\ r\ l p = 2\ l t f,$$

or the bursting pressure $p = f t \div r$.

Hence it would take twice as much pressure to burst the boiler if we assumed it to burst endwise. We always calculate the strength of a pipe or boiler on the second assumption therefore, and we have the rule :—The bursting pressure in pounds per square inch is equal to the tensile strength of the metal in pounds per square inch, multiplied by the thickness of the metal in inches, divided by the radius of the boiler or pipe in inches.

149. When the press is *thick*, as we find it in an hydraulic machine, it is rather more difficult to calculate the bursting pressure, because the tensile strain is not distributed uniformly over the section at which there is a tendency for rupture to occur. The inside portions of the metal near the water are subjected to more pulling forces than the outer portions.

The mathematical part of this subject will come before us in Art. 275. The subject is important and needs to be introduced here also. **The result** of the mathematical investigation is this:—Neglecting the strength of the ends, the bursting pressure, multiplied by the sum of the areas of the outer and inner circles of the cross section of the cylinder, is equal to the tensile stress which the metal will stand multiplied by the area of the metal exposed in such a cross section. This rule you will find applicable to *thin* cylinders as well; it is the same rule as the one already given for thin cylinders. You see that if the fluid pressure is equal to the tensile strength of the metal, no thickness of the cylinder can prevent its bursting.

The investigation might perhaps have some use if iron when it leaves the foundry had all through its thickness the same qualities, and a perfect absence from **strain**. It is good to remember that an iron casting when it leaves the foundry, although not quite so curiously strained as a Prince Rupert's drop or toughened glass, is yet in a state of strain which we know very little about.

The above investigation shows that in a **thick cylinder** there is an exceedingly great difference in the tensions of the inner and outer portions of the metal of a **gun** or hydraulic press. For the purpose of producing a more uniform state of stress, the inside portions of the metal are often **chilled**; that is, the metal in the inside is very quickly cooled after it is cast. The result is that these portions are virtually in a state of great compression, and hence, when the regular strain occurs through water pressure, there is a considerable tensile yielding in the inner portions of metal before the bursting tensile stress is reached there; and hence, with a proper amount of chilling, it is possible to have the tensile stress in the metal nearly uniform when fracture is ready to occur. In such a case as this, the calculation of the bursting pressure of a press is just the same as if it were thin. We shall return to this subject in Art. 275.

It will sometimes surprise a student to see how much pressure is withstood by a **wooden pipe** wound round its outside with hoop-iron, or a hollow cylindric masonry or **concrete** reservoir with strong tightened chains or rings of iron on its outside. [The engineer need not fear deterioration of iron imbedded in cement. My brother, who has great experience, tells me that burying even one end of a bar of iron in cement has the power of preserving all of it from ordinary weathering or galvanic action.]

150. Cast steel is now getting common for the cylinders of small jacks and bears. The large presses used in warehouses are, however, still made of loam-moulded cast iron, as all presses used to be in my apprenticeship days. But in a great deal of large press-work cast steel has come largely into fashion, its tensile strength being so much greater than the strength of cast iron as to make a most extraordinary difference in the outside sizes of the cylinders. Thus, for example, in a certain **cotton-press**, the use of steel enabled three cylinders to be placed side by side in a space which would only have allowed one to be used if it had been made of cast iron.

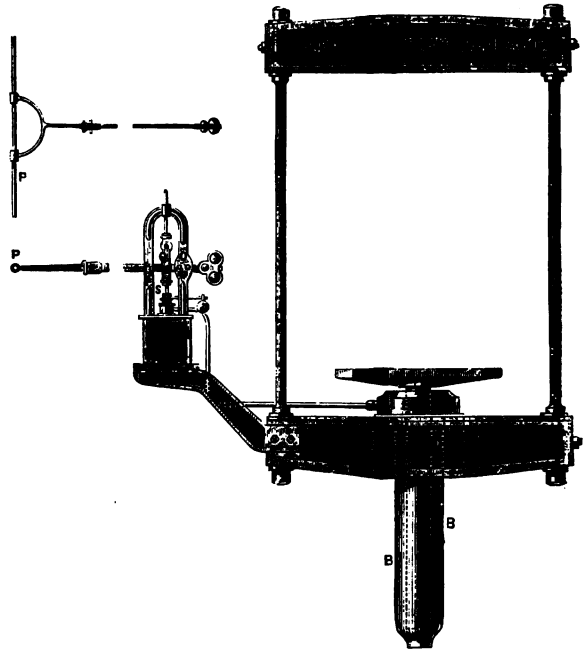

Fig. 125.

Fig. 125 is a drawing of an hydraulic press, which gives a fairly good idea of the press used in warehouses for bales of linen and Manchester goods, linen-yarn, etc. The ram is usually about 10 inches in diameter, and the working pressure of the water from 2 to 3 tons per square inch. As the area of the cross section of the ram is 78 square inches, this means a total pressure of upwards of 230 tons. When used for **packing hay**, for **expressing oil** from seeds, and for other purposes, alterations are made in the shape, length of standards, and, indeed, in the size of all the parts. In expressing oil, for example, there would be an alteration in the arrangement of the table or platten and the head. Sometimes the table becomes a piston, fitting into a cylinder attached to the head. Usually, however, the arrangement is what you see. If **yarn** has to be pressed, it is placed in a great oak box, bound with iron, running on wheels. It is run in between the columns. The pumps work, a movable bottom of the box rises up, the pressure on the yarn gets greater and greater, and when about 230 tons is the total force, the yarn, very much diminished in bulk, is tied up, the ram descends, the great box is rolled out, and another loosely-filled one is rolled behind it to undergo the same process.

For **warehouse use** the pumps are often worked by hand, but even thirty years ago I remember that orders were generally for pumps worked by power; that is, driven from shafting and worked by cranks. In this case it was common to attach a series of these presses to one set of pumps, all the presses in a warehouse being fed from a long pipe, each having its separate valve.

151. You will see that, although in hydraulic jacks for lifting heavy weights the weight on the ram is the same during the whole of any operation, this is not the case in baling presses. Thus, for example, in the Indian cotton trade, governed as it is by the Suez Canal regulations, it is necessary to press cotton so compactly that it is like a piece of oak, and it can be planed up like oak.

During the early part of the operation the pressure on the ram is therefore *small*, becoming very *great* towards the end; and it is the *greatest* pressure, of course, which decides the relative sizes of plunger and ram. If the ram were made to rise quickly at the beginning, and more slowly towards the end, it is obvious that there would not only be a saving of time, but a more regular doing of work.

In **hand presses** it is usual to change the fulcrum of the lever, so that a labourer may work more rapidly at the beginning. It is also common to use a *large* pump plunger in the early part of a pressing operation, changing it to a *small* one at the end; or to use two of equal size, throwing one of them out of gear towards the end of the operation; or to use water from an accumulator (Art. 152) in the early part of the operation, finishing with water from an **intensifier** (Art. 152).

In an early form of **cotton baling** machine, instead of using *one* large ram, *three* were used. At first only the centre press was connected with the pumps, and of course it rose quickly; meanwhile the other two were filling with water from a tank, but the pressure of the water in them was insignificant. As the operation proceeded one of the side presses was disconnected from the tank, and was fed from the pump. The operation proceeded more **slowly** now, as the pump had to supply *two* presses instead of *one;* but the possible total force was doubled. Towards the end of the operation the third press was disconnected from the tank, and was connected with the pump; the operation proceeded **more slowly still,** although the pump might be working at much the same speed as in the beginning. But the possible total force was just *three times* what it would have been with only one ram.

In a later form there are twelve pump plungers attached directly to the cross-heads of the steam-engines. At the beginning all twelve are working, as the pressure is small. Then as the pressure gets greater, one set of four pumps is detached, so that they do not pump water into the press, but merely pump water back to the cistern from which they draw it. Thus eight pumps are now pumping, forcing into the press less water than the twelve did before; but as they have the whole steam piston force acting on them, they are able to force the water in against a very much greater ram pressure. Later on in the operation four more pumps cease to act.

A **further improvement** is to be noticed. The head or platten is made with two long columns attached to its under side, hanging down. When the first operation is finished, the bottoms of these columns are just above the base, and may be locked firmly, so that the head cannot fall back again. Now the finishing stroke is made; two 19-inch rams are pressed downwards on the upper end of the bale with a much greater pressure than it was possible to apply with the bottom 11-inch ram.

Other forms of cotton-press are used. Sometimes fewer pumps are used, and two bottom presses and rams, instead of *one* with larger rams, for pressing downwards from the top of the press to finish the operation. In some cases the finishing pressure operation may be performed when the bottom ram is being withdrawn, so that one bale is being finished in the upper part of the press when the box is being filled with cotton for a new bale. Diagrams have been obtained giving the nature of the pressures to which a cotton bale is subjected. In one which is before me, when the lower ram has risen 11 feet, the force is only 8 tons; but in another foot the force increases to 16 tons; in another foot, to 33 tons; and in another to 59 tons. The upper rams begin to operate when the total force is about 160 tons; but on moving through 3 inches, they have to exert 200 tons; three inches further, 300 tons; three inches further, 500 tons; and three inches further —that is, when the bale was finished—they were exerting a force of 900 tons.

We see, then, that capability of working very rapidly when the forces are small, and working very slowly when the forces are great, on the supposition that the steam-engine is always working at the same speed—these are the **important things** to be looked for in hydraulic presses.

152. There are **hydraulic power companies** now in many towns, whose 4- and 6-inch, and other sizes of pipes are laid along the streets like gas-pipes, so that any customer may be supplied with pressure water to work pressing, lifting, and other machinery, paying merely for the amount of water used. The force-pumps are usually worked directly from the cross-heads of steam-engines; they must be capable of delivering the maximum supply required by customers. If the area of the pump plunger is a square inches, the pressure p lb. per square inch, the force on the plunger must be $a\,p$ lbs. As there ought always to be a fly-wheel on the engine, it is the average effective pressure of the steam, *minus* frictional forces, which is determined by this. If the length of a stroke is l feet, and there are n effective strokes per minute, the horse-power expended in the pump is $plan \div 33{,}000$. Usually there are three pump plungers working, so that the flow of water shall be more nearly uniform. There is also always an **accumulator**, or heavy weight w, of perhaps 112 tons, resting on the top of a ram, of 20 inches diameter, its press communicating freely with the hydraulic mains. In such a case, the pressure in the accumulator would be 800 lbs. per square inch. When there is a small demand, w will be seen to rise, its press taking the surplus supply. When the demand is great, W will be seen to fall, its press providing for the extra demand. Very often the engines will be seen working fast; W will be seen rising. When W reaches a certain height it acts on a lever which shuts off steam from the engines; these work less rapidly, and W will be seen to fall. When W descends far enough it moves a lever, and admits more steam. The **total store of energy** in W lifted is often not more than half a minute maximum supply, yet it is sufficient for regulation. In the above case the maximum store, if the maximum lift is 12 feet, is 112 × 2,240 × 12 foot-pounds. The same answer will be obtained if we remember that **every pound** of water means a store of 2·3 × 800 foot-pounds. Or, again, **every cubic foot** means a store which is numerically equal to the pressure in pounds per square foot. When, as on board ship and elsewhere, a great weight would be inconvenient, an **intensifier** (Fig. 126) is used. If water from a tank at the pressure p communicates with one side of a piston A of area A, it acts just like a weight of the amount p A. C is the accumulator, whose ram a is so much smaller than the piston that the pressure P in the accumulator is that necessary in the hydraulic mains D E. When a tank of water is not available for A, the pressure of steam is employed.

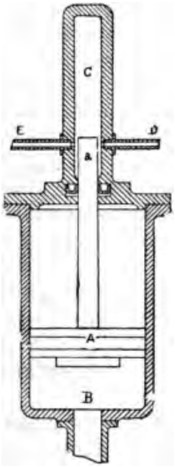

Fig. 126.

Example.—If we wish to get 700 lbs. pressure from a ram 5 inches diameter, and we have a supply from a tank 88 feet high, what size of piston is needed in the intensifier? Here p is 88 ÷ 2·3, or 38·3 lbs. per square inch. (See Art. 410.) The total force needed is 700 × 5^2 × ·7854, or 13,744 lbs.; so that the area of the piston, or large ram, is 13,744 ÷ 38·3, or 359·13 square inches; or it is 21·4 inches in diameter.

Exercise.—A press has two stuffing boxes in line, and a vertical rod protrudes through both, the lower end being the smaller. The diameters being 8 inches and 5 inches, and the water pressure 1,400 lbs. per square inch, what is the load on this differential accumulator? If it may rise 10 feet, what is its store of energy, and what is its store of water in cubic feet, and in gallons? *Ans.*, 42,882 lbs.; 428,820 foot-lbs.; 2·127; 13·25.

153. Pressure water, as sold by the hydraulic companies, would be cheaper if a better load factor (average output ÷ maximum output from the central station) existed, as the engines could work continuously. They charge about 2d. per horse-power per hour, and the cost of production is about 0·6 pence. The cost of the actual power given out by the motors used by customers varies greatly.

Exercise.—At a pressure of 800 lbs. per square inch, what is the charge for 1,000 gallons of water at 2d. per horse-power hour? *Ans.*, £4 13s. 5d.

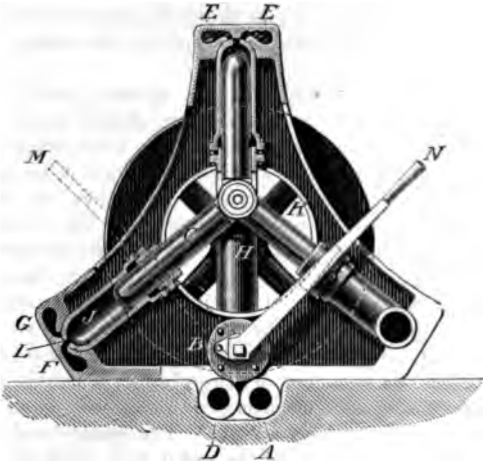

Fig. 127.

154. A **water-pressure engine** may be looked upon as the inverse of a reciprocating pump. If we neglect the shocks, which are always due to imperfect construction, when a water-pressure engine or pump works at a certain speed, the loss of energy by friction in the engine seems to be the same at high and at low pressure, and hence there is greater efficiency at high pressure. In all cases, wherever kinetic energy is produced in water, it is almost altogether wasted except in turbines, and to some extent in certain punching-machines.

Fig. 127 shows one of the simplest forms of engine. Water enters the arrangement by the pipe A when the cock B is opened by means of the handle N. There are three rams here, in oscillating presses, gearing on the same crank pin; but I mean to confine my attention to one of them. The supply of water enters at A, and finds its way to the space F, by means of a passage in the framework of the machine. There is another passage leading to the exhaust spaces G, and allowing water to flow from these spaces through the discharge pipe Q. By reversing the handle, F may be made the exhaust space, and G the supply space, and when the handle is in the middle position it acts as a brake, so that by means of this handle we can make the engine work in opposite directions, or stop it altogether.

Water enters at L, and fills the space J, and it presses on the ram C, causing it to leave more empty space behind it. You know now how to calculate the force with which the plunger is being pressed. The plunger cannot go out without turning the crank, H K. Observe, too, that as there is no connecting rod, the cylinder J turns just like the oscillating cylinder of certain steam-engines. When the crank reaches its dead point, the plunger can go out no further; but when this happens the orifice L is just ceasing to let water enter from F, and is beginning to let the water escape into G. As three plungers act on the same crank pin, the crank does not stop anywhere, and as it moves on, the plunger C comes back again, driving the water from J through L into G and away by Q.

This is all exceedingly simple. The quantity of water used in one stroke of C is simply the volume of C which leaves J in one stroke, assuming that the water can come in quite freely. Each pound of this water has a certain amount of pressure energy, which it gives up to the plunger, and which you may take to be the pressure energy in F minus the pressure energy in G. This is only true if we assume that the water has no kinetic energy where it is in contact with the plunger; but as the plunger is itself moving, we know that we are here making a small error. This is the same as saying that the pressure on the plunger is less than the pressure in the mains. We are also assuming no loss by friction at the passage, L, which is again an error.

Neglecting friction, if p is the total pressure difference between the supply and exhaust, and a is the area of cross section of the plunger, then $p\,a$ is the force upon it in pounds. If l is twice the length of the crank in feet, and there are n

revolutions per minute, each plunger gives out $p\,l\,a\,n \div 33{,}000$ horse-power, and the three plungers give out three times this. Notice that this engine is almost too simple; there is considerable wire-drawing, and therefore loss of pressure due to friction at the valves.

155. Fig. 128 is a section of another three-throw water-pressure engine. The pistons in the three cylinders, A B C, are trunked and connected to one crank-pin, D, O being the centre of the crank-shaft. The method of packing is clearly shown. The water presses only on one side of each piston, and as between the rods and crank-pin only pushing forces act, the bearing is only on a small portion of the circumference of the pin. It can easily be imagined how the pressure water is admitted to each cylinder at the end of the stroke of its piston, and how it is allowed to escape at the *other* end. The engine will start

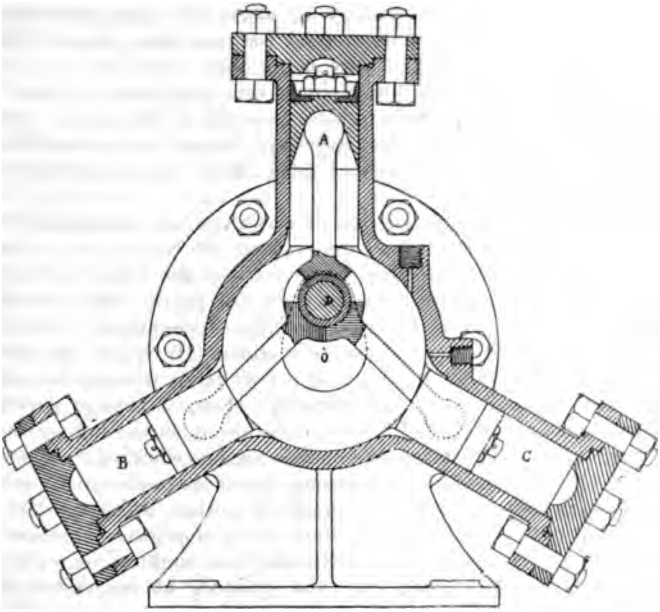

Fig. 128.

from any position, and the flow of water to it is fairly regular, because three cylinders are used.

156. It is evident to anyone who has studied air- and steam-engines that water-pressure engines may be of a very great number of shapes.

In most engines of this kind the work to be done per stroke may be very different at different times, and yet the pressure water used—that is, the energy—is always the same, and so there is considerable loss. One method for remedying this evil is to shorten the crank as the work being done is less.

Another method which has been suggested is that of admitting pressure water for less than the whole stroke, simply taking water from the discharge-pipe for the remainder. When engines have a fixed sort of duty, there is no need for any adjustment.

The common construction of water-pressure engines will be readily understood if you understand the construction of the steam-engine. Remember, however, that the velocity of water ought never to be great in the engine or pipes. Wire-drawing leads to serious loss by friction in the steam-engine; it is far more serious in water-pressure engines. In these the valves ought to be quite open, giving a very large passage for water to flow through almost immediately. Hence, although the slide arrangements of Fig. 127 are allowable in small engines, they cannot be used in large economical engines working constantly. Remember, too, that all frictional losses are made much greater by quick motion, and by reversals of motion, and hence that it is very important to have a long stroke of piston or plunger.

Lastly, remember that, although there ought to be no waste space between steam-piston and cylinder at the end of the stroke (very little clearance), this is of almost no importance in water-pressure engines, because of the incompressibility of water.

157. We have described (Art. 136) the **force-pump** used by hydraulic engineers. All force-pumps are much the same in principle, but in ordinary force-pumps for water (see Art. 406), and for use in feeding boilers there is usually less trouble with the valves and packing than there is in hydraulic pumps. Students will pay attention in Chap. XXIII. to the blow caused by **sudden stoppage** of the flow of water.

Direct-acting steam-pumps are much used in draining mines. *Sometimes* water-pressure engines have been used instead of steam—that is, water from an accumulator on the surface goes down the mine to a water-pressure engine which drives a pump; the mine water and the exhaust water of the engine are sent to the surface. In such a case, pressures are very great, and the effects due to sudden stoppage of flow in pipes may produce damage unless the valves are so arranged that when they close or open the general circulation is not much affected.

The **common lift-pump** is described in Art. 406. Its form as an air-pump is usually described in books on the steam-engine, and, indeed, it is in such books that pumping machinery more naturally finds a place than in books on applied mechanics. In Art. 433 we find that in a stream line the total energy of one pound of fluid remains constant if there is no friction; a pump adds to this store; a turbine or water-wheel takes away from this store; friction also diminishes the store, converting mechanical energy into heat.

158. What now are the conditions under which transmission of power by hydraulic action is most suitable?

1st. Intermittent action, because the accumulator is so nearly perfect, giving out energy simply in proportion to the quantity of water used, and yet allowing an engine of small power to be storing continually.

2nd. Action requiring not a very great quantity of power.

3rd. Action of a comparatively slow kind, the water never being allowed to flow so fast that its store of kinetic energy is great, since the kinetic energy is nearly all wasted; slow action, with considerable force.

4th. Action which is greatly continuous in one direction, not requiring much stoppage or reversal of the water motion.

You will see from this that the conditions required in pressing machinery, cranes, hoists, and lifts are better satisfied by hydraulic transmission of power than they can be by any other method of power transmission which is known to us.

159. In Fig. 129 we have one specimen of a **hydraulic crane**, whose action it is very easy to understand. Suppose water at 700 lbs. pressure per square inch admitted to the cylindric space A, and that the space B, on the other side of the piston, although filled with water, has only a comparatively small pressure, and communicates with a low-lying tank; neglecting the small pressure in B, we see that the piston is pushed forward with a force of $700 \times A$ lbs. if A is the area of the piston

in square inches. Now the motion of the piston is multiplied eight times by the chain, which passes over blocks, each containing four sheaves, attached at M and N. The block at N gets the motion of the piston, and the chain at P must be drawn in eight times more quickly than this. You know that the pull in the chain may be one-eighth as much as the total force on the piston, and it can therefore lift through eight times the distance a weight of one-eighth the amount, or 700 A ÷ 8 lbs.

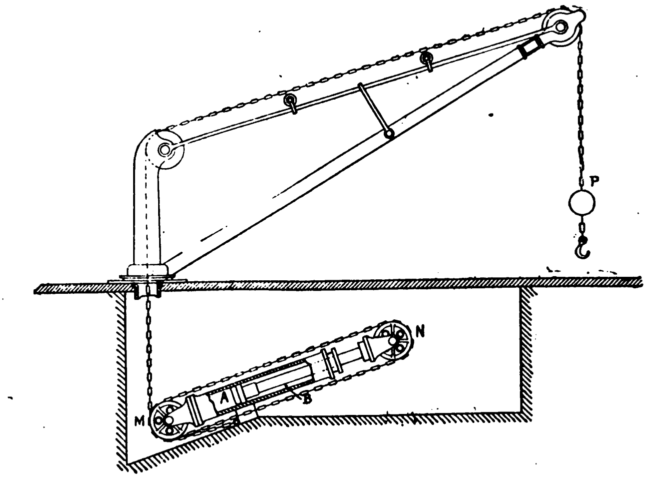

Fig. 129.

This is the original form of **Lord Armstrong**. Whatever defect there is lies in the use of chains passing over numerous sheaves giving rise to a great amount of friction. Cranes require so little horse-power to work them, however, that mere economy of coal is barely worth considering, and the risk of accident, which might be done away with very greatly by direct hydraulic action, is not important either. You see that if A and B both receive pressure water there is less water used than before; for although as much comes into A as before, B is sending water back to the pump or into A. In fact, the total pressure on the pisto is 700 lbs. multiplied by the difference between the areas of th two sides of the piston exposed to pressure—that is, the me *area of* cross-section of the thick piston-rod. Hence, we can wo this crane so that it lifts heavy loads or light loads—that is, it is double-powered. Unfortunately, however, when working any

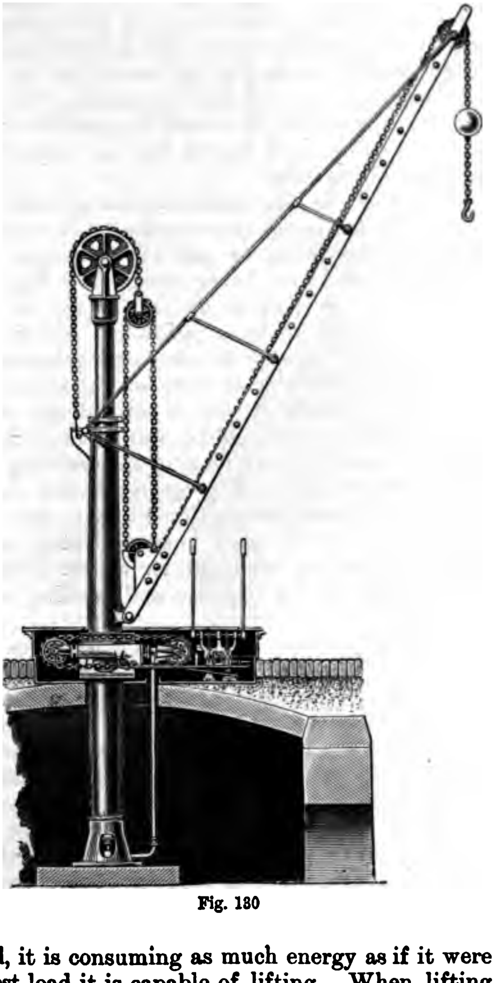

Fig. 130

heavy load, it is consuming as much energy as if it were lifting the heaviest load it is capable of lifting. When lifting on its second power, and lifting a light load, it is using as much energy as if it were lifting the heaviest load this second power is capable of lifting.

Thus, let us suppose it lifting eight tons to a height of 20 ft., and that one cubic foot of water is used in the operation. Then, on the same power, suppose it to be lifting four tons to the same height, it still uses one cubic foot of water, and in both cases there is the same energy used for the pumps. Of course, by a combination of cylinders, it would be possible to vary the work expended as the load varied, but the expedient is not of a very practical character.

Such a crane as this has usually another cylinder and chain for turning it round. In many cranes the action is more direct; a two-sheaved pulley-block, or one with a single sheave, or even no pulley-block at all, being used. (See Fig. 130.) In many cases loads are pushed up on the top of a ram, in others they are pulled up by a piston rod directly; in fact, there is an endless variety in the arrangements, but the principles of the hydraulic press parts are the same in all.

160. Now, supposing it is easy to get a supply of pressure-water, what, besides cranes, hoists, and pressure-engines, can be worked by means of it? All **forging and welding machines**, which with moulds and dies properly shaped and pressed together with enormous steady force, seem, for objects of settled shapes, to be a very great improvement on any method of hammering; stamping machines, for all sorts of purposes; and bending machines, for joggling and bending angle-irons, rails, and beams.

Students must examine such bending-machines for themselves, and notice how the travel of the press-block is determined by tappets, which open and close the valve for water supply at any point in the stroke we please. By means of such a bending-machine any number of curves may be made identically the same. In many **punching, forging, stamping, and shearing machines** we also have the inlet and exhaust valve worked by hand or foot levers, the stroke being regulated with great nicety by tappet motions. Thus, in a riveter, a short ram of eight inches in diameter may be used. Used with an accumulator and pumps of its own, the pressure is usually 1,400 lbs. per square inch; so that the total force available is 31 tons.

Next we have tools which are **easily portable.** **In Fig. 131** *the ram gets* a motion about its centre, its axis and line of action being an arc of a circle. By this compact arrangement we do away with connecting-rods and many other complications, and we get a great increase in stiffness. Instead, then, of bringing work weighing many tons to a machine, we bring a little machine, weighing 5 cwt., to the work; we can punch holes and finish the riveting; that is, we can finish most of it—for there must always remain rivets in difficult positions which require to be done by hand—with no other intermediate gearing between this little riveter and the steam-engine than a small jointed pipe, the two forms of the universal joint which are used being made water-tight by leather collars. Instead of a jointed pipe, a pipe formed into a spiral forms a good yielding connection. It is, perhaps, in looking at these little riveters, rather than in any other examples, that you will be struck by the simplicity of hydraulic working. The flexible pipe transmits power more faithfully than huge beams and cog-wheels would do.

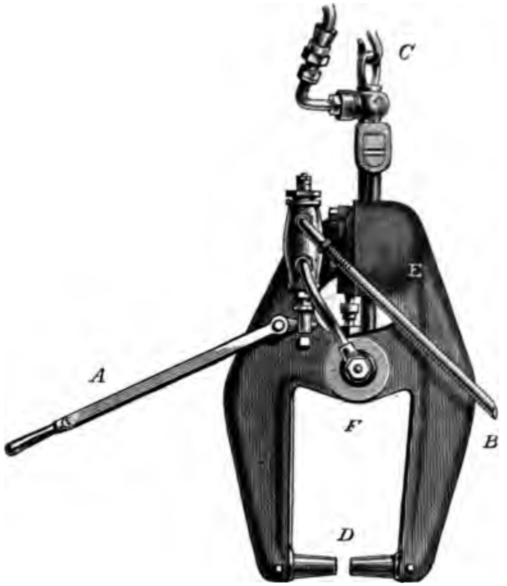

Fig. 131.

161. The machines which I have described are well suited to hydraulic work. A punching, or shearing, or stamping, or riveting machine, driven by shafting, repeats its stroke at regular intervals, and the workman cannot arrest a stroke. He wastes time in waiting for a stroke, but when a stroke is being made, and he sees that his plate has been wrongly placed, or has shifted its position, he must just as patiently watch the inevitable completion of the stroke. With the hydraulic punching-machine, on the contrary, the workman can stop the motion at any instant, even if the punch has made its mark on the surface of the plate. He can start instantly from a condition of rest; he has not to wait till he pulls the belt on to the fast pulley, or starts the donkey-engine, and he knows that when there is no stroke being made there is no power being wasted. Then, again, all the shafting and other machinery of a large shop have not to be set in motion, or varied in their motion, to punch a five-eighth inch hole. A man wastes just the same energy in punching this one hole as if it were one of a hundred

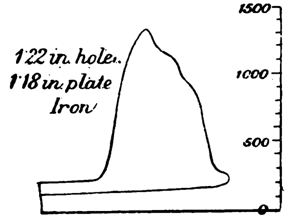

Fig. 132.

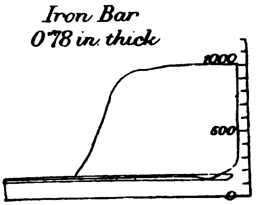

Fig. 133.

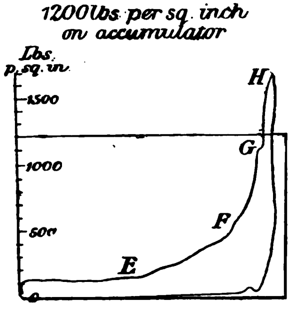

Fig. 134.

he was punching. Suppose, again, that a man carelessly puts a 1¼-inch plate under a punch arranged for a five-eighth inch plate. The sudden blow of an ordinary punching-machine, with its fly-wheel, would produce a fracture somewhere. In a hydraulic machine there is a simple stoppage. Think, too, of the strength of roof and columns needed to carry shafting; of the trouble in the use of overhead cranes when there are many shafts and belts; and, above all, think of the noise, in comparison with the invisibility of pressure mains, and the dead silence of hydraulic tools. Observe, too, that these hydraulic machines need but little foundation.

The diagrams of Figs. 132, 133, and 134 (p. 198), showing the water-pressure in the cylinders of various tools at Toulon during their stroke, are exceedingly interesting. Fig. 132 shows a curve from a punching-machine. The pressure is not equal to that in the accumulator, unless the tool has so much resistance to overcome that it is moving very slowly indeed. The sudden rise shows what occurs when the punch is just beginning to act on the plate. In practice you must understand that this early part of the diagram has no existence; Mr. **Tweddell's** tappet motion prevents such waste. The area of each diagram is roughly the amount of energy utilised. The area of the rectangle in Fig. 134 shows the energy taken from the accumulator. In all these diagrams, then, it must appear to you that there is a large amount of waste. This is greatly reduced by the tappet motion, and in any case it is very much greater in such tools when driven by shafting.

Fig. 133 shows a diagram from a plate-shearing machine. The uniformity of the pressure is due to the angle which the edges of the shears make with one another.

The riveting-machine diagrams are most interesting. Neglecting the useless part of the stroke in Fig. 134, we see the rise of pressure, E F, due to the setting up of the rivet, F H, the clinching of the rivet and closing of the plates. The sudden stoppage of motion of the water gives a blow, shown by the pressure becoming half as much again as the accumulator pressure, and we rely on this blow as perfecting the filling of all cavities.

162. In Fig. 135 two rams, A and *a*, are equal, one **carrying a gun** which is to be quickly lifted and lowered, the other *a* a counterweight. As it is not convenient to vary the counterweight, and as there is really balance only in one position (for either A or *a* getting lower causes the pressure acting on it upwards to get greater), a plan taken is this:—A and *a* have two presses communicating by means of a small pipe; and E, which is greater than R, is partly supported on a small extra ram of area a' in a press of its own, which may either be made to communicate with the other two presses, or with a neighbouring tank where the pressure is small. In the one case, E rests on what is practically a ram of area, $a + a'$, and it is lifted; in the other case on a ram of area *a*, and it falls. The supply of pressure water to a' is effected by means of a pump and accumulator. When pressures are already very great, change of pressure due to rise and fall of a ram is not very important, but in long rams it is often important.

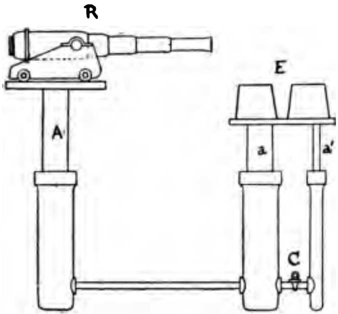

Fig. 135.

163. In most of the machines which we have described, although water usually changes its level during the action, this change of level has been so small as to be negligible. But in nearly all lifting operations we have to consider the work done in the lifting of water as the ram rises.

We all know the conditions required in an ordinary **hotel or chambers hoist**; those conditions are absolutely the same for warehouse hoists, because a hoist which carries goods occasionally carries men in charge of these goods. Long ago, I had some designing and carrying out of **mill hoists**, in which the cage was lifted by a rope passing over an elevated pulley, driven from the main shafting of the mill, and stopped at any point of ascent or descent by automatic disengaging apparatus which also braked the pulley. The cage was balanced by counter-weights, as a window is balanced. Our greatest trouble was in the arrangement of safety apparatus, which would stop the cage in falling should the rope break. Now, it is well known that such safety apparatus can never be thoroughly depended upon, however ingenious its design may be, because the ordinary working of the hoist does not keep the safety apparatus in action; immunity from accidents causes it to be neglected, and when an accident does happen it will not work.

There is nothing so safe as a hoist whose rapid motion is resisted by a considerable amount of friction. But, unfortunately, if the friction is that of solids on one another, there is as much frictional resistance to the ordinary working of the hoist as there is when an accident occurs, and hence assurance of safety by friction means tremendous loss of power at all times.

Now, you remember that the frictional resistance of water was of quite a different kind. There is almost no resistance to the flow of water, if the flow is slow. There is only a moderate loss of power in the ordinary use of a hydraulic hoist; but the motion cannot become too rapid for safety, for the frictional resistance is exceedingly great at high speeds.

Although, therefore, serious accidents cannot be wholly prevented—water may leak away by valves and leave the cage wholly unsupported, for example—yet there is more safety possible with hydraulic than with other hoists.

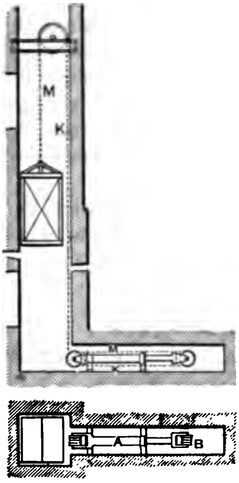

Fig. 136.

164. In a great many hydraulic hoists the action is precisely the same as in Armstrong's cranes. Fig. 136 shows such a construction, used by Armstrong himself. A is a pressure cylinder, with its ram carrying at B the movable block with sheaves, which pull the chain or wire rope, M, K. There is a loss of effect, due to the altering weight of the chain, as the cage rises or falls. This difficulty may be got rid of by letting the ram move vertically, when the altering weight of the ram itself may be made to balance the altering weight of the chain. All such hoists as this can be readily balanced, so that the dead weights may balance at all points in the ascent and descent. They are, however, subject to the risks inseparable from the use of chains or ropes, and must be regarded as unsatisfactory for this reason. That the lifting of every load means the expenditure of the same amount of energy is not a consideration of any importance in these hotel hoists. Of course, there is a slightly greater speed when the load is small, as the water pressure is capable of lifting the heaviest probable loads; but you know enough already about water friction to see that the increase of speed must be insignificant. This condition is the same for all hydraulic hoists hitherto constructed. Very often we use a direct acting hoist. Here the ram moves, pushing the cage up directly. When the pressure of water is very considerable, say 200 lbs. per square inch, and the lift is not too high, this form of hoist is good; for although rather wasteful of power, it is exceedingly simple. The press is sunk so far beneath the basement that there is room for the whole length of the ram when the cage is in its lowest position.

165. It is necessary now to consider the **diminution of lifting force as a ram rises**, and we return to Fig. 135.

Let the total load upon the ram of area A, including its own weight, be called R, and let the total load on the ram of area a, including its own weight, be called E, and for easier calculation assume the ends of the rams to be flat and horizontal. When a is just about to descend, let the end of A be h feet above a. When E falls one foot we found how high R must rise if we could only neglect the weights of the water. But we shall now consider the weight of the water, and take the areas a and A to be in square feet, and the forces R and E to be in pounds.

When a falls a very short distance, x feet, further displacing $a\,x$ cubic feet of water, or 62·3 $a\,x$ lbs. of water, this water is lifted through h feet. Now A will rise $x \times \frac{a}{\mathrm{A}}$ feet, and the work done by E in falling being E x, is equal to the work $\mathrm{R} \times x\,\frac{a}{\mathrm{A}}$ done in lifting R, and also to 62·3 $a\,x\,h$, the work done in lifting the water. Writing this down algebraically, we find—

$$\frac{\mathrm{E}}{a} = \frac{\mathrm{R}}{\mathrm{A}} + 62{\cdot}3\,h,$$

or the pressure at the end of a, which is E ÷ a, is greater than that at A, which is R ÷ A, by the amount 62·3 h lbs. per square *foot*. *We find* this to be an increase of 2,116 lbs. per square foot, or 14·7 lbs. per square inch, or one atmosphere as it is called, for every 34 feet of difference of level.

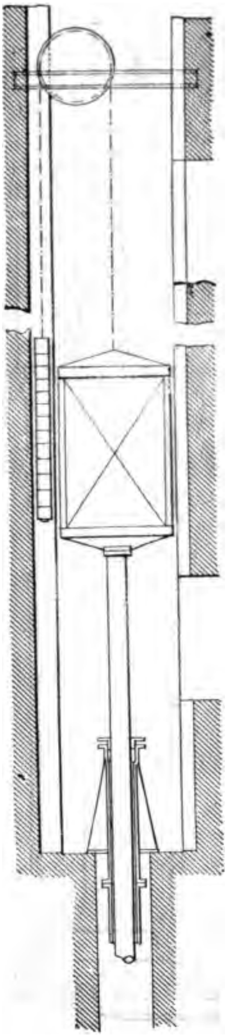

Fig. 137.

When a ram rises 34 feet, if the supply pressure to the press remains the same, the lifting force on the ram diminishes 14·7 lbs. per square inch, and for a 68 feet rise the diminution of pressure would be twice as much. These changes of pressure are not very important when we are dealing with pressures in the press of 200 or 300 lbs. per square inch, but they may be very important at lower press pressures, and in some cases we make the supply pressure increase as the ram rises. Notice that you may look upon the phenomenon in two seemingly different ways. You may either say to yourself, "As a stone is lighter when surrounded by water, so this ram is lighter when it is at the bottom, for more of it is surrounded by water in the press"; or you may put it in this form, "The pressure on the bottom of the ram must **just balance** the weight of ram, cage, etc., but as the bottom of the ram rises, this means that we ought to have a constant pressure at the bottom of the ram wherever it may be, and consequently a gradually increasing pressure in the cylinder everywhere as the ram rises." Now, I do not care which of these two views you take, but you must not mix them, and say that "not only does the ram get heavier, but it needs a greater pressure at its lower end as its lower end rises." I prefer always to say, "The ram appears to get heavier just in proportion to the amount it has been raised, and this must be balanced by increasing the pressure of the supply water."

Now, remember that our supply water in the direct-acting hoist is at a constant pressure, and you will see that it is quite impossible, with such a simple arrangement, to have perfect uniformity of action, although it is approximated to more and more nearly as the pressure is greater. In this kind of hoist it is usual to let the water escape from the cylinder to a discharge cistern considerably above the cylinder, so that in its descent the ram and cage may not fall too rapidly. Here, again, we have the same want of uniformity of action, since the apparent weight of the ram gets less as it falls.

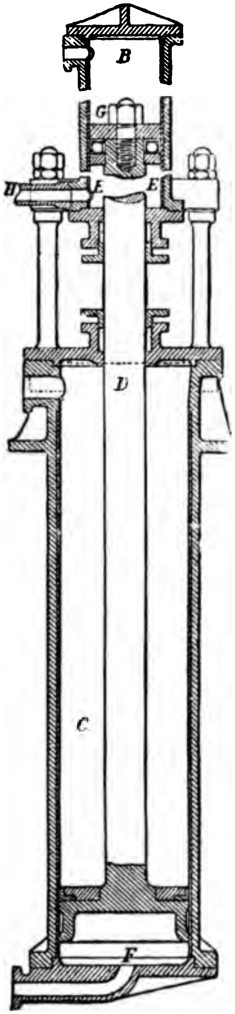

Fig. 138.

166. The usual practice has been to nearly **balance the dead-weight** of ram and cage by a weight, as in Fig. 137 (p. 203), so as not to require too high a lift in the discharge-pipe, and to so arrange that the varying weight of chain shall just balance the apparent change of weight of the ram. It is evident that if the ram rises one foot, the counterweight increases by the weight of two feet of chain; hence, the weight of two feet of the chain ought to be equal to that of water occupying the volume of one foot length of the ram. Unfortunately, these chains and counterweights destroy the simplicity and absolute safety of the hydraulic hoist. If the ram were to break near its upper end, the cage would be drawn violently upward by the chain. The upper part of the ram is in tension, and the lower part in compression.

It is obvious, then, that there must, for a complete and perfect hydraulic lift, be such a regulation of the pressure of the water as it enters the cylinder of a hoist that the only force to be overcome shall be the variable weight placed in the cage, whether that of passengers or goods, together with the necessary friction. The hydraulic balance hoist satisfies this condition. It can be worked with either high or low pressure water; the ram is always in compression, supporting the load, and no part of the machinery is above the cage, and there is no part of the machinery likely to break in such a way as to cause an accident.

This **hydraulic balance lift** is shown in Fig. 138 (p. 204). The hydraulic cylinder, ram, and cage are as usually made, except that the ram is somewhat smaller in diameter. Its size is determined by the strength required to carry the load, and not by the working pressure of water available. The lift cylinder is in hydraulic connection with a second and shorter cylinder, E, below which there is a cylinder, C, of larger dimensions. There is a piston in each, connected by the rod. The capacity of the annular space E below the upper piston is equal to the displacement of the lift ram. The annular area of the lower piston is sufficient, when subject to the working pressure of the supply water, to overcome friction and lift the net load; and the full area of the upper piston is sufficient, when subjected to the same pressure, to balance within a small amount the unalterable weight of the ram and cage.

Assuming the cage at the bottom of its stroke, the valve is opened by a man in the cage pulling on a rope, by a system of levers, and pressure water is admitted. The pressures on the two pistons cause them to descend, forcing water from the annular space to the hoist cylinder. The hoist ram ascends, and in doing so gets heavier, but the pistons are descending, and the total pressure on them is getting greater just in the same proportion. When the ram reaches the top of its stroke the valve is closed and the lift stops. Now open the exhaust valve, which lets the water pass away from C—only from C, remember—and the weight of the ram and cage presses the water from the lift press into E, causing the pistons to rise.

To make good any possible leakage, provision is made for admitting pressure water under F, and so raising it, the lift ram being at the bottom of its stroke, that water will flow into the space.

167. We see that the hydraulic hoist has—(1) The great element of safety from the absence of possible breakage of chains or ropes. But in *some forms it* is not without a danger of its own—namely, the danger that when the cage is remaining caught in a fixed elevated position for a time, the cylinder may be emptying of water through a leakage of the valves. (2) That the expenditure of energy depends very little upon the dead load. But there is still the drawback that every load, however small, requires the same expenditure of energy as the greatest load which the hoist can lift. This drawback is common to all hydraulic hoists such as I have been describing.

168. You can now have no difficulty in understanding the construction of all warehouse and hotel lifts. The hydraulic principles involved in all lifts are the same, but with larger weights to be raised there are peculiarities of construction which ought to be studied from actual drawings of such lifts, as, for example, the pair of lifts at the Seacombe Pier, on the Mersey, to take carts and waggons from the **floating landing-stage** to the high level. The height of lift is 32 feet, and the net load 20 tons. Here we have simply direct acting rams and presses sunk in the river-bed, the presses being surrounded by cast-iron protecting cylinders. In this case there is no attempt to balance the increasing weight—that is, the displacement of the ram as it rises. Each cage, or platform, is supported on one ram, and the designer has to take care that the weight of any waggon shall be so carried on the top of the ram that no part of the structure is unduly strained. The waggon rests on a platform, which can slide on the bottom of the cage, so that although the cage rises vertically, the platform may everywhere be in the position in which it would be on the landing-stage, if the landing-stage rose and fell. The landing-stage is attached, as we know, by girders, which do not alter in length, so that it does not rise and fall vertically, but really in arcs of circles. In this case, the lift is of a variable amount, depending on the height of the tide. There is a connecting valve between the two presses, so that a descending load in one lift may raise a less load in the other when necessary.

169. This idea of having two lifts side by side, so that the lowering of one may cause the rise of the other, you will understand better in the original example of its use, the canal lift of **Messrs. Clark and Standfield**, on the **River Weaver**, in Cheshire. Figs. 139 and 140 show the **canal** and the river, one 50 feet above the other. We want to raise or lower canal boats from the one to the other, and we want to avoid the expenditure of water, and the delay which occurs when there is a chain of locks.

There are two great wrought-iron troughs, each 75 feet long and 15½ feet wide, each being carried on the top of a ram 3 feet in diameter. Now, when I tell you this, you will at once see how incomplete my description is. Each tremendous trough

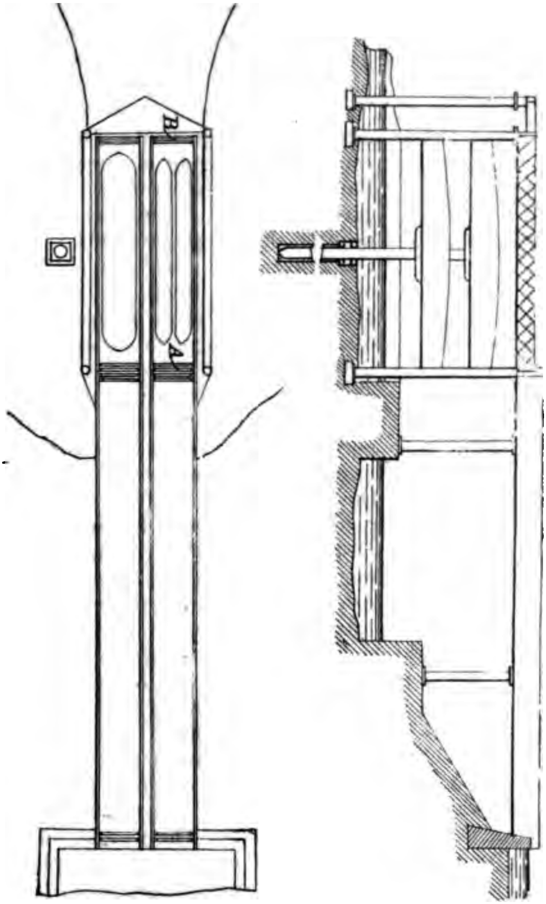

Fig. 139.

is carried easily by one ram acting at its centre. You can, in your imagination, go into all the details of girder-work necessary for the safe carrying of such a load in such a manner. The weight of each trough, with the water and barges in it, is 240 tons, and this gives a pressure of about one quarter of a ton per square inch in the press. At each end of each trough there is a gate.

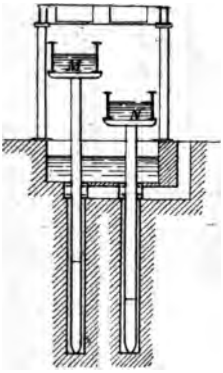

Fig. 140.

When the trough is up, and gate A is lifted, the trough forms part of the aqueduct. A barge floats into it from the canal, and the gate is closed again; also the aqueduct end is itself closed with a gate. Now the trough is lowered containing the barge, and when it reaches the lower level, gate B is lifted, when, of course, the trough really forms part of the river. You must describe to yourselves how this press is sunk—how we have a tunnel which enables us to examine the packing of the cylinders, how these great columns are firmly supported, so that we may have guides to prevent the tilting of the troughs. All this is constructive detail which you can read about in the Proceedings of the Institution of Civil Engineers. You might fear also that the joints of the gates at the end of the trough might be leaky, and, above all, that the joint of the trough with the aqueduct end might be leaky; but even from the figure you can see how perfectly these difficulties can be got over.

Suppose I have a boat in any vessel of water, and I know that the water is at a certain level, and I take out this boat and put in another boat, and add water or take it away until the level is just what it was before. You know that the weight of boat and water is always the same. The total weight simply depends on the level of the water. The weight of the boat alone is always equal to that of the water it displaces, and, therefore, the trough filled to a certain height with water, if there is no boat, will just weigh the same as if there is a boat in it and the water is at the same level as before.

Now, suppose that the trough M is part of the canal, and *there is or is* not a barge in it, and suppose that N is down, and is in communication with the river, and there is or is not a barge in it. Now close the gates. Suppose there is five feet of water in M, and that, if there is not four feet six inches of water in N, we let water into or out of it from the river till the water is at this level. Now let the valve be opened; water will flow from one press to the other, for M is heavier than N, and M will fall, causing N to rise. Suppose the lifts were very high indeed, it is evident that this falling will not stop till M is well below the level of N; in fact, till the lightening of the ram M, as it displaces more water, and the increased weight of ram N as it leaves the water in its press—till these two added together are equal to the real difference in the loads of M and N. In the present case, the lift is not great enough for such an effect to occur. But as soon as M enters the river it gets very much lighter in weight. Indeed, as soon as it sinks six inches into the river, the weights of M and N are equal, so that leaving out of account the displacements of the rams, N cannot be raised any further by the falling of M. There is exact balance, and no further motion, then, when N and M have still more than four feet six inches further to travel. The valve is now closed; there is no further communication between the presses. The press of N is put in communication with an accumulator, which lifts N, pressing it home at its water-tight joint with the aqueduct. The water in N is six inches lower in level than it ought to be, however. Water is allowed to pass from the aqueduct through a valve into the space between the two gates of the trough and of the aqueduct, when the latter are easily lifted; the former are also easily lifted now, and water passes freely into N, as it is now part of the aqueduct, and the barges can be taken off into the canal. Meanwhile we have left M still resting in great part on its ram. The water from its press is now allowed to escape, and M sinks in the river, its river-gate is lifted, and the barge is taken out.

Twelve siphons in M were put in working order during its immersion, and when it is being lifted again they empty it down to the level of their free ends, which are adjusted to leave exactly four feet six inches of water in M. Thus there is an automatic adjustment of the water-levels in M and N, so that whatever be the weights of the floating barges they contain, the total weight is always the same for every operation.

The waste in an operation comprises, first, six inches depth of water in one of the troughs. The falling of this 50 feet is 1,800,000 foot-pounds. Secondly, the accumulator raises N four feet six inches, and as the weight of N is about 240 tons, the waste here is 2,419,200 foot-pounds. The total waste is, then, about 4,219,200 foot-pounds.

Now, suppose the same operation were performed down a flight of ordinary canal locks, under favourable circumstances, they would require that 14½ feet depth of water of the area of one lock should fall 50 feet, or there would be an expenditure of 51,500,000 foot-pounds, or twelve times as much as with the lift. If, however, a canal has a plentiful supply of water, this is not of such great importance as it seems. It becomes of great importance when, as in the present case, there is but a small supplyof water. The advantage of such a system as the present is rather, in my opinion, in the fact that the operation is finished in eight minutes, whereas in a similar ascent or descent, but by means of a flight of locks, at Runcorn, the operation of letting one barge through requires one hour and a half.

170. We are, however, more interested, just now, in the hydraulic question, the saving of energy, the saving of water from the canal, and accumulator energy, and so we may consider a somewhat similar canal lift which has been constructed at Fontinette, Belgium, to replace a flight of five locks, with a total fall of 43 feet. The troughs are of double the length, and are of greater depth than the last. Each ram is six feet six inches in diameter, and there is the improvement that we have what are called compensating reservoirs. Water flows from one of them to the descending trough, thus increasing the weight of it, just in proportion as its ram becomes immersed in its press; and water flows back again from the ascending trough to the reservoir, just in proportion as its ram comes out of its press. Thus the ram displacement is balanced. But there is a further improvement: the descending trough does not descend into water, for this made it get light too soon; it descends into a dry chamber, and only becomes a portion of the lower canal when the gates are lifted. Thus the falling of one trough can lift the other all the way to the upper level, and the accumulator is only needed to supply leakage from the presses. A single operation causes a loss of 20 tons of water from the upper level, or less than 2,000,000 foot-pounds of energy altogether, and yet the troughs have more than twice the *capacity of those* at Anderton.

171. We have noticed here one of the methods adopted for balancing the change due to displacement of rams. But many other methods have been adopted to suit special cases, in which the falling of an accumulator ram causes the rise of another ram. Thus, for example, we let a ram rise vertically above the accumulator, passing water-tight into a tank of water. Its increased apparent weight (it is always covered), as the accumulator ram descends, compensates for the displacement effects of all the other rams in the arrangement. Again, we may have a tank of water as part of the load of the accumulator, and its water-level is kept the same as that of a neighbouring fixed tank, by means of a siphon, so that the weight increases as the accumulator ram sinks. By properly shaping this tank we are able to get any variation of pressure that is wanted during an operation, for such purposes as cotton-pressing, etc. It is easy to see that we have here a means of balancing accurately the dead weight of any platform and its ram, which has to be raised and lowered, at every point during an operation, by the use of an ordinary accumulator.

172. Another very important improvement in lifts is this. Suppose that a bridge has to be lifted by rams at its ends. Suppose the presses of these rams are connected with an accumulator, they are not likely to rise equally fast, and the bridge would be tilted; indeed, one of them may not rise at all, and the other may be rising quickly. Again, suppose we have a separate accumulator for each separate press, so that there is no water communication between them, great care has to be taken at the valves to make the ends of the bridge rise equally fast. This difficulty is got over by having two presses and two rams on the accumulator. Now, let each accumulator press be connected with each bridge press. The frame carrying the accumulator weight is guided so that it cannot tilt; each equal ram, therefore, falls the same amount as the other, and the equal bridge rams must rise one as much as the other.

When a large structure has to be lifted, the lifting presses which get this synchronous action are so distributed in supporting the structure that there is no tilting of it possible.

Fig. 141 shows an **hydraulic grid.** The strong girder, with its projecting ribs, rests on the ends of a number of rams, whose presses are sunk in the bed of the dock or tidal river. The keel of a vessel is brought directly over it, and secured in position by the bilge blocks, and side-shoring frames; the presses are worked, and the rams lift the grid and the vessel above the level of high water. Now a number of struts, which were previously held up horizontally, are liberated, and hang from the grid alongside the rams; on lowering the rams, the lower ends of these struts fit into the tops of the presses, forming a support for the grid, and the rams are withdrawn into their presses.

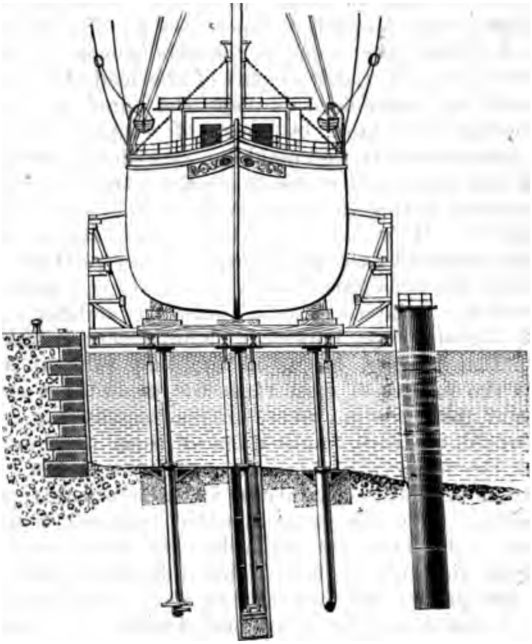

Fig. 141.

There are only a few guide columns needed for the grid to slide against as it rises and falls, because the presses are arranged in three equal groups on the above-mentioned principle, supplemented by an automatic safety valve, which lets the water escape from a press when its ram has risen more than the others. Hence the grid must, when rising and falling, *remain perfectly* level.

EXERCISES.

1. Hydraulic jack; velocity ratio of lever, 30; ram, 2¼ inches diameter; pump-plunger, ¾ inch diameter. If it is experimentally found that, E being effort on handle, W the weight lifted, there is a straight-line law connecting E and W; and if, when $W = 1{,}605$ lbs., $E = 10$ lbs.: when $W = 6{,}805$ lbs., $E = 50$ lbs.; show that $W = 305 + 130\,E$. If when $W = 7{,}000$ lbs. the pressure of the water is found by a pressure-gauge to be 1,932 lbs. per square inch, what is the loss by friction at the leather? Assuming the same percentage loss at the two leathers, what is the law connecting E and the force P with which the lever acts on the plunger?

Ans., 9·1 per cent; $P = 41 + 17.5\,E$.

2. A steel hydraulic press 13 inches internal diameter is 3 inches thick. What is the greatest tensile stress when there is a fluid pressure of 2¾ tons per square inch? *Ans.*, 7·6 tons per square inch.

If a steel pipe is 1 inch in diameter inside, and the greatest tensile stress is to be 5 tons per square inch when there is a fluid pressure of 3 tons per square inch, what is the thickness of the metal? *Ans.*, ½ inch.

3. In an accumulator the average pressure is to be 700 lbs. per square inch; ram, 12 inches diameter. What is the necessary weight? If the ram rises 10 feet, what energy is stored up? Neglect change of pressure due to lifting. If the pressure is found to fluctuate between 720 and 680 lbs. per square inch between slow lifting and slow falling of the weight, what is the force of friction? If the pressures are settled for a position half-way up, what is the real fluctuation, taking friction and change of level into account?

Ans., 78,400 lbs.; 784,000 ft.-lbs.; 1 ton; 2,484 lbs.

4. In a three-cylinder single-acting pressure engine like that shown in Fig. 128 each piston is 4 inches diameter and 3 inch stroke; the average acting pressure is 700 lbs. per square inch. What is the indicated horse-power at 50 revolutions per minute, assuming an average back-pressure due to friction, etc., of 200 lbs. per square inch? The coil of a rope on a drum on the crank-shaft is 18 inches diameter; what is the average pull in the rope if the brake horse-power is 0·7 of the indicated? If the pull is to be only 300 lbs., what ought the stroke to be altered to?

Ans., 7·14; 700 lbs. 1·3 inch.

5. The area of the piston of a hydraulic crane is 90 square inches on one side and 40 on the other; it pushes a three-sheave pulley-block. If the water-pressure is 700 lbs. per square inch, what weights can be lifted —(1) when pressure-water is admitted on one side only, (2) when admitted on both sides. Take the efficiency of the hydraulic parts as 0·9, and of the pulley and crane parts as 0·4. If the full loads in the two kinds of working are being lifted, what work is done per cubic foot per pound, and per gallon of water?

Ans., 1,680 lbs.; 3,780 lbs.; 36,290 ft.-lbs.; 5,826 ft.-lbs.

6. A hydraulic punch has a ram 8 inches diameter; ⅝-inch holes are being punched, each requiring a force of 70,000 lbs. What is the water-pressure? This water comes from a steam intensifier; the area on which the steam acts is 300 square inches; the area of the ram is 30 square inches. What is the pressure difference of the steam, neglecting friction?

Ans., 1,393 lbs. per square inch; 139·3 lbs. per square inch.

7. In a pipe from which a press is supplied, the pressure is in one case 400 lbs. per square inch and in another it is 100 lbs. per square inch; the

end of the ram of a hoist is 70 feet below the level of the pipe, and gradually rises to be 10 feet above the pipe. What is the change of pressure? If the rams are 3 and 6 inches in diameter respectively, what are the lifting forces at the bottom and top? What are the fractional changes in the two cases? If chains go from the cages vertically over a pulley to a counter-weight, what ought to be the weight of the two chains per foot of their length?

Ans., 34·8 lbs. per square inch; 3,043 lbs.; 2,797 lbs.; 3,687 lbs.; 2,703 lbs.; 8 per cent., 28 per cent., 3·06 lbs., 12·23 lbs.

8. Show that if a tank has water in it to the same level, whether there are floating objects in it or not, its total weight is the same. If the weights of two tanks (with rams) are 160 and 144 tons, and they rest on rams A and B, each 3 feet diameter, what are the pressures at the bottoms of the rams? If the presses communicate with one another, what is the difference of levels of the bottoms of the rams when there is balance? Disconnect the presses; connect the press B with an accumulator whose ram is 21 inches diameter, its end being now 30 feet below the end of B. What is the load needed for the accumulator if B is to be lifted another 5 feet?

9. The areas of vertical cross-sections of the immersed part of a ship at intervals of 10 feet are A square feet, given in the following table. Find approximately the weight w_1 of water displaced between every two neighbour sections. The weight of the portions of the ship bounded by the same sections are w_2; what are the resultant loads $w_2 - w_1$ acting downward on the ship between every two sections? Draw a curve showing these values.

A	0		30		75		100		100		100		98		90		78		40		0
w_1		9,300		33,000		55,000		62,000		62,000		61,700		59,000		52,000		37,000		12,000	
w_2		23,000		37,000		47,000		54,000		57,000		56,000		53,000		47,000		39,000		30 000	

10. Fifty rams, each of 14 inches diameter, begin to lift a gridiron from the bottom of a harbour, and so lift a ship which was floating directly above. The plane areas in square feet bounded by the water-line of the vessel, after the following numbers of feet of lift, are shown as the numbers A. If the total apparent weight of the gridiron is taken to be constant at 860 tons, find the pressure in the presses for every position, and compare with the answers given in the table.

Lift in feet.	0	1	3	6	9	12	15	18	21	24	27	30
A	4,050	4,090	4,120	4,100	3,970	3,400	3,000	2,200	1,400	600	200	0
p	250	283	350	450	547	637	717	780	824	848	858	800

173. Force due to Pressure of Fluids.

Exercise 1.—Prove that if p, the pressure of a fluid, is constant, the resultant of all the pressure forces on the plane area A is Ap, and acts through the centre of the area.

2. The pressure in a liquid at the depth h being wh, where w is the weight of unit volume, what is the total force due to pressure on any immersed plane area? Let D E (Fig. 142) be the surface from which the depth h is measured, and where the pressure is 0. Let B C

be an edge view of the area; imagine its plane produced to cut the level surface of the liquid C in D. Let the angle EDC be called α; let the distance DP be called x, and let DQ be called $x + \delta x$: and let the breadth of the area at right angles to the paper at P be called z. On the strip of area $z \, . \, \delta x$ there is the pressure wh if h is PH, the depth of P, and $h = x \sin. \alpha$; so that the pressure force on the strip is $wx \, . \sin. \alpha \, . \, z \, . \, \delta x$, and the whole force is

$$F = w \sin. \alpha \int_{DB}^{DC} x \, . \, z \, . \, dx \, . \, . \, . \, . \, (1).$$

Also, if this resultant acts at a point in the area at a distance X from D, taking moments about D,

$$F X = w \sin. \alpha \int_{DB}^{DC} x^2 \, . \, z \, . \, dx \, . \, . \, . \, . \, (2).$$

Observe in (1) that $\int_{DB}^{DC} x \, . \, z \, . \, dx = A\bar{x}$, if A is the whole area and $\bar{x}$ is the distance of its centre of gravity from D. Hence the **average pressure over the area is the pressure at the centre of gravity of the area.**

Observe in (2) that $\int_{DB}^{DC} x^2 \, . \, z \, . \, dx = I$, the moment of inertia of the area about D. Letting $I = k^2 A$, where k is called the radius of gyration of the area about D, we see that

$$F = w \sin. \alpha \, . \, A \bar{x}, \qquad F X = w \sin. \alpha \, . \, A k^2.$$

Hence $X = k^2/\bar{x} \, . \, . \, . \, . \, (3)$, **the distance from D at which the resultant force acts.**

Example.—If DB = 0 and the area is rectangular, of constant breadth b, then

$$I = b \int_0^{DC} x^2 \, . \, dx = \frac{b}{3} DC^3,$$

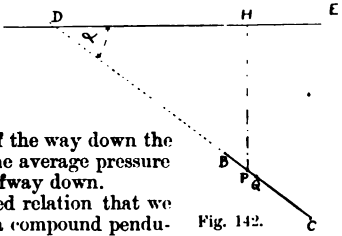

Fig. 142.

and $A = b \, . \, DC$; so that $k^2 = \frac{1}{3} DC^2$, also $\bar{x} = \frac{1}{2} DC$. Hence $X = \frac{2}{3} DC$; that is, the resultant force acts at $\frac{2}{3}$ of the way down the rectangle from D to C, and the average pressure is the pressure at a point halfway down.

It is an easily-remembered relation that we find in (3). For if we have a compound pendulum whose radius of gyration is k, and if $\bar{x}$ is the distance from the point of support to its centre of gravity, and if X is the distance to its point of percussion, we have the very same equation (3). Again, if X is the length of the simple pendulum, which oscillates in exactly the same time as the compound one, we have again this same relation (3). These are merely mathematical helps to the memory, for the three physical phenomena have no other relation to one another than a mathematical one.

EXERCISES.

1. Find the whole pressure of water upon a vertical dam 10 feet deep, 30 feet wide. What would be the pressure of water of the same depth against a dam inclined at 45°? *Ans.*, 93,750 and 132,200 lbs.

2. A water-tank is 13 feet square and 4 feet 6 inches deep. Find the pressure upon one of the sides when the tank is full. *Ans.*, 8,226·56 lbs.

3. In the vertical plane side of a tank which holds water there is a rectangular plate whose depth is 1 foot and breadth 2 feet, the upper edge being horizontal, and 8 feet below the surface of the water. Find the pressure on the plate. *Ans.*, 1,062·5 lbs.

4. A rectangular tank, 5 feet square, is filled with water to a height of 7¾ feet. A rectangular block of wood weighing 312·5 lbs., and having a cross-section of 5 square feet, is placed in the tank, where it floats in the water with this section horizontal. How much does the water rise in the tank, and what is the increase in pressure on a vertical side of the tank? *Ans.*, 2⅖ inches, 489 lbs.

5. A sluice-gate is 6 feet broad and 8 feet deep, and the water rises 3 feet on one side above the lower edge, and 7 feet on the other side. Find the resultant pressure and centre of pressure for each side of the gate. What is the resultant of these and its position?

Ans., 1,682 lbs., 9,158 lbs.; 2 ft. and 4⅔ ft. below surface; 7,476 lbs., 4·36 ft. from surface.

6. A vertical wall, 10 feet high and 2 feet thick, weighing 153 lbs. per cubic foot, has to support the pressure of water (weighing 62·3 lbs. per cubic foot). How high may the water rise against one side of the wall without causing the resultant force at the base to pass outside the outer edge? *Ans.*, 5·65 feet.

7. A rectangular opening A B C D is made in the outer vertical face of a reservoir embankment and closed by a door, hinged along the lower horizontal edge A B. Find the pressure to be applied at the upper edge C D of the door, in order to shut against the pressure of the water. Breadth of door, 3 feet; depth, 4 feet; depth of upper edge below surface, 8 feet. *Ans.*, 3,489 lbs.

8. In a hydraulic jack the ram is 5 inches, and the pump plunger ¾ inch diameter; the leverage for working pump, 10 to 1; what is the velocity ratio? Experimentally we find that a force of 20 lbs. lifts 8,200 lbs.; what is the efficiency? *Ans.*, $444\frac{4}{9}$; 92¼ per cent.

9. Water at 750 lbs. per square inch acts on a piston 8 inches diameter; the velocity is multiplied by 8 by a four-sheaved pulley block; what weight may be lifted by the chain if the efficiency of the mechanism is 55 per cent.? In a lift of 40 feet, how many gallons of water are used? *Ans.*, 2,592 lbs.; 87.

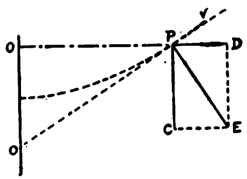

Fig.

174. Whirling Fluids.—If fluid rotates like a rigid body about the vertical axis O O (which is in the plane of the paper) with the angular velocity a radians per second, at a point P let there be a particle of weight w lbs.; let O P $= x$ feet. The centrifugal force is $\frac{w}{g}a^2x$ in the direction P D; the weight is w in the vertical direction P C; and the resultant force is in the direction P E of the amount $w\sqrt{1 + a^4x^2/g^2}$, the tangent of the angle D P E being $w \div \frac{w}{g} a^2x$ or g/a^2x.

Imagine many curves drawn such that their direction at any point represents the direction of the resultant force there. Such curves are

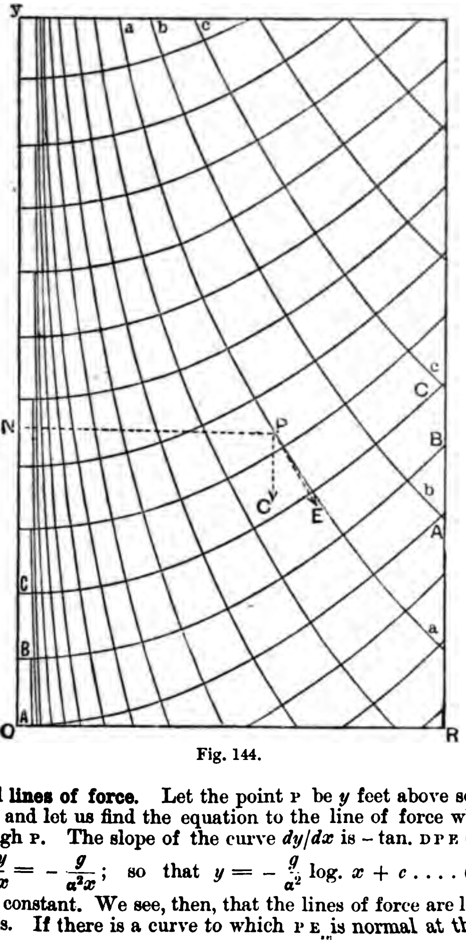

Fig. 144.

called **lines of force.** Let the point P be y feet above some datum level, and let us find the equation to the line of force which passes through P. The slope of the curve dy/dx is $-$ tan. D P E (Fig. 143), or $\frac{dy}{dx} = -\frac{g}{a^2x}$; so that $y = -\frac{g}{a^2} \log. x + c \ldots . (1)$, where c is a constant. We see, then, that the lines of force are logarithmic curves. If there is a curve to which P E is normal at the point P, its slope is positive, being tan. V P D, or $\frac{dy}{dx} = \frac{a^2}{g} x$; so that the equation to the curve is $y = \frac{a^2}{2g} x^2 + c$, where c is a constant, depending upon the datum level from which y is measured. This is a parabola, and if it revolves about the axis O O we have a paraboloid of revolution. Any surface which is everywhere at right angles to the force at every point is called a level surface, and we see that the level surfaces in this case are paraboloids of revolution. These level surfaces are sometimes called **equi-potential surfaces**. It is easy to prove that the pressure is constant everywhere in such a surface, and that it is a surface of equal density; so that if mercury, water, oils, and air are in a whirling vessel, their surfaces of separation are **paraboloids of revolution**.

The student ought to draw one of the lines of force and **cut out a template** of it in thin zinc, O O being another edge. By sliding along O O he can draw many lines of force. Now let him cut out a template for one of the parabolas, and with it draw many level surfaces. The two sets of curves cut each other everywhere orthogonally. Fig. 144 shows the sort of result obtainable where *a a*, *b b*, *c c* are the **logarithmic lines of force**, and A A, B B, C C are the level paraboloidal surfaces.

175. The engineer seldom deals with other volumetric forces than those due to gravity and centrifugal force. By dealing in this way with centrifugal force he is able to treat a rotational problem as a statical problem. Whatever be the system of volumetric forces, we are supposed to know the lines of force in the fluid and a series of level or equi-potential surfaces cutting these lines at right angles. In Fig. 145, if A B C and A′ B′ C′ be lines of force, and B B and C C be two level surfaces; if F is the force on the fluid at O per unit volume and B C $= \delta s$; if the pressure at B B′ is p and at C C′ it is $p + \delta p$, consider the equilibrium of the prism whose end of small area a is at B B′ and other end of equal area is at C C′ (we take the ends of equal area because we afterwards assume a and δs to be smaller and smaller without limit). Evidently, as the volume is $a . \delta s$, the volumetric force is $F . a . \delta s$, and we have

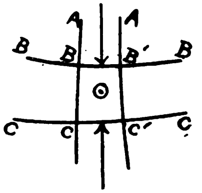

Fig. 145.

$$p a + F a . \delta s = (p + \delta p) a \; . \; . \; . \; . \; (1).$$

Hence $\quad F . \delta s = \delta p, \quad$ or $\quad F = dp/ds \; . \; . \; . \; . \; (2).$

Example 1.—If gravity alone acts and δs is called δh, so that h is measured vertically *downwards* from A towards C, and we take $F = w$, the weight of a cubic foot of fluid, to be constant, being 62·3 lbs. per cubic foot for water, then $\frac{dp}{dh} = w \; . \; . \; . \; . \; (3)$, or $p = wh + c$ where c is a constant. If we measure h from a level where we take the pressure to be p_0, then $c = p_0$ and $p - p_0 = wh$.

Example 2.—Take w to be variable, say $w \quad cp$, a rule that holds for gases at constant temperature, then (3) becomes $\frac{dp}{dh} = cp$, or $\frac{dp}{p} = c \,.\, dh$, or log. $p = ch + \text{c}$ where c is a constant.

Let $p = p_0$ where $h = o$, then $\text{c} = \log.\, p_0$, and we have log. $p/p_0 = ch \ldots.$ (4). The actual value of c depends upon the constant temperature supposed to be maintained.

Example 3.—Take w to be variable, say $w = cp^{1/\gamma}$ where $\gamma = 1{\cdot}414$ for air, being the ratio of the specific heats. This is the law which is much more likely to hold in a mass of gas than the constant temperature law.

Then $\frac{dp}{dh} = cp^{1/\gamma}$, or $p^{-1/\gamma} \,.\, dp = c \,.\, dh$, or $p^{1\,-\,1/\gamma}/(1 - 1/\gamma) = ch + \text{c}$. It is easy to show that this leads to the result that the temperature increases in proportion to the increase of depth in the atmosphere.

Example 4.—In our whirling fluid it is easy, since $\text{F} = w\sqrt{1 + a^4x^2/g^2}$ and $\frac{ds}{dx} \quad \sqrt{1 + g^2/a^4x^2}$, to find from (2. the law of y, if one knows how to integrate. Take the simpler case, in which x is so great that the lines of force may be regarded as horizontal, and the level surfaces vertical circular cylinders. Then letting δs be called δx, x being the radius of the path in which a particle revolves,

$$\text{F} = \frac{w}{g}a^2x, \text{ and } \frac{w}{g}a^2x \,.\, dx \quad dp.$$

(1) If w is constant, $\frac{wa^2}{2g}x^2 = p + \text{c}$ where c is some constant.

Let $p \quad p_0$ where $x = x_0$, and we have $p - p_0 = \frac{wa^2}{2g}(x^2 - x_0^2) \ldots.$

(2) If $w \quad cp$, then $\frac{a^2x^2}{2g} = \frac{1}{c}\int p^{-s}\, dp$, or $\frac{a^2x^2}{2g} = \frac{p^{1-s}}{c\,(1-s)} + \text{c}$,

or $\frac{c\,(1 - s)\, a^2\,(x^2 - x_0^2)}{2g} \quad p^{1-s} - p_0^{1-s}$.

If we take $s = \frac{1}{\gamma}$ where $\gamma = 1{\cdot}414$ for air, and the easily found value of c for any given conditions, we can find the increase of pressure due to centrifugal force in a mass of whirling gas, as in the wheel of a fan when it is not delivering much fluid.

CHAPTER X.

MACHINERY IN GENERAL.

176. Mechanism.—When the power of a steam-engine is distributed through a factory, the distribution is performed by means of shafts, spur and bevil wheels, belts and pulleys, and other kinds of gearing. As we are writing for men who have observed such transmission of energy, it is no part of our object to describe here what can be seen in any workshop. Perhaps no study is more useless from books alone than the study of mechanism; whereas it is very useful and easy if we examine the actual thing, or make a skeleton model or a skeleton drawing.

At the same time it is necessary to read books. The present book deals with the **kinetics of mechanism**; but there is another part, a preliminary part, and students must read some book giving descriptions of contrivances and the mere kinematics of the subject. We give a short sketch of certain important principles here, and in Chap. XXVI., because they are necessary for the proper understanding of our own division of the subject.

177. Velocity Ratio.—In any machinery the velocity of a point may be calculated when the velocity of any other point is known. The number of revolutions per minute made by a shaft tells us the velocity of any point on any wheel or pulley fixed on the shaft; the circumference of the circle described by such a point, multiplied by the number of revolutions per minute, is evidently the distance moved through by the point in one minute. Now, when one shaft drives another by means of spur or bevil wheels, or by two pulleys and a strap, it is evident that the number of revolutions per minute made by one of the shafts, multiplied by the number of teeth of the wheel, or by the circumference or diameter of the wheel or pulley, is equal to the number of revolutions per minute made by the other shaft, multiplied by the number of teeth, or by the circumference or diameter of the other wheel or pulley. This is evidently true, supposing that the strap does not slip on the pulley. Hence the rule—to find the speed of a shaft, driven from another by means of any number of wheels and pulleys, *multiply the speed of the driving shaft by the product of the* *diameters or numbers of teeth in all the driving wheels or pulleys, and divide by the product of the diameters or numbers of teeth in all the driven wheels or pulleys.* By the diameter of a spur wheel we mean the diameter of its *pitch circle.* Two spur wheels enter some distance into one another, and the circle on one which touches the circle on the other (the diameters of these circles being proportional to the numbers of teeth on the wheels), is called the pitch circle. The circumference of the pitch circle, divided by the number of teeth, gives the *pitch* of the teeth.

178. Shapes of Wheel Teeth.—We know that if two spur wheels gear together, however badly their teeth are formed, so long as a tooth in one drives past the line of centres of a tooth in the other, their average speeds follow the above rule. But if we want the speed ratio at any instant to be the same as at any other instant, it is necessary to form the teeth in a certain way. *The curved sides of teeth ought to be cycloidal curves.* The proof of this is not very difficult; it is given in Art. 462. It is not usual to employ these cycloidal curves, for it is found that certain arcs of circles approximate very closely to the proper curves. The method of drawing rapidly the curved tooth of a wheel you will find taught by every teacher of mechanical drawing, you will find described in a great number of books, and you will see it in use in the workshop.* You must remember that no study of books, and I may also say, no fitter's or turner's work that you may engage in, will make up for want of the experience which you would gain by actually drawing to scale a spur or bevil wheel, a bracket or pedestal with brasses, and a few other contrivances used in machinery. **A worm and worm-wheel,** that is, a screw, every revolution of which causes one tooth of a wheel to be driven forward, is sometimes used when we wish to drive a shaft with a very slow speed. If the worm-wheel has 30 teeth, it evidently makes one-thirtieth of the number of revolutions of the driving shaft.

179. Skeleton Drawings.—When we consider the relative motions of, say, a piston and the crank which it drives, we come to something which it is not so easy to state without some little knowledge of mathematics. It is the same with all sorts of combinations of link work, and with cams. Even a good knowledge of mathematics is only sufficient to give one a rough general idea of the relative motion in such cases; and for the study of any special case there is nothing so good as a skeleton drawing or a model. I give one example of the use of skeleton drawings—**a crank and connecting-rod.** Let A and B (Fig. 146) be the ends of a connecting-rod. As A moves from a_1 to c and back again, B describes the complete circle, b_1 d b_1. Set off equal distances, $b_1 b_2$, b_2 B, B b_4, etc., and make $b_2 a_2$, $b_4 a_4$, etc., equal to the length of the connecting-rod. Then the points $a_1 a_2$, etc., and $b_1 b_2$, etc., show in a very good way the relative motions of A and B. When you have finished this exercise, work others in which, with the same length of crank, you have longer or shorter connecting-rods. You will get some such results as are shown in the upper part of the figure. In every case, if we imagine the crank to revolve uniformly, the motion of A, the end of the connecting-rod, is shown; the distance from any one point to the next is passed over by A in the same interval of time. **Simple Harmonic Motion** (see Chap. XXV.) is the name given to the motion of the piston-rod, when we imagine the connecting-rod to be infinitely long; or rather, as we make the connecting-rod longer and longer, we get more and more nearly to this sort of motion. You see, then, that by skeleton drawings I mean drawings which show successive positions of the different parts of a mechanism whose motions we want to study. You will find that an **eccentric** and its rod may be regarded as a crank (the length of a crank is the distance between the axis about which the eccentric is revolving and its true centre), and a very long connecting-rod (the *length of* the connecting-rod being the length of the eccentric

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* Consult Professor Unwin's "Machine Design" on the teeth of wheels.

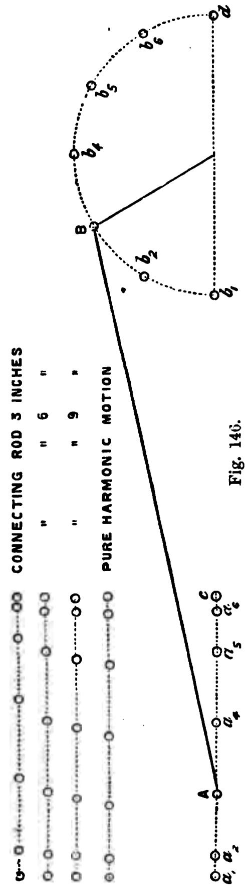

Fig. 146.

rod measured to the true centre of the disc). The advantage derivable from skeleton drawings will be more obvious if you consider, in the above case of a crank and a connecting-rod, that A need not be the cross-head at the end of the piston-rod; it may be the end of a lever, and so move in the arc of a circle; it may be a slide moving in a slot of any curved form. One of the most instructive cases of skeleton drawing is a link-motion. Taking any good drawing of a link-motion to start with, find the relative motions of piston and of slide valve for various positions of the link. In the study of the motion of a slide valve it is much too usual to assume that the piston's motion is what is shown in Fig. 146 as simple harmonic motion. The reason of this lies in the ease with which it can be stated in mathematical language; but it is incorrect.

180. In this early part of our work we wish to confine our attention to energy and power calculation in the simplest mechanisms; the transmission of power by rotating shafts; and the kinematic principles involved are very simple. In Chap. XXVI. we have something more to say about mechanism in general.

EXERCISES.

1. Two parallel shafts, whose axes are to be as nearly as possible 2 feet 6 inches apart, are to be connected by a pair of spur wheels, so that while the driver runs at 100 revolutions per minute, the follower is required to run at 25 revolutions per minute. Find the diameters of the wheels and also the number of teeth on each, if the pitch is 1¼ inches.
Ans., 48 inches, 12 inches; 120 teeth, 30 teeth.

2. A main shaft carrying a pulley of 15 inches diameter and running at 60 revolutions per minute communicates motion by a belt to a pulley of 12 inches diameter, fixed to a countershaft. A second pulley on the countershaft of 8½ inches diameter carries on the motion to a revolving spindle which is keyed to a pulley of 4¼ inches diameter. Find the number of revolutions per minute made by the last pulley. *Ans.*, 150.

3. On the crank-shaft of an engine there is a pulley 2 feet 6 inches in diameter. By means of a belt this drives a pulley 25 inches in diameter on a second shaft, on which is another pulley, 24 inches in diameter, which drives another pulley, 15 inches in diameter, fixed on a third shaft. On this shaft is a pulley 25 inches in diameter, which drives one of 10 inches diameter on a fourth shaft. On this shaft is another pulley, 20 inches in diameter, which drives a pulley 8 inches in diameter fixed on a dynamo shaft. If the engine runs at 100 revolutions a minute, what will be the speed of the dynamo? *Ans.*, 1,200 revs. per minute.

4. In a screw-jack where a worm-wheel is used, the pitch of the screw is ⅝-inch, the number of teeth is 20, and the length of the lever which works the worm is 12 inches. What is the velocity ratio? *Ans.*, 2413·1.

5. A wheel of 40 teeth is turned by a winch handle 14 inches long, and gears with a rack having teeth of 1 inch pitch. If the axis of the

wheel is fixed, what is the travel of the rack for two turns of the handle? *Ans.*, 80 inches.

6. The table of a drilling-machine is raised by a worm-wheel in combination with a rack and pinion. The handle which works the worm is 12 inches long, the worm wheel has 30 teeth, and the pitch circle of the rack pinion is 4 inches in diameter. What is the lift of the table for one turn of the handle? If the table and accessories weigh 500 lbs., what weight on the table would be balanced by a force of 12 lbs. applied at the handle, if 45 per cent. of the force applied be lost? *Ans.*, 0·42 inch 688 lbs.

7. If the two wheels in the back-gear of a lathe have 63 teeth each, and the pinions 25 teeth, what is the reduction in the velocity ratio of the lathe spindle due to the back-gear? *Ans.*, 6·35 ; 1.

8. The slide-rest of a screw-cutting lathe moves along the bed 14 inches while the leading screw makes 56 revolutions. What is the pitch of the screw thread? *Ans.*, ¼ inch.

9. It is desired to cut a screw of ⅜ inch pitch in a lathe with a leading screw of 4 threads to the inch, using four wheels. If both screws be right-handed, what wheels would you employ?

10. It is required to cut a left-handed screw of 5 threads to the inch in a lathe fitted with a right-handed guide-screw of ½ inch pitch. Show how the change wheels might be arranged, and state the numbers of teeth on them.

181. How a Shaft transmits Power.—I have refused to describe for you what you may see for yourselves at any time in workshops—how spur and bevil wheels and belts transmit power; how there are arrangements for disengaging such gearing, and stepped cones for giving change of speed when belts are used; how shafts are carried near walls or columns; how machine tools work, and a hundred other matters about which a little observation and drawing are of more importance than a large amount of reading. But there are some matters connected with machinery of great interest to you which you are not likely to observe unless I direct your attention to them. When a shaft transmits power it is in a state of strain; it is in a twisted condition. The twist is not perceptible to the eye, of course, but methods have been arranged to show it to the eye, and measure it; and it is found that *the twist in a shaft is proportional to the horse-power transmitted by the shaft divided by the number of revolutions per minute.*' Now to explain what I mean by a *twist.* Let *a straight* line be drawn along the shaft when power is not being

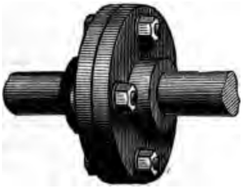

Fig. 147.

transmitted, then, if power be transmitted, the shaft will receive a *twist*, and this line will become a spiral line. The inclination, at any point, of the spiral line to its old position, is a measure of the twist.[*] When, instead of the ordinary coupling, Fig. 147, in which the two halves are connected by means of bolts, we use one, Fig. 148,[†] in which the two halves are connected by means of spiral springs, these springs get extended when the shaft transmits power. The yielding of the springs cannot be observed unless we make some arrangement like that shown, where the motion of A relatively to C causes the arm E to move and bring the bright bead B towards the axis. If everything is made dead black except the bead it will be seen describing a circle of greater or smaller radius, and a scale with a sliding pointer enables us to measure accurately the distance moved inwards by the bead. *The reading on the scale multiplied by the number of revolutions of the shaft per minute tells us at once the horse-power actually passing through the coupling.*[‡]

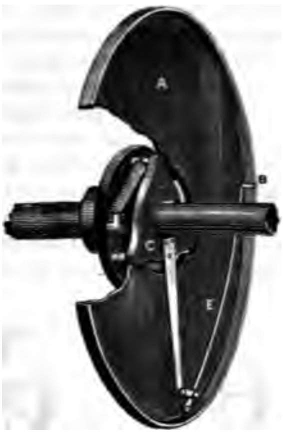

Fig. 148.

At Finsbury, the scale comes out from a wall, the shaft is about ten feet above the floor; the observer stands on a ladder with a gas-jet behind him. It is quite easy to avoid error due to parallax, and to read with a very considerable amount of accuracy. It is exceedingly interesting to watch the bead when the loads are suddenly altered.

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[*] The best measure of the twist is this angle of the spiral divided by the radius of the shaft, and the quotient is called the *angle of twist*. See Art. 294.

[†] Ayrton and Perry's Dynamometer Coupling.

[‡] The total moment of the forces of the springs in pound-feet, or, as it has been called by Professor James Thomson, *the torque*, multiplied by the angular velocity in radians per minute, divided by 33,000, is the horse-power. Suppose that when one of the lengths of shafting is held fast we find the position of the bead when we hang weights on levers or round pulleys or wheels fastened to the other length; a torque of 52·5 pound-feet will cause the bead to move radially inwards by a distance which we call ·01 on our scale; a torque of 105 pound-feet causes the bead to move inwards a distance which we call ·02 on our scale, and so on. Such a coupling ought to be graduated by actual experiment. We generally have done it statically, and it is then necessary to eliminate friction by vibration, &c. In actual practice, friction is eliminated, because of the continual vibration of the parts.

It is a great pity that there should not be at least one such dynamometer coupling on every length of shafting in factories. We have it from a man who has made careful measurements that the loss of power in ordinary shafting is very great indeed. If the fact were continually before our eyes great improvements would certainly be effected. This is a function of measuring instruments (keeping defects prominently before us) which is very important. Mechanical engineers are largely in the habit of treating indicated horse-power of an engine as if it were the actual power given out by the engine. Steam-engine construction has improved enormously of late, mainly because electricians have been able to measure electrical power with great accuracy, and so the mechanical losses of power were brought prominently into notice.

182. Belts.—If the pulley A, Fig. 149, is driven from B by means of a belt, you must remember that there is a pull in the part of the belt M, as well as in the part N. These two pulls are generally pretty great, as you know, but if you could measure them accurately you would find that there is more pull in N, else A would not turn. It is the *difference of these pulls* which concerns us. You may, perhaps, understand this better from Fig. 150. The pull in A M is the weight of M, say, 20 lbs. The pull in A N is the weight of N, say, 50 lbs. If N falls two feet, M rises two feet, and the work done upon the pulley and which it transmits through the shaft somewhere else is 50 × 2, or 100 foot-pounds, minus 20 × 2, or 40, the difference being 60 foot-pounds. In fact, it is the difference of pull in the two cords, 30 lbs., multiplied by the space passed over by the cord, 2 feet; result, 60 foot-pounds.

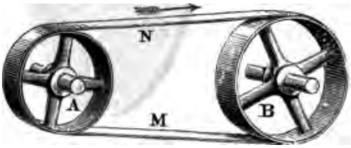


Fig. 149.

The horse-power given by a belt to a pulley is, then, the difference of pull in the belt on the two sides of the pulley, multiplied by the speed of the belt in feet per minute, divided by 33,000.

This is only a particular case of the general rule. *If* M *is the sum of the moments of a number of forces tending to cause* *rotation, about the axis, in pound-feet, and if a is the angular velocity in radians per minute, then* Ma *is the work done in foot-pounds per minute, so that* Ma ÷ 33,000 = H, *the horse-power* . . . . (1).

Example.—In our dynamometer coupling if there are four springs, each exerting a force of 160 lbs., the distance of the axis of each spring from the axis of the shaft being 0·7 foot, the turning moment is 4 × 160 × 0·7, or 448 pound-feet. If the shaft makes 150 revolutions per minute, this means 150 × 2π, or 942 radians per minute; and hence 448 × 942 ÷ 33,000 = 12·8 horse-power.

Example.—An ordinary flange coupling has six bolts at 0·7 foot from the axis; what force is resisted by each bolt (it tends to break by shearing, Art. 281), when 60 horse-power is being transmitted at 120 revolutions per minute? *Answer.*—If F is the force, 6 F × 0·7 is the moment in pound-feet; this, multiplied by 120 × 2π ÷ 33,000, is equal to 60: and hence, F = 33,000 × 60 ÷ (6 × 0·7 × 120 × 2π), or F = 625 lbs.

When we know the maximum horse-power at the minimum speed (observe that torque or turning moment depends upon power ÷ speed), we can calculate the maximum F for each bolt, and our knowledge of the strength of materials (Art. 284) enables us to say if the bolt is strong enough.

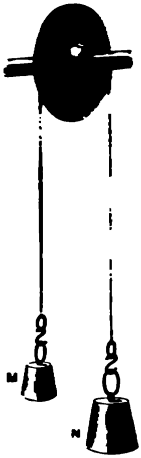


Fig. 150.

The general rule appears in many special forms, of which we now have one example in belting; and just as it enables us to calculate the strength of bolts in a flange coupling, or the size of the shaft itself (Art. 296), or tells us how to order the springs of a dynamometer coupling, so it here tells us how to find the proper size of a belt. We have here (N − M) V ÷ 33,000 = H, the horse-power. The greatest pull in the belt is N lbs., and it is this which determines how strong the belt must be. Hence, we must answer the question:—

If it is the difference of pull that produces turning, why is there so great a pull even in M, Fig. 149, as we usually find? Refer again to Fig. 150. If we want the difference between M and N to be 30 lbs., why not make M have no weight at all, and

N may then be only 30 lbs.? Evidently we should not b
to get friction enough, and the weight N would fall, ca
the cord to slide on the pulley; in fact, the friction be
the cord and pulley must be more than 30 lbs., else ther
be slipping; and to produce this friction it is necessary to
a weight at M as well as at N. If we allowed the cord
round more of
pulley, the nec
friction might b
duced with a
weight at M. T
an idea of the fr
between a cord
pulley, arrange a p
or other round c
P, as in Fig. 151.
it firmly. Pla
weight at M, say
Now place weigl
the scale-pan at N,
the cord just
slowly. Say we
3 lbs. to be nece
The difference be
N and M, or 2 lbs.,
friction. Now put
the former weight
you will find that
twice the former N will just cause slipping, so that the fr
is doubled. In fact, we have our old law, "friction is p
tioned to load." But now let us see **how friction depen**
the amount of lapping of the cord. In your first exper
measure the cord actually in contact with the post P. Su
it to be 4 inches: now, keeping M, 1 lb., let the cord lap
more of the post P, say 8 inches this time, and find the w
N, which will just produce a slow sliding. You will fi
to be 9 lbs. If the cord touches on 12 inches of the
P, you will find that 27 lbs. at N will be necessary to s
overcome the friction. It is only by actually trying
experiment for yourself that you will get a clear idea of
rapidly the friction increases with the amount of lapping
is on this account that one man can check the motion c

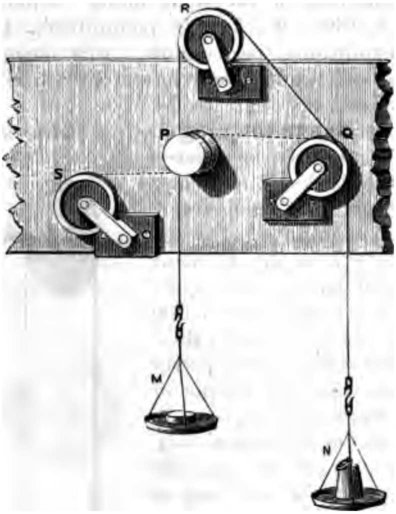


Fig. 151.

largest vessel by simply coiling a rope a few times round a post.

The apparatus, Fig. 151, is so arranged that any required amount of lapping may be given to the cord round the fixed post P. In an actual experiment, the fixed weight M was 50 grammes. By means of the pulleys the amount of lapping round P was varied, and weights were placed in N, in each case just sufficient to overcome the friction and raise M slowly, as above described. The following are the results of the whole series of experiments :—

Number of times the cord laps round.	The weight required at N to overcome friction and the weight of M.	Logarithms of the ratio of N to M.
$\frac{1}{2}$	80	0·2041
$\frac{3}{4}$	105	0·3222
1	150	0·4771
$1\frac{1}{4}$	200	0·6021
$1\frac{1}{2}$	255	0·7076
$1\frac{3}{4}$	330	0·8195
2	400	0·9031
$2\frac{1}{4}$	500	1·0000
$2\frac{1}{2}$	700	1·1461
$2\frac{3}{4}$	1,000	1·3010
3	1,150	1·3617
$3\frac{1}{4}$	1,500	1·4771

Plotting the first and third columns on squared paper, we find that a straight line passes nearly through all the points. From this line we deduce the equation—

$$n = 2{\cdot}2 \log \frac{\text{N}}{\text{M}},$$

where n is the number of times the cord laps round. From this it is easy to show that the coefficient of friction, μ, between the cord and the pulley is ·166.

You must then remember that the tension in M, Fig. 149, is necessary to produce as much friction as will prevent slipping. If ever the excess pull in N is greater than the friction, there will be slipping. If the belt slips, there is energy wasted, which you can calculate if you know the force of friction, and multiply by the distance *through* which slipping occurs.

183. The law is this. If μ is the coefficient of friction between the cord or belt and the pulley: if l is the length of the cord or belt which touches the pulley, say in inches: and r the radius of the pulley in inches, then

$$\text{Log}_e \frac{N}{M} = \mu \frac{l}{r},$$

N and M being the pulls in the belt or cord on the two sides of the pulley. l/r is the angle of lapping stated in radians.

This rule is arrived at mathematically in the following way.

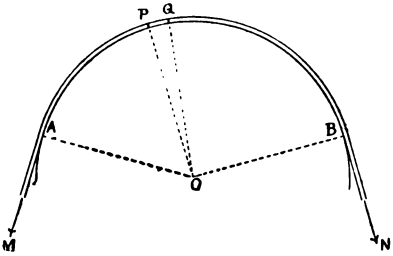


Fig. 152.

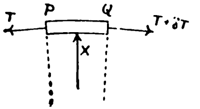


Fig. 153.

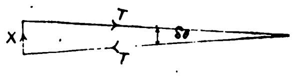


Fig. 154.

Let A P Q B be the part of a pulley touched by the belt M A P Q B N. Imagine the pulley fixed, and slipping occurring because the tension at N is greater than at M. Let the angle A O P be called θ. Let the tension in the belt at P be T. We study what occurs at the place P Q, and we greatly magnify P Q in Fig. 153. What are the forces acting upon the short piece of belt P Q? We have a pull $T + \delta T$ at Q, and a pull T at P, and forces acting radially from the pulley, their resultant being X: we have also friction whose amount is μX if μ is the coefficient of friction, and this friction is what we overcome by the excess tension δT. To find X, assume $\delta T = 0$, and no friction (it will be found that these terms become less and less important as the distance P Q is made less and less). Let the angle A O Q be called $\theta + \delta\theta$, so that $POQ = \delta\theta$. The three forces T, $-X$, T being in equilibrium, let them be represented in direction and sense by the sides of the triangle shown in Fig. 154, and it is evident that $X = T \cdot \delta\theta$. Hence the force of friction is $\mu \cdot T \cdot \delta\theta$, which is overcome by δT. When slipping is just occurring,

$$\mu \cdot T \cdot \delta\theta = \delta T \; .\;.\;.\;.\; (1);$$

or rather, because the theory is true only when $\delta\theta$ is thought to be smaller and smaller without limit,

$$\frac{dT}{d\theta} = \mu T \; .\;.\;.\;.\; (2).$$

This is an example of the compound interest law. "The rate of increase of T as θ increases is proportional to T itself." In any elementary book on the calculus it is shown that (2) is the same as

$$\log. T = \mu\theta + \text{constant}.$$

Putting T = M when $\theta = 0$, and T = N when θ = A O B, we find

$$\log. N - \log. M = \mu \, . \, AOB \, . \, . \, . \, . \, (3).$$

This is the rule given above. Assuming that there is a constant μ, and that we know its value, this rule enables us to design belts and ropes to be strong enough to transmit a given amount of horse-power H at the belt velocity V feet per minute. For we have already used (N − M) V ÷ 33,000 = H . . . . (4). If the angle A O B be called θ for shortness, it is easy to deduce from (3) and (4)

$$N = \frac{33{,}000\,H}{V} e^{\mu\theta}/(e^{\mu\theta} - 1) \, . \, . \, . \, . \, (4).$$

This is the tension in the belt where it is greatest, and it is from this value that we calculate the strength of the belt. Thus if b and t are the breadth and thickness of the belt in inches, and if f is the safe load per square inch of section for the material, we make btf equal to the calculated N. We take f about 330 lbs. per square inch in leather belting, and this leads to the easily remembered rule, giving bt in square inches,

$$bt = \frac{100\,H}{V} e^{\mu\theta}/(e^{\mu\theta} - 1) \, . \, . \, . \, . \, (5).$$

184. In a great number of cases the least angle of lapping when one shaft drives another by belting is $\frac{2}{5}$ths of the circumference, and μ may be taken as 0·3; and so we are led to the common, easily remembered shop rule, $bt = 200\,H \div V$ . . (6). It is to be remembered that b is the breadth, t the thickness of the belt in inches; H is the horse-power transmitted, V the velocity of the belt in feet per minute; μ the coefficient of friction, and θ the angle of lapping in radians.

Leather belting is seldom more than 4 feet wide, even in America, where the widest belting is employed. A *single* leather belt is about a quarter of an inch thick, and the material will stand a pull of from 700 to 1,200 lbs. per inch of its width, before breaking. A single belt at an ordinary laced joint may be taken to stand a greatest working pull of about one-third of the strength of the leather. An ordinary laced joint, with splice, has about twice the strength of an ordinary grip-fastened joint, and not quite twice that of a butt laced joint. Many engineers take 80 to 90 lbs. per inch of width of a single belt as the usual pull on the tight side.

Strips from the best parts of the tanned hide are cemented at long splices to form belts; vulcanised indiarubber alone, or with intermingled canvas cotton, and various other kinds of waterproof belting are used. There is always slipping or creeping, so that it is not quite accurate to take the velocity of the belt as being equal to that of the pulley. (See Art. 70.)

Belts are in use which are made of a great number of little links of thick leather on wire pins, forming a chain. Hemp and cotton ropes are now often used, instead of belting. Many V grooves in the rim of a drum receive each its own rope, which is usually lying wedged in the groove, with not much tension when there is no motion, except what is due to its own weight, the span being 20 to 60 feet. As in belts conveying a considerable amount of power, the tight, or N side, is underneath, as this gives more lapping. The strengths of ropes and chains are given in Art. 266, but in rope-driving our ability necessitates some such rule as (see Art. 183)

$$\text{N} - \text{M} = 8\,g^2,$$

when g is the girth of the rope in inches, and N and M are in pounds. The wedging of the ropes in grooves causes the coefficient of friction to be virtually of its usual value divided by the sine of half the angle of the groove, and as this is usually 45°, we take μ to be about 0·6, and sometimes more. Splices are about 10 feet long. The speed is usually about 4,000 feet per minute. The bending and unbending of the rope is what seems to shorten its life, which is usually one of a few years. There is probably considerably more waste of energy by friction than when toothed gearing is used.

With **wire ropes** (strands of wire round a hempen core) the wedging in grooves is much more hurtful, and hence reliance is placed upon greater velocities, even to 6,000 feet per minute. Under usual conditions the life of a rope is one of less than a year. In calculations of the tensions due to the weights of ropes, we may take it that the hanging curve is practically parabolic (see Art. 133). The bottoms of the grooves on the pulleys are usually of leather, wood, or gutta-percha, and μ may then be taken as 0·25. Each relay being about 1,000 yards, we may take its efficiency as about 95 per cent.

Exercise.—If there are nine relays, what is the total efficiency? Answer.—0·95 raised to the 9th power, or 0·63.

At Oberursel a rope is used whose diameter is 0·6 inch, of 36 wires each 0·06 in diameter; pulleys, 12 feet in diameter, placed 400 feet apart; 94 horse-power is transmitted 3,000 *feet,* the rope travelling at 4,400 feet per minute.

185. Chain Gearing.—The use of tricycles and bicycles has caused a very great development in this non-slipping, certain, and efficient method of transmitting power. It is likely to have a very large further development for large powers in factories on account of its success for small power, not merely in cycles, but in electric, motor, tramcar and locomotive work. No tension is needed on the slack side.

EXERCISES.

1. A pulley of 3 feet diameter receives 10 horse-power at 150 revolutions per minute; find the difference of pull between N and M. If the lapping is 0·4 of a circumference, and $\mu = 0{\cdot}25$, find N/M. Now find N.
Ans., 233·4 lbs.: N=501·3 lbs.

2. Use the above workshop rule to find the breadth of a single belt ($t = 0{\cdot}4$ inch, say) to transmit 20 horse-power at a velocity of 1,300 feet per minute. *Ans.*, 15·4 inches.

3. A rope is wrapped three times round a post, and a weight of 12 lbs. is hung from one end. Find the least pull applied at the other end necessary to raise this weight if $\mu = {\cdot}3$. What weight would be required to just prevent the 12 lbs. from slipping down? *Ans.*, 3,440 lbs.: 0·042 lb.

4. The pulley on an engine shaft is 5 feet in diameter, and it makes 100 revolutions per minute. The motion is transmitted from this pulley to the main shaft by a belt running on a pulley, and the difference in tension between the tight and slack sides of the belt is 115 lbs. What is the work done per minute in overcoming the resistance to motion of the main shaft? What is the horse-power transmitted?
Ans., 180,714 ft. lbs.: 5·47.

5. 50 horse-power is being transmitted by a belt moving at a speed of 70 feet per second. What width of belt will be required if its thickness is 0·6 inch, assuming the maximum working stress to be 330 lbs. per square inch, and the tension on the tight side being double that on the slack side?
Ans., 8 inches.

6. A pulley 3 feet 6 inches in diameter, and making 150 revolutions a minute, drives, by means of a belt, a machine which absorbs 7 horse-power. What must be the width of the belt so that its greatest tension shall be 70 lbs. per inch of width, it being assumed that the tension in the driving side is twice that on the slack side? *Ans.*, 4 inches.

7. A rope pulley, carrying 20 ropes, is 16 feet in diameter, and transmits 600 horse-power when running at 90 revolutions per minute. Taking $\mu = 0{\cdot}7$ and the angle of contact 180°, find the tensions on the tight and slack sides of the rope. *Ans.*, 246 lbs.: 27·2 lbs.

186. Transmission and Absorption Dynamometers.—I have already described to you an instrument which allows us to measure the horse-power transmitted by a shaft. I am in the habit of employing a somewhat similar arrangement for measuring the power transmitted by a belt to any machine. It is shown in Fig. 155, and is easily understood from the description of Fig. 148. I can take it near any machine, and drive the machine through it, using two belts instead of one. G is a loose pulley. A belt *drives* H, *which* drives the plate E through four spiral springs B. The plate E is keyed to a shaft carried on the frames C and D, and the pulley F is keyed on the shaft. A belt from F, therefore, will drive any machine. When much torque is acting, the springs B become extended, causing a relative motion of E and H, and this motion is shown by the bright bead A, at the end of the lever I A, approaching the axis of rotation. A fixed scale attached to the frame C allows the motion of A to be measured.

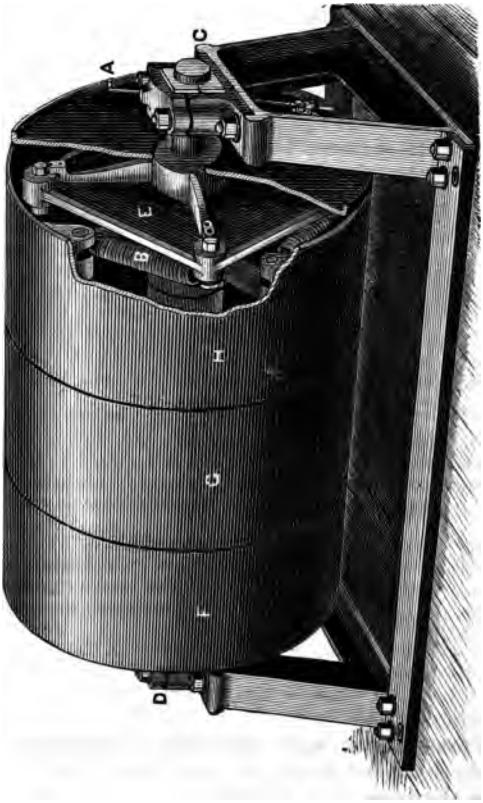


Fig. 155.

In Fig. 159, showing the Froude or Thorneycroft Dynamometer, the pulley D drives the pulley F by a belt which passes also round the two pulleys A and B. These pulleys are on a frame A B L, which is pivoted at E. The tension N, on the tight side, is greater than M, and a measurable force P must be exerted to keep the frame in the position shown. Evidently $2\,\text{N}.a = 2\,\text{M}.a + \text{P}.b$ so that $\text{N} - \text{M} = \frac{\text{P}.b}{2a}$.

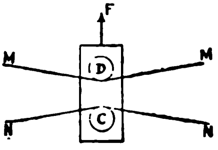


Fig. 156.

This transmission dynamometer is specially useful when small powers are to be measured in the laboratory. In the Hefner-Alteneck form, the long and nearly horizontal belt driving a pulley passes round two guide pulleys D and C on a suspended and balanced frame. When the angle made by the slack parts M M is the same as that made by the light parts N N, a measurable force F is required to maintain the frame in its symmetrical position.

Exercise.—If the angle between M and M is $180 - \theta$, and there is the same angle between N and N, and the pull F is symmetrical, show that $\text{F} = (\text{N} - \text{M})\,\theta$ if θ is small. In truth, $\text{F} = (\text{N} - \text{M})\,2 \cdot \sin \frac{1}{2}\,\theta$.

187. Absorption Dynamometers are used to measure the power given out by steam-, or gas-, or oil-engines, or electric or other motors. They measure the power, consuming it as they do so. One of **Professor James Thomson's** forms, as arranged for measuring the power given out by a small electro-motor, is shown in Fig. 157. The motor drives the grooved pulley A, and the pulley B turns along with A. A cord hangs lapping round part of B, and carries at its one end a scale-pan M, containing a weight. The other end N′ is pulled by means of a piece of metal fastened to the rim of a loose pulley C, which has a weight N always acting upon it, tending to turn it round.

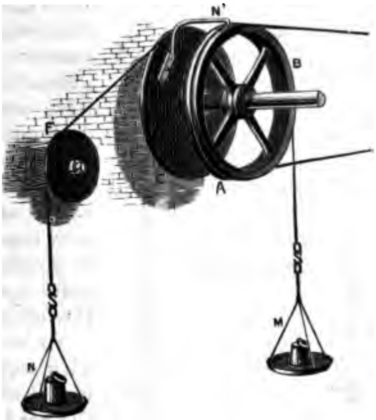


Fig. 157.

Evidently the cord is pulled with a weight M at one end, and a weight N at the other. If now there is slipping between the cord and B, the friction is measured by the difference of the weights N and M. If M is 100 lbs. and N is 400 lbs. the friction is 300 lbs. If the pulley has a circumference of 2 feet, and makes 80 turns per minute, the amount of slipping is × 80 2, or 160 feet per minute, and the work done against friction is 160 × 300, or 48,000 foot-pounds per minute—that is, 1·45 horse-power. In this case *all the power is wasted in friction*, and this is called an Absorption Dynamometer because it measures the power but absorbs it in doing so; whereas the coupling of Fig. 148 and the dynamometer of Fig. 155 are called Transmission Dynamometers, because they measure the power transmitted through them whilst working any machines.

Should we try to use **merely a cord** or belt with two different weights N and M at its ends (as in Fig. 158) as an absorption dynamometer, the cord or belt slipping on the pulley, we should find that, after adjusting the weights, when we leave them hanging on, one of them will gradually overcome the other till it touches the floor, and after that we are ignorant of N — M. In fact, the co-efficient of friction gradually alters. In Fig. 157 this is **automatically** counteracted by a change in the amount of lapping. Sometimes, instead of the lighter weight M, we have used a spring balance and a dash-pot to still the jerky motion which occurs, but in this case we must take readings of the changing values of M; and again, it is usually very important during a test to keep N–M constant. We have met with quite wonderful success by adopting the following very simple expedient:—A B is a grooved pulley, and a cord with scale-pans hangs round it as in Fig. 158. At P there is a **knot in the cord**, or an excrescence of any kind which, when it is drawn up by the gradual falling of N, will just slightly jam itself *in the groove*. When measuring the power from a three-

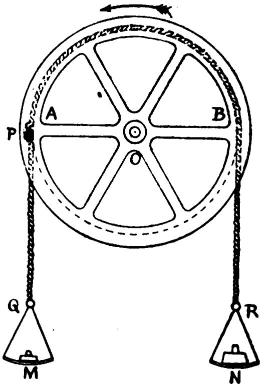


Fig. 158.

quarter horse electric motor we have kept the same weights M and N in the scale-pans for an hour at a time, a slow gentle rise and fall taking place now and again. Before we discovered the value of the knot we went to great trouble in having a thick rough cord from P to Q spliced to a thinner one, so that when N fell a greater lapping of the rougher cord made up for a lessening of the friction on the smoother one, but as a matter of fact the knot is excellent. We have used the same expedient with a canvas belt on a large drum when measuring 26 horse-power; here a leather lace passed carelessly a number of times through the belt introduced sufficient resistance at P to prevent N falling.

In all these cases **soapy water** is kept trickling over the rubbing surfaces to carry off the heat, for all the mechanical energy which we measure is converted into heat. The water ought to be soapy, because when pure water acts as a lubricant there is considerable, and seemingly spasmodic, variation in the friction. A break on the principle of N, a weight, and M, the pull of a spring balance, is sometimes used even for as much as 50 horse-power, N being attached to two ropes going about three-quarters of the way round an ordinary pulley, and kept apart on the smooth rim by distance pieces of wood which fit the edges of the rim, one rope completing the round so as to

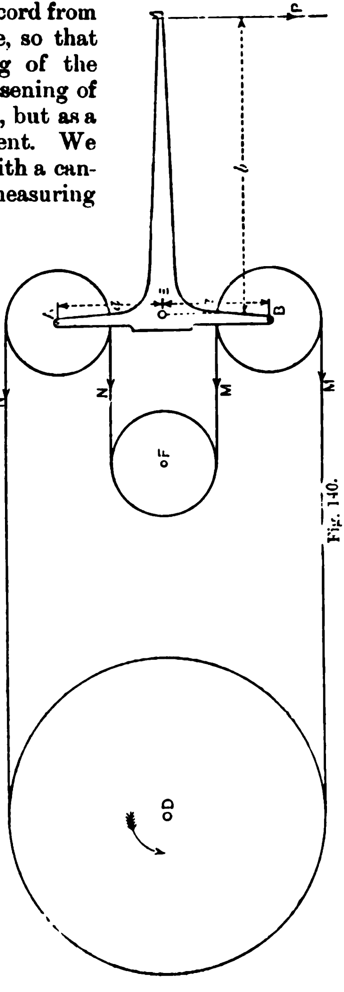


Fig. 140.

come between the first two on leaving, M is then an upward pull, and N downward.

Fig. 160 shows a brake which may be used for large powers, and we have used it with satisfactory results. It consists of blocks of wood on hoop iron, forming a brake strap, the two ends of which D and E are fastened to the lever D H, a small angular change in the position of which produces great increase in tightness. A load W lb. is hung on at C and a spring balance acts at H. The pull in the spring balance, w lb., is usually small. It is evident that $W . R - w r$ is a moment in pound feet which, multiplied by the angular velocity in radians per minute, and divided by 33,000, gives the horse-power. Lastly, **Froude's liquid dynamometer** may be employed up to very high powers; the heat is developed in a considerable mass of water which keeps cool. For the latest and best form of this, students are referred to a paper by Prof. Reynolds in the Transactions of the Royal Soc., May, 1897.

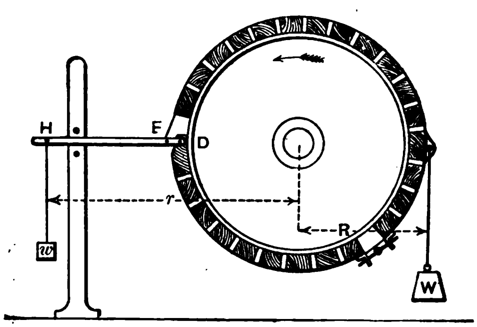


Fig. 160.

If any student has trouble in seeing why we assume a virtual surface of rubbing coinciding with the centre of the rope in Fig. 158, let him return to the consideration of Fig. 149. Here it is obviously the velocity of the centre of the rope that is taken, and we calculate this from angular velocity and virtual diameter of the pulley. Now suppose the pulley of Fig. 149 to be at rest and the rope slipping; the same amount of power is now wasted that used to be given to the shaft; there will be exactly the same forces of friction and waste through friction if the pulley moves and the weights do not.

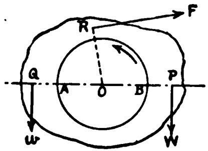


Fig. 161.

Generally, let A B be a revolving wheel with centre O, and let forces F, W and w keep the brake block P Q R from moving with the wheel. Let there be no other forces acting on the brake block which exercises moment about O.

How much power is being wasted at the rubbing surfaces? It is the sum of all the forces of friction multiplied by their respective rubbing distances (we need not here speak of pounds, feet, &c., or the weight in which power is measured). But this is the same as the sum of all the moments about O of all the forces of friction multiplied by the angular velocity, and the sum of the moments of the forces of friction is measured by $\mathrm{W.OP} + \mathrm{F.OR} - w.\mathrm{OQ}$., because the block is in equilibrium under the action of the forces of friction and W, F and w (see Art. 98).

188. *Example.*—In an **Ayrton-Perry Coupling**, the bright bead does not begin to move inward till a certain horse-power is reached. If a radial motion of 10 inches corresponds to a lengthening 0·1 inch of each spring: if there are eight springs, whose axes are 10 inches from the centre, and the shaft runs at 200 revolutions per minute; if the maximum horse-power is 250, and if it is to alter from 200 to 250 for an 18 inch motion of the bead, how ought the springs to be ordered? If F is the pull in each spring at H horse-power, $8 \times F \times 10 \div 12$ in pound feet, multiplied by $200 \times 2\pi \div 33{,}000$ gives H, or $F = 3\cdot 94\ H$. Hence the range of pull in each spring is from $3\cdot 94 \times 250$, or 985 lbs., to $3\cdot 94 \times 200$, or 788 lbs., and for this the bead has a range of 18 inches; and therefore each spring will yield by $18 \times \frac{0\cdot 1}{10}$, or 0·18 inch. Thus the spring yields 0·18 inch for a difference of pull of $985 - 788$, or 197 lbs., and will therefore yield $985 \times 0\cdot 18 \div 197$, or 0·9 inch for the maximum pull. We therefore order eight springs, each of which has a greatest working pull of 985 lbs., and is then to have an axial lengthening 0·9 inch. It will also be necessary to tell the spring-maker what the exact length of the spring is to be between the end pins, and possibly its outside diameter. It is usual to arrange adjustable stops to keep the springs stretched, so that they shall not elongate for any force less than 788 lbs. When the springs are long we use diametral wires to prevent any action due to centrifugal force, and in some cases of very fluctuating loads we have used **dash-pots**.

Example.—The middle of a key is 2 inches from the centre of a shaft; the key is the only fastening of a wheel which makes 100 revolutions per minute, and gives out 10 horse-power. What is the force transmitted by the key? *Answer.*—The force is F if $F \times 2 \div 12$ (the moment in pound feet), multiplied by $100 \times 2\pi \div 33{,}000 = 10$. That is, $F = 3{,}016$ lbs.

Example.—A 6-feet drum at 150 revolutions per minute drives through a **Hefner Alteneck** Dynamometer. A spring balance registers F as 0 when no power is being transmitted, and registers it as 60 lbs. when we know that 25 horse-power is being transmitted. Find θ if $(N - M)\,\theta = F$. We know that the velocity of the belt is $6\pi \times 150$, or 2,827 feet per minute; so that $N - M$ is $25 \times 33{,}000 \div 2{,}827$, or 292 lbs. Hence, θ is $60 \div 292$, or 0·206 radians, or 11·8 degrees.

Example.—A dynamo machine going at 420 revolutions per minute is supported **on knife edges**. The driving-belt causes the machine to tilt, and the tilting tendency is prevented by a weight W of 15 lbs. suspended from an arm 5 feet from the axis, measured horizontally. Find the horse-power given to the machine. The machine gives out 39·2 ampères at 98 volts. What is its efficiency? *Answer.*—The turning moment is 15×5 pound feet, and therefore the horse-power is $15 \times 5 \times 400 \times 2\pi \div 33{,}000$, or 5·71. The electrical horse-power given out is $39{\cdot}2 \times 98 \div 746 = 5{\cdot}15$. The efficiency is therefore $5{\cdot}15 \div 5{\cdot}71$, or 0·902, or 90·2 per cent.

Example.—A brake block is kept at rest by certain forces, 500 lbs. acting at 2 feet from the axis, 30 lbs. acting at 8 feet from the axis, 50 lbs. acting at -5 feet from the axis (we mean here that the moment of this last force is opposite to the others). The shaft makes 120 revolutions per minute. Find the horse-power consumed. The moment is $500 \times 2 + 30 \times 8 - 50 \times 5$, or 990 pound-feet. This, multiplied by $120 \times 2\pi \div 33{,}000$ gives 22·6 horse-power.

Example.—The following readings were taken when testing an electric motor. Find the electric power supplied and the mechanical power given out in each case. The knot dynamometer was used; each string was found to be almost exactly 6·1 inches horizontally from the centre. If n is the number of revolutions per minute $(N - M) \times \frac{6{\cdot}1}{12} \times 2\pi \times n \div 33{,}000 = H$, or $(N - M)\,n \div 10{,}000 = H$ nearly. The student will remember that 1 ampère × 1 volt = 1 watt, and 746 watts = 1 *horse-power.*

Ampères.	Volts.	N	M	n	Electrical Horse-power.	Brake Horse-power.	Efficiency.
8·4	28·1	4	1·6	1,504	·318	·162	·51
7·5	30·4	4	2·4	1,796	·304	·129	·42
13·1	19·8	18	13	572	·349	·128	·37
16·4	31·0	19	12	1,008	·681	·316	·46
15·3	30·7	17	10	1,048	·632	·329	·52

EXERCISES ON BRAKES.

1. In an Ayrton and Perry knot-brake (Fig. 158) the centres of the cords to N and M lie respectively 6 and 6·2 inches horizontally from O (it is usual to take these distances equal and equal to the radius of the circle of the centre line of the cord on the pulley, but very careful measurement will detect slight differences due to stiffness of cord and effect of knot). The weights are N = 30 lbs., M = 5 lbs. The pulley makes 400 revolutions per minute; find the horse-power. Here we had better take moments, as the distances are unequal. The moment is (30 × 6 ÷ 12) − (5 × 6·2 ÷ 12), or 12·42 pound-feet. The speed is 400 × 2 π, or 2,514 radians per minute, and 12·42 × 2,514 ÷ 33,000 = 0·86 horse-power.

2. In a rope-brake on a flywheel of 8 feet diameter, the ropes being 1 inch thick (so that the moment due to friction is as if the rope had no thickness, but the wheel is 8·083 feet in diameter), the load is 500 lbs. The pull in the spring balance varies from 10 to 20 lbs. during the test. The wheel makes 105 revolutions per minute; find the horse-power. The velocity of the virtual wheel is 8·083 π × 105 feet per minute, and this, multiplied by 490 or 480, divided by 33,000, gives a power varying from 39·6 to 38·8 horse-power.

3. The power of an engine is tested by passing a belt over the fly-wheel, which is 5 feet in diameter. One end of the belt is secured to a spring balance, and a weight of 300 lbs. hangs on the other end. What is the brake horse-power when the balance registers 180 lbs. and the fly-wheel makes 150 revolutions per minute? *Ans.*, 8$\frac{4}{7}$.

4. In Exercise 3 the pull in the spring balance gradually alters during the test to 200 lbs. This is due to a change in μ, the co-efficient of friction. Find the two values of μ. *Ans.*, 0·159; 0·129.

If the extreme tensions 300 lbs. and 180 lbs. were maintained constant, as in the James Thomson dynamometer (Fig. 157), and the lapping was 180° to begin with, what would it be at the end? *Ans.*, 222°.

5. The diameter of a steam cylinder is 8 inches, the length of crank 9 inches, the number of revolutions per minute is 150, and the mean effective pressure of the steam is 35 lbs. Find the indicated horse-power.

The same engine is tested by a brake (like Fig. 160) on the crank shaft, R being 2½ feet, W being 310 lbs., w being 15 lbs., and r being 4½ feet. Find the brake horse-power and the working efficiency of the engine. *Ans.*, 24; 20·2; 0.84.

6. Find the horse-power given out by a steam-engine driving a Froude dynamometer which discharges 700 gallons of water per hour with a rise of temperature of 18·5 Centigrade degrees. *Ans.*, 92.

CHAPTER XI.

KINETIC ENERGY.

189. We sometimes assume that our readers know quite well the fundamental principles of mechanics, and then, again, we assume that they do not. We hope that they agree with us that we are right in proceeding in this way.

When a weight, A Fig. 22, in falling lifts a weight B by the use of a machine inside the box C, let us consider the store of energy at any instant. The store of energy consists in—first, the *potential energy* of A—that is, the weight A in pounds, multiplied by the distance in feet through which it is possible to let it fall to some datum level; second, the potential energy of B, which is the weight of B multiplied by the distance through which it is possible to let B fall; third, the *energy of motion*, or **kinetic energy**, of everything which is moving—namely, A, B, and the parts of the mechanism. We are supposing that there are no other weights which can fall or rise, and that there are no coiled springs or other stores of energy in the mechanism. Now, if A is just heavy enough to maintain a steady motion, the kinetic energy remains the same; so that, whatever energy is given out by A in falling is in part being given as potential energy to B, and is in part being wasted in friction. But suppose A to be heavier than this, then there is more potential energy being lost by A than is being stored by B or wasted in friction, and it must be stored up in some other form. The surplus stock shows itself in a quicker motion of everything; it is being stored up as kinetic energy.

190. We have now to consider an important question. When a certain amount of potential energy (measurable in foot-pounds) disappears, and becomes kinetic energy, how quickly must all the parts of the machinery move to store it all up? This problem is very troublesome, because everything in Fig. 22 is in motion in a different way; some parts of the mechanism are moving slowly, others quickly. It is, however, easy to find out how much kinetic energy a small body has if we know its weight and its speed. Let there be a small ball hung from the point O, Fig. 162, by a silk thread, *so that when* it vibrates we can call it **a simple pendulum.**

Now, you know that when it reaches the end of its swing at A it is, for a very short interval of time, motionless, and has no kinetic energy. It falls from A to B; and as there is almost no friction, we may suppose that the potential energy which it loses in falling through the vertical height from A to B is all stored up as kinetic energy when the ball reaches B. The body has a certain velocity in feet per second when it reaches B, and it is on account of its having this velocity that we say it has a store of kinetic energy. If the vertical height from B to A is h feet and the weight of the body is w lb., we have $w\,h$ foot-pounds, which it had at A, converted into kinetic energy at B. Now comes the question. We know that this kinetic energy is proportional to w, but how does it depend upon the velocity?

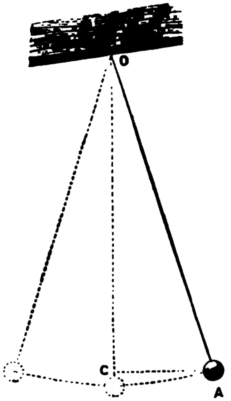


Fig. 162.

It evidently does not depend on the *direction* of the velocity at B, and some mathematicians would say that this shows that it must depend upon an even power of v, but this is metaphysics.

Now, experiment shows that when a body falls freely at London without friction, its velocity when it has fallen freely without friction at any place through distances proportional to 0, 1, 4, 9, 16, 25, 36, &c., is proportional to the numbers 0, 1, 2, 3, 4, 5, 6, etc. Experiment shows that these numbers are quite independent of whether we measure in feet or centimetres or yards, in minutes or seconds. Hence we say that the velocity is proportional to the square root of the height, and we assume that we are stating our result when we say $v^2 \propto h$, or $v^2 \div h =$ a constant, which we shall call b, or $v^2 = bh$ . . . (1).

Notice that when we write our result (1) down we have written down something which is beyond mere arithmetic; it is on the very much higher subject—physics, which in its mathematical form is called algebra. v, b and h are not mere numbers; they are *quantities*, and if v and h were merely to be considered *numbers*, the dimensions being feet and seconds, the assumption in (1) involves $v \left(\frac{\text{feet}}{\text{second}}\right)^2 = b\,h$ (feet), or $b = \frac{v}{h}$

$\frac{(\text{feet})}{(\text{second})^2}$. That is, b is *not* a mere number; but we may take it to be the mere number $\frac{v}{h}$ multiplied by $\frac{\text{feet}}{(\text{seconds})^2}$. In truth, we find out in various ways that b had better be written $2\,g$ where g is an acceleration of 32·2 feet per second per second in London.

We have, then, the rule for bodies falling freely from rest, $v = \sqrt{2\,g\,h} \ldots .$ (1).

Now, h is $v^2/2\,g$, and hence, when a body has fallen h feet, and its velocity is v, the potential energy, $w\,h$, lost by it, may be written $w\frac{v^2}{2\,g}$ or, as it is more usually written, $\frac{1}{2}\frac{w}{g}v^2 \ldots$ (2). w is the gravitational force or weight of the body in pounds, and g is 32·2 feet per second per second in London.

191. It was by experiments on falling bodies that **Galileo** was led to formulate the law that at any place a body's w/g, which we call the mass or inertia of it, is constant; and we have been led by astronomical observations to believe that the mass or inertia of a body is a constant everywhere. The gravitational force on a body, w, may vary, and g, the acceleration due to gravitational force, may vary; but the ratio between them seems everywhere to be constant. We generally denote the mass or inertia w/g by the letter m. A body kept in London, defined by law to have a weight of 1 lb. if weighed in vacuo, has a mass $1 \div 32{\cdot}2$ in the absolute units which are most convenient for engineering purposes. Any body whose weight in London is w lb. has a mass numerically expressed by $w \div 32{\cdot}2$. Any body whose weight in any kind of units is w at a place where the gravitational acceleration in any units is g, is said to have a mass w/g.

If we wish to be very exact, we notice that if we speak of a force as w lbs., the w is a mere number; if we speak of a force as w, then w is much more than a mere number. A body whose weight in London is 32·2 lbs. has a mass 1.

192. We have now seen that half the mass of a body, multiplied by the square of its speed in feet per second, is its kinetic energy. When the bob, Fig. 162, is at B, let us say that its *total store* of energy is kinetic. When it is at A, the energy is all potential. When it is anywhere between, the total amount of energy is exactly the same as before, but part is potential and part is kinetic. During the swinging of a pendulum there is a constant change going on, potential energy changing into kinetic or kinetic into potential, and the sum of these two would always remain the same only that friction is constantly reducing this sum by converting part of it into energy of another order—namely, heat.

Exercise.—Imagine no frictional resistance to the motion of a projectile. A projectile of 100 lbs. with a muzzle velocity of 1,000 feet per second. What is its kinetic energy when it is at heights of 10, 50, and 100 feet? When its whole velocity is 707 feet horizontally per second, what is its height? This being the horizontal component of the velocity throughout, what was the angle of elevation of the gun?

Ans., 1,551,000, 1,547,000 and 1,542,000 ft.-lbs. [illegible]

It is to be noticed that we assume that the potential energy of a body is *w*, multiplied by the vertical height through which it can fall (that is, to some fixed datum level), so that at a certain height we know its potential energy, and if we know the total energy, the remainder is kinetic, and this kinetic energy being always $\frac{1}{2} m v^2$, the velocity is known when the height is known.

193. Test of our Law.—We now have our rule to calculate the energy stored up in a moving body, every part of which is moving with the same velocity, and we can test it in the following way. Get a pulley (Fig. 163) as light and frictionless as possible, because we must, at the beginning, neglect both the energy stored up in the pulley itself and the loss by friction. Fasten the pulley at a considerable height above the floor. Let two equal weights, A and B, balance one another at the ends of a long silk cord, passing over the pulley; and let there be a wooden scale, close alongside which A passes as it ascends and descends. Let us be able to fix to this scale, at any place, a plate which will suddenly stop A, and, above this, a ring which will just allow A to pass through. You will find such an arrangement as I speak of in almost every little collection of apparatus in the kingdom, and it is called an **Attwood's Machine.** Now let A be as high as possible at the beginning: place on it a little weight *w*, such as will be lifted off when A passes through a ring; and place a ring so that it will lift the little weight off A when A has fallen, say, 3 feet. You know that, so long as the little weight lies on A, the speed of A downwards and B upwards must become greater and greater. In fact, the potential energy lost by the little weight becomes converted into kinetic energy of the whole arrangement. Now, as soon as the little weight is stopped, A and B move with a steady motion; and if the table is placed by trial so that one second after A passes the ring it is suddenly stopped by the table, the distance between the ring and table shows the velocity which A, B, and the little weight had when the little weight was removed. In one experiment—A being 1 lb. and B the same, and the little weight 0·25 lb. — the velocity was measured after A had fallen 3 feet, and was found to be about 4·5 feet per second. Now, the potential energy lost by the little weight was 3 × ·25, or ·75 foot-pound. The kinetic energy was stored up in 2·25 pounds, moving with the velocity of 4·5 feet per second, and, according to the above rule, its amount is

$$2{\cdot}25 \div 64{\cdot}4 \times 4{\cdot}5 \times 4{\cdot}5,$$

or ·71 foot-pound, or ·04 foot-pound too little. If we consider that there was some friction, that the pulley retained some kinetic energy, and that it was difficult to fix the table, so that exactly one second elapsed from A's passing through the ring until it was stopped, we see that the experiment is a fairly good illustration of the rule. You ought, with your own hands, to make a number of such experiments. In exercise work we shall call $A = B = W$ lb., and we shall call the little extra weight w.

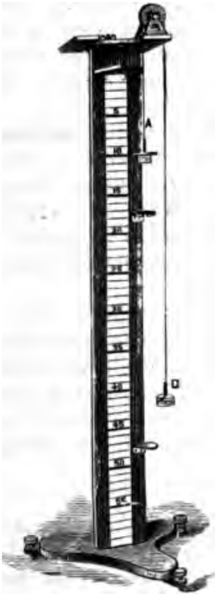


Fig. 163.

In Art. 50 we found the sort of corrections which had to be introduced by friction at the wheel pivots and by the bending of the cord. Their study led us to the great subject of friction. But in a well-made Attwood's machine, a far more important matter to consider is the energy (so far neglected) in the pulley. The experimental study of this leads us to a *consideration* of angular motion.

194. Energy in a Rotating Body.—Suppose now that the pulley is so massive that its kinetic energy is considerable, and may not be neglected, is there any way of finding from its speed how much energy it has stored up in it? We can easily calculate the energy in any little portion of a wheel if we know its velocity and mass, but those portions near the centre are moving more slowly than portions near the circumference, so that we have to calculate the energy in each little portion separately, and add all the results together. There is one thing which all portions of a wheel have in common—they all go round the centre the same number of times per minute. Suppose now that the number of revolutions of a wheel is doubled, the real velocity of every point in the wheel is doubled, whether that point be near the axis or not, so that the kinetic energy of the whole wheel is quadrupled; in fact, then, we find that the kinetic energy stored up in a wheel depends on the square of the number of revolutions which it makes per minute, so that *the energy must be equal to a constant number multiplied by the square of the number of revolutions per minute.*

195. To find experimentally how much energy is possessed by a wheel when it is rotating, let the wheel be mounted on an axle supported on very frictionless bearings. If the centre of gravity of the wheel is not exactly in the axis, then it is better to place the wheel as in Fig. 164. Now let a cord with a loop from the pin B be wound round the axle and pulled by a weight W. Suppose the weight to be 1,000 lbs., and that we only allow it to fall 8 feet from rest before the cord drops off the pin, so that when it has fallen this distance it no longer acts on the wheel, which will then rotate with a constant speed. Roughly speaking, the wheel possesses 1,000 × 8, or 8,000 foot-pounds of energy stored up in it. This is not quite true, because the weight itself possessed a certain amount of energy of motion which must be subtracted. Suppose that at the instant before being stopped the weight was moving with a velocity of 1·5 foot per second, then we must subtract

$$\frac{1{,}000}{64\cdot4} \times 1\cdot5 \times 1\cdot5, \text{ or about 35 foot-pounds.}$$

If there were no friction, and we find that a speed of 10 revolutions per minute has been given to the fly-wheel, we know that we have to find a constant number, **M**, which, when multiplied by the square of 10 or 100, will give 7,965 foot-pounds. Evidently **M** = 79·65, and hence, if ever we find this fly-wheel rotating, we know that it has stored up in it the amount of energy in foot-pounds 79·65 × square of number of revolutions per minute.

196. In the above calculation we have neglected friction; but, as a matter of fact, in experiments the **friction never is negligible.** As for the friction of the wheel itself, on a cord similar to that which you have already used, hang a small weight such as will merely overcome friction, so that when you give the wheel a jerk for the purpose of starting the motion, this weight will just suffice to prevent friction reducing the speed. Suppose this weight to be 5 lbs., then it is quite evident that 5 lbs. of the original 1,000 were really employed in overcoming friction and not in storage. Hence our calculation gives

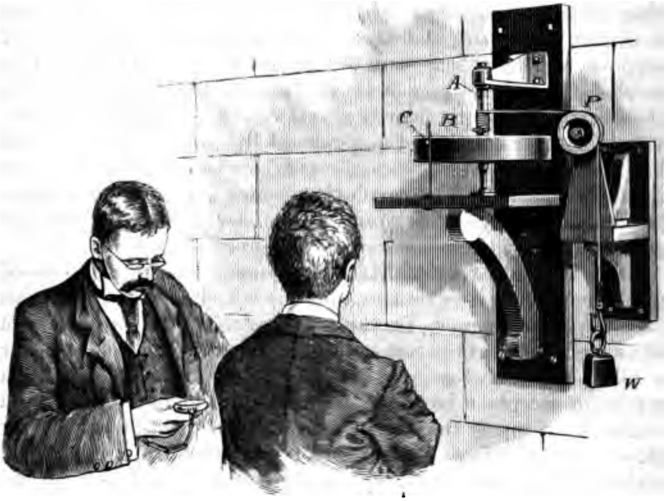


Fig. 164.

995 × 8 - 35, or 7,925 foot-pounds as the total storage.

This is at ten revolutions per minute. When it makes one revolution per minute the storage is 79·25 foot-pounds, and at any other speed we multiply 79·25 by the square of the number of revolutions per minute; 79·25 *is called the* **M** *of the wheel.*

As for the **friction of the pulley** P and the energy wasted *in bending* the rope, the pulley must be tested like the pulley of Art. 50, and a correction made on account of the two parts of the rope being at right angles to one another. The two pulleys may be tested at the same time, the cord going over both of them to two weights.

197. It is obvious that we must be pretty quick in **counting the number of revolutions** of the wheel produced by the falling of the weight. Indeed, we ought to observe, if possible, the time taken in part of one revolution, using some special form of time-measurer, because the speed will now continually decrease on account of friction.

But there is another way in which it is easy to find the speed at the instant when the weight ceases to act. Find the total number of revolutions made by the wheel till the cord drops off its pin, and let someone observe this time in minutes. Then, as we know that the speed increases uniformly during this interval of time, the mean speed is just half the speed at the end of the interval; that is, *divide the number of revolutions by the number of minutes in which they were performed, and twice the quotient will give the number of revolutions per minute made by the wheel when the weight just ceases to act.* You can test your result by counting the number of revolutions from the time the cord drops off until the wheel is stopped by its own friction and dividing by the time which elapses; twice this quotient ought also to be the speed you want to know.

198. It is not necessary even to measure the friction directly, for we found that 7,965 foot-pounds were given out by the weight in falling; now *if we count the total number of revolutions made by the wheel from the time of starting until stopped by its own friction, and divide 7,965 by the total number, we shall find the loss of energy due to friction during one revolution,* since there is just as much energy wasted by friction in any one revolution as in any other [a statement to be tested by the student]. Ten times this must be the same amount of energy as 5 × 8, or 40 foot-pounds, for we measured the friction during 10 revolutions of the wheel as equivalent to 5 lbs. falling 8 feet. This, then, is the method we ought to employ.

I know of no exercise in my laboratory which is so useful as this. A thoughtful student revives his knowledge of nearly all the important dynamical principles in making the corrections, which enable him to get nearer and nearer to the correct answer *for M.*

199. You see that **M** is a number which ought to be known for every fly-wheel; it is just as important to know the **M** of a fly-wheel as to know the weight of an ordinary body. We have only to multiply the **M** by the square of the number of revolutions per minute, and we find at once the energy in foot-pounds stored up in the wheel. I have shown you how to find the **M** of a fly-wheel by experiment; I will now give you an idea of **its value in different cases.** Imagine a **grindstone** whose diameter is 4·5 feet, whose breadth is 1·4 foot, the weight of its material per cubic foot being 132 lbs.; then we can calculate its **M** by first finding

$$132 \times 1{\cdot}4 \times 4{\cdot}5 \times 4{\cdot}5 \times 4{\cdot}5 \times 4{\cdot}5,$$

and dividing this answer by 59,800. For any rotating object of cylindrical shape, the shape of a grindstone, this rule will always find **M**. *Multiply the weight of the material per cubic foot by the breadth or width; multiply this by the fourth power of the diameter, and divide by the constant number* 59,800. Whether the material is wood or stone or metal, this will give **M**, and this multiplied by the square of the number of revolutions per minute will give the energy in foot-pounds stored up in the rotating body. For the above grindstone, on calculating out, you will find the **M** to be 1·26. So that when it makes 1 revolution per minute, there is stored up in it 1·26 foot-pound of energy; when it makes 2 revolutions per minute, there is stored up in it 1·26 × 4, or 5·04 foot-pounds.

					Foot-pounds.
At	3	revolutions,	2·53 ×	9, or	22·68
„	20	„	2·53 ×	400, „	1,012
„	50	„	2·53 ×	2,500, „	6,325
„	100	„	2·53 ×	10,000, „	25,300

200. If we fix a small weight of 20 lbs. on a wheel, at 12 feet from the axis, this adds to the **M** of the wheel the amount

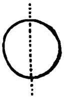
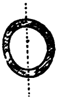
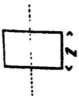
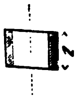
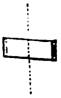
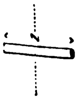
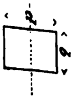
$$20 \times 12 \times 12 \div 5{,}873, \text{ or } 0{\cdot}49;$$

or the weight multiplied by the square of its distance from the axis, divided by 5,873.

If we add a very *thin* rim to a wheel, the addition to **M** is found by multiplying the weight of the rim by the square of its average radius, and dividing by 5,873.

Note.—In Table II. all dimensions are supposed to be in **feet.** See *Art. 203.*

TABLE II.

	Nature of Rotating Body.	I	M	k
	Sphere of diameter d, rotating about diameter as axis. . . .	$wd^5 \times \cdot001626$	$wd^5 \div 112{,}166$	$\cdot3162d$
	Spherical shell, whose outside diameter is d and inside is d_1, rotating about diameter as axis	$w\,(d^5 - d_1^5) \times \cdot001626$	$w\,(d^5 - d_1^5) \div 112{,}166$	$\cdot3162\sqrt{\dfrac{d^5 - d_1^5}{d^3 - d_1^3}}$
 	Cylinder, diameter d, length l, rotating about its axis . . .	$wld^4 \times \cdot00305$	$wld^4 \div 59{,}814$	$\cdot3535d$
 	Hollow cylinder, outside diameter d, inside diameter d_1, length l	$wl\,(d^4 - d_1^4) \times \cdot00305$	$wl\,(d^4 - d_1^4) \div 59{,}814$	$\cdot3535\sqrt{d^2 + d_1^2}$
	Thin rim, mean radius r of weight w . .	$\mathrm{w}r^2 \div 32{\cdot}2$	$\mathrm{w}r^2 \div 5{,}873$	r
 	Thin rod, of length l, rotating about axis through its middle point, at right angles to its length. Weight of rod w	$\mathrm{w}l^2 \cdot00258$	$\mathrm{w}l^2 \div 70{,}474$	$\cdot2882l$
 	Thin rectangular plate, rotating about axis through its centre parallel to side b, the side d being at right angles to axis. Weight of plate w	$\mathrm{w}d^2 \cdot00258$	$\mathrm{w}d^2 \div 70{,}474$	$\cdot2882d$

It will be found that if a fly-wheel has light arms and a heavy rim, as we often see on such wheels, *a fairly good approximation to its* **M** *is found by multiplying the weight of the wheel by the square of the mean diameter of the rim, and dividing by* 23,000.

Example.—The rim of a fly-wheel weighs 15 tons; its mean diameter is 20 feet. Calculate approximately what energy is stored up in it when it makes 60 revolutions per minute. Here you will find the **M** of the fly-wheel to be about 584, and hence the stored energy is 584 × 60 × 60, or 2,102,400 foot-pounds.

The mathematicians do not use our **M**. They use a quantity called **I**, the moment of inertia of a rotating body, which is numerically equal to twice the energy stored in the body when it has an angular velocity of one radian per second. See Art. 203. I give the values of **I** as well as of **M** in the above table.

In the case of any rotating body, if we imagine a fly-wheel made of the same weight and same **M** or **I**, with an exceedingly thin rim, the average radius of this rim is called the radius of gyration of the body. In fact, the mass of a body multiplied upon the square of its **radius of gyration** is its moment of inertia. The radius of gyration of a fly-wheel rim is usually taken to be the average radius of the rim.

201. Steadiness of Machines.—A fly-wheel is put upon a riveting- or shearing-machine, or other machine, because the supply of energy to the machine is not given regularly, or else because the demand for energy from the machine is irregular. The fly-wheel enables the machine to maintain a more constant speed. In calculating the proper size of a fly-wheel for any machine we must know two things: first, what is the greatest alteration of speed allowable in the case; and secondly, the greatest fluctuation of the demand and supply of energy. Thus, suppose we wish never to have the speed of the fly-wheel more than 51 nor less than 49 revolutions per minute, and that during some interval of time the fly-wheel has to give out 20,000 foot-pounds more than it receives during that time; then, although the fly-wheel will afterwards have this deficiency made up to it by some steady supply, it is obvious that its speed must diminish. We wish its speed to diminish only from 51 revolutions to 49 revolutions per minute in *this* interval of time. Now, when the fly-wheel runs at 51 *revolutions, it* has stored up an amount of energy equal to its **M** × 51 × 51; and when it runs at 49 revolutions, its store is **M** × 49 × 49, and the difference between these two ought to be 20,000. Hence, subtracting 49 × 49, or 2,401 from 51 × 51, or 2,601, we get 200; and dividing 20,000 by 200, we find 100 as the required value for **M**. *Subtract, then, the square of the least speed from the square of the greatest, and divide the greatest excess of demand or supply by this remainder; the quotient is the* **M** *of the fly-wheel.* Having found **M**, the question is, how can you tell from it the size and weight of the wheel? Find the **M** of any wheel of the same shape and material as that which you want to use. It is obvious that the *diameters of the wheels are as the fifth roots of their* **M**s.* We want a wheel whose **M** is 100. Suppose I find a wheel of the shape I wish to use whose outer diameter is 8 feet, and I calculate its **M**, and find it to be 11; then

The fifth root of 11 : fifth root of 100 :: 8 : answer.

Log. 11 = 1·0414; divided by 5 it is 0·2083, which is the logarithm of 1·615.

Log. 100 = 2·0; divided by 5 it is 0·4, which is the logarithm of 2·512. Hence

1·615 : 2·512 :: 8 : answer.

This is an easy exercise in simple proportion. I find my answer to be 12·44 feet, or 12 feet 5¼ inches, the diameter of the required fly-wheel, which is to be similar in form to the smaller specimen used by me for calculation.

202. **The total kinetic energy stored up in any machine** is found by calculating the energy in every wheel and in every

---

* If we have any two similar wheels, or other rotating bodies of the same material; if we consider any similar small portions of them; it it evident that their weights are proportional to their cubic contents, or to the cubes of any similar linear measurements. Hence, if one is, say, twice the diameter of the other, as every dimension of the one is twice that of the other, the weight of one must be 2 × 2 × 2, or eight times that of the other. Now, the **M** of any rotating body depends not merely on the weight of each portion of the body, but on the square of its distance from the axis, so that the **M** of one must be 8 × 2 × 2, or thirty-two times the **M** of the other. Similarly, if the linear dimensions were as 3 to 1, the values of **M** would be as 243 to 1 for a pair of similar wheels.

Example.—We want a wheel which will have a store of 1,000 foot-pounds when rotating at twenty revolutions per minute, and it is to be of the same shape as that of an already existing wheel, which is 4 feet in diameter, and which contains a store of 1,350 foot-pounds when running at 30 revolutions. Evidently the **M** of this second wheel is 1,350 ÷ 900, or 1·5, and the **M** of the first wheel is to be 2·5. Using logarithms, we find that the fifth root of 1·5 is to the fifth root of 2·5 as 4 feet is to 4·4 feet, the answer.

moving part, and adding all together. But suppose that in the machine there is some shaft of more importance than any other, it is usual to give the speed of this shaft only, because if its speed be doubled, the speed of every other is doubled. Thus, in a steam-engine we state the number of revolutions per minute of the crank shaft, and this tells us the speed of every part of the engine. Let, then, the number of revolutions of some such principal axle of a machine be found. If this number of revolutions is doubled, the kinetic energy stored up in the machine is quadrupled; and, in fact, *the kinetic energy stored up is equal to a certain number which can be found for the machine, and which we shall call its* **M**, *multiplied by the square of the number of revolutions of this particular axle per minute.* The **M** of *any* machine may be experimentally determined in exactly the same way as we have shown above.

If we know the **M** of any machine, then the **M** of any other machine made to the same drawings, and of the same materials, but with all its dimensions twice as great, is thirty-two times as great, because the **M**'s of the two machines are proportional to the fifth powers of their corresponding dimensions.

203. The **energy stored up** in a rotating body is equal to $\frac{1}{2}\mathrm{I}a^2$, where I **is moment of inertia** about the axis; that is, the sum of all such terms as mass of a little portion multiplied by the square of its distance from the axis. a is angular velocity in radians per second. Hence, as $a = \frac{2n\pi}{60}$, if n is number of revolutions per minute, and π is 3·1416, the energy is $\mathrm{I}\frac{n^2\pi^2}{1,800}$; so that our **M** is $\frac{\mathrm{I}\cdot\pi^2}{1,800}$. In Table II. we give values both of **M** and of **I**. In both cases w is in lbs. per cubic foot, and all dimensions are supposed to be in feet.

We ask for the advice of students in an important matter. Is it good to use the idea of our **M**, the energy stored in a body when it makes one revolution per minute? To the weaker vessel, the beginner, or the man who dislikes algebra, we know it to be useful. Indeed, to any practical engineer, however mathematical he may be, it is convenient to have such a quantity. But is its convenience overpowered by the inconvenience of having a new quantity to think of? We do not know, and we should like to receive advice.

As for the idea of moment of inertia, it comes in in this way. If we have a small mass m moving round an axis at the distance r feet with angular velocity a, its linear velocity is ra feet per second; its kinetic energy is $\frac{1}{2}mr^2a^2$ foot-pounds. Now, if we have many small masses and make this calculation, we see that every little term has $\frac{1}{2}a^2$ in it, common to them all. Hence we multiply every little *m* by [illegible] say mathematically. [illegible] energy. In any book [illegible] the calculation of [illegible] Every student who [illegible] these, and also the value [illegible] shapes as may be [illegible]

EXERCISES.

1. A fly-wheel of cast iron [illegible] section 8″ × 10″, has a volume [illegible] a cubic inch weighs 0·26 lb. the weight [illegible] inertia is, approximately, 7.540 [illegible] **M** [illegible]

The **M** of a fly-wheel is [illegible] lution per minute. This is [illegible] radians per second: so that **M** is ½ [illegible]

2. What energy will this fly-wheel [illegible] revolutions per minute? *Ans.* [illegible]

3. A gas engine has [illegible] minute: what is the indicated work [illegible]

Ans. [illegible]

4. If one-half of this is stored [illegible] revolutions per minute, what is the **M** [illegible] moment of inertia? If it is made [illegible] of Exercise 1, what is the average [illegible]

Ans. [illegible]

5. If the gas-engine of Exercise [illegible] and if there is a complete [illegible] is stored, what is the **M** of the [illegible] the same as in (4)? [illegible]

6. In Attwood's machine [illegible] applied for a height of 2 feet, what [illegible]

Ans., Total mass is [illegible] ½ × ·1025 × v^2. This is equal [illegible] $v^2 = 1{\cdot}2 \div 1.025$, or $v = 3{\cdot}42$ feet per second.

7. If in Exercise 6 the wheel weighs [illegible] which happens to be equal to the radius [illegible] cord as it goes round, we have [illegible] have 0·17 lb., or rather 0·17 ÷ 32·2 [illegible] corrected value of *v* at the time when [illegible]

Ans., The whole mass is now [illegible] that $v = 3{\cdot}34$ feet per second.

8. If the radius of the [illegible] radius of gyration of the pulley is [illegible] 0·17 lb. and radius of gyration [illegible] radius of gyration 0·24; so that [illegible] $x = 0{\cdot}13$. We therefore imagine a mass [illegible] ÷ 32·2, [illegible] to the original weight. In this case find *v*.

Ans., $v =$ [illegible] feet per second.

It is this sort of calculation, when experimenting with an Attwood's machine, which gives a man a practical knowledge of mechanics.

9. A homogeneous cylinder of mass *m* and radius *r* rolls down an

inclined plane. If the linear speed of its centre is v, show that its angular velocity a about the centre is v/r. What is its kinetic energy?

Ans., $\frac{1}{2}mv^2 + \frac{1}{2}\text{I}a^2$.

Table I. shows that its radius of gyration, k, is $\cdot 3535d$; and as $\text{I} = mk^2$, we have the kinetic energy $\frac{1}{2}m\ (v^2 + k^2a^2)$, or $\frac{1}{2}m\left(v^2 + k^2\frac{v^2}{r^2}\right)$, or $\frac{1}{2}mv^2\left(1 + \frac{k^2}{r^2}\right)$. Notice that in a cylinder $k^2/r^2 = \frac{1}{2}$; so that the total kinetic energy is $3mv^2/_4$. If the cylinder is cast iron, $r = 0{\cdot}17$ foot, its length 0·3 foot; then its weight is 12·257 lbs., its mass is 0·3808, and its kinetic energy is $0{\cdot}2855v^2$.

If this cylinder has rolled without friction along a switchback path through a vertical height of 1·5 foot, show that $v = 8{\cdot}02$.

If the path was an inclined plane, such that for a vertical fall of 1·5 feet the distance traversed is 20 feet, and if the average velocity was half the final velocity, find the time taken to reach this point. *Ans.*, 4·99 seconds.

On a plane at the angle a with the horizontal, show that the time taken for a sloping distance l feet, being $l \div \frac{1}{2}v$, is $\sqrt{3l/g}$ sin a.

10. A riveting-machine needs 3 horse-power; a fly-wheel upon it fluctuates in speed between 80 and 120 revolutions per minute; an operation occurs every two seconds, and this requires $\frac{7}{8}$ths of all the energy supply for two seconds. Find the **M** and **I** of the fly-wheel.

The energy supply for one minute is 3 × 33,000, and $\frac{7}{8}$ths of the supply for two seconds is $3 \times 33{,}000 \times \frac{7}{8} \times \frac{1}{60} \times 2$, or 2,875 foot-pounds. This is equal to $\text{M}\ (120^2 - 80^2)$; so that **M** is 0·36, **I** is $\cdot 36 \times 1{,}800 \div \pi^2$, or 65·6.

11. A machine (consisting of a grindstone and the shafting of a shop) has an $\mathbf{M} = 12{\cdot}13$. Suppose the loss of energy by friction in one revolution to be always 1,550 foot-pounds, whatever the speed. If the machine (the grindstone spindle) is making 140 revolutions per minute and ceases to receive energy, in how many revolutions will it come to rest?

Let the answer be x; then $x \times 1{,}550 = 12{\cdot}13 \times 140^2$, so that $x = 154$ revolutions. As the average speed will be 70 revolutions per minute, the stoppage will take 2·2 minutes.

12. Find the **M** and **I** of a fly-wheel which stores 25,000 foot-pounds of energy when changing in speed from 50 to 70 revolutions per minute. It is to be similar, and of the same material, to an existing fly-wheel whose **M** is 0·047. What is the ratio of their sizes?

Ans., 10·4 ; 189·7 ; 2·93 : 1.

13. If the earth described a circular path of 92×10^6 miles radius in 365¼ days, if it were a sphere of 8,000 miles diameter, of uniform density 5·6 times that of water, what is its mass in engineer's units? What is its kinetic energy because of the motion of its centre? What is its kinetic energy because it rotates on its axis once in 23 hours 56 minutes 4 seconds?

14. An engine is running at 200 revolutions per minute. Suppose that after the steam is shut off and the load removed the engine made 250 revolutions before coming to rest. Assume a constant moment of resistance, and calculate its amount if the moving parts are capable of

storing as much energy as a fly-wheel weighing 1 ton and having a radius of gyration of $2\frac{3}{4}$ feet. *Ans.* [illegible]

15. The rim of a fly-wheel is 9 inches broad and $4\frac{1}{2}$ inches deep, the external diameter being 9 feet. Find the moment of inertia of the rim. If an engine be supposed to drive this wheel, overcoming no resistance, how many revolutions will have been made before the speed acquired from rest by the wheel is 120 revolutions per minute, the diameter of the cylinder being 18 inches, the stroke 2 feet 6 inches, and the mean effective pressure 15 lbs. per square inch? What time would be required?

Ans. [illegible]

16. The fly-wheel of an engine of 4 horse-power, running at [illegible] revolutions per minute, is equivalent to a heavy rim [illegible] in mean diameter, weighing 500 lbs. Estimate the ratio of the kinetic energy in the fly-wheel to the energy developed in a revolution, and determine the maximum and minimum speeds of rotation when the fluctuation of energy is [illegible] the energy of a revolution. *Ans.* 1 : [illegible]

17. A fly-wheel is made to rotate by means of a weight of [illegible] attached to the end of a cord passing round the shaft, the diameter of which is, together with thickness of cord, $1\frac{1}{2}$ inches. Find the moment of inertia of the wheel if the weight falls a distance of [illegible] feet from rest in 1 minute, friction being neglected. *Ans.* [illegible]

204. *Exercise.*—In the Attwood's machine, if the pulley of moment of inertia I is supported on four friction wheels each of moment of inertia I_1 about its own axis, and if the spindle of the pulley has a diameter one-twentieth of the diameter of the rim of each friction wheel, show that the pulley has a virtual moment of inertia $I + \cdot01\ I_1$.

In a good Attwood's machine calculate carefully the virtual moment of inertia of the pulley. Find experimentally for any values of A and B the lessening of the preponderating force which represents the friction and stiffness of the cord. If you can work with a discarded old machine with massive wheels and much friction and hemp cords, you will find it far more instructive than an expensive, nearly frictionless contrivance. In measuring time, exercise a little ingenuity of your own in getting accuracy; but if your teacher has made everything very easy for accurate measurement, you will learn nothing. If you have read too much about other people's methods, you will learn very little. You will learn most when you try to get accuracy of measurement with very rough apparatus. Teachers who do not think will introduce roughness and inaccuracy in the wrong places; they will give you a bad timekeeper, perhaps, or an inaccurate scale for measuring distances.

After having worked with the Attwood's machine, in which the velocity ratio is 1, you will find it very interesting if you have time enough to experiment with the apparatus shown in Fig. 22, in which the velocity ratio may have any value. In the experiments described in Art. 51 you aimed at keeping speed constant. Now let there be an excess weight B, and measure the kinetic energy as with the Attwood machine. We need not here describe more fully the obviously interesting and instructive experiments which may be made.

205. In a Bull engine (*see* Fig. 166) steam-pressure in the space B, p lb. per unit area (in excess of the pressure from above the piston) acts on a piston of area A with a total upward force p A.

This causes a total weight W to be lifted (consisting of steam-piston, rod, pump-plunger, etc.). If p varies when the piston rises through a height h, the total work E done by the steam must be calculated for small changes of level, and the results added up. Thus in Table IIA. we give the pressure of the steam as the piston rises, the pressure being measured by an *indicator*. The area of the piston is 900 square inches, so that 900 p is the lifting force. It is therefore evident that the fourth column shows at each place (approximately) the work already done upon the piston when W (W = 20 tons, or 44,800 lbs.) has risen h feet. If it is rising with the velocity v feet per second, it possesses W h foot-pounds of potential and $\frac{1}{2}\frac{W}{g}v^2$ foot-pounds of kinetic energy. But we know the total E; and if we subtract W h, we know the kinetic energy, and we can therefore calculate v.

TABLE IIA.

h Feet.	p Pounds per Square Inch.	A p Pounds.	E, the total Work done in Foot-pounds.	W h.	$\frac{1}{2}\frac{W}{g}v^2$.	v.
0	100	90,000	0	0	0	0
0·5	100	90,000				
1·0	100	90,000	90,000	44,800	45,200	8·06
1·5	100	90,000				
2·0	100	90,000	180,000	89,600	90,400	11·4
2·5	80	72,000				
3·0	66·7	60,030	252,000	134,400	117,600	13·0
3·5	57·1	51,390				
4·0	50·0	45,000	303,390	179,200	124,190	13·36
4·5	44·4	39,960				
5·0	40·0	36,000	343,350	224,000	119,350	13·1
5·5	36·4	32,760				
6·0	33·3	29,970	376,110	268,800	107,310	12·41
6·5	30·8	27,720				
7·0	28·6	25,740	403,830	313,600	90,230	11·38
7·5	26·7	24,030				
8·0	25	22,500	427,860	358,400	69,460	9·98
8·5	23·5	21,150				
9·0	22·2	19,980	449,010	403,200	45,810	8·10
9·5	21	18,900				
10·0	20	18,000	467,910	448,000	19,910	5·33
10·25	19·5	17,550				
10·5	19	17,100				
10·75	18·6	16,740	481,600	481,600	0	0

206. The earnest student will work out the numbers in the above table, and draw curves showing h and p and h and v. In Fig. 165 O G H I J shows p the pressure, O F being the distance travelled by the piston and weight. O P N represents the velocity at every instant. Notice that if we divide the force pA and the weight W by A, we get p and W/A to compare. The straight line K M represents W/A. The area of O K M N is equal to the area O G H I J N; that is, W/A is the average value of the pressure until h = O N. The student must see clearly that the force increasing the velocity is represented to scale by the ordinate of K G H L I J M K, and this force becomes negative after L, so that the point L tells us where v reaches a maximum and begins to diminish. The positive area K G H L K is equal to the negative area L I J M L. It is worth while spending a good deal of time over such curves and such an investigation as this.

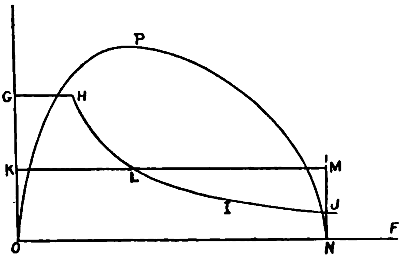


Fig. 165.

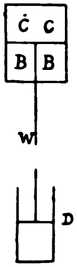


Fig. 166.

The student ought now to assume that besides the force W there is a constant force of friction (say 2,000 lbs.) to be overcome. He will therefore subtract 2,000 h foot-pounds from E, as well as Wh, to find the kinetic energy, $\frac{1}{2}\frac{W}{g}v^2$, and he will calculate v, again obtaining a new curve. In this engine, when the steam is allowed to escape, the weight W falls, and performs the actual pumping operation.

207. Should the advanced student care to take up a problem that more nearly approaches the actual case (namely, assume that the force of friction F is not constant, but is some function of the velocity), he must find out for himself some graphical method of working. Our own plan is this:—We calculate v at the end of an interval from a rough notion of the velocity, and therefore of F, the force of friction during the interval. We can now approximate more closely to the actual friction and calculate more exactly. We advise the student to proceed in this way in the above case, taking F as following the law

$$F = 500 + 100\, v.$$

When a **torque M turns a body** through the angle θ radians, it does work of the amount $M\theta$. Thus, if a turning moment of M pound-inches is acting on a shaft which revolves at n revolutions per minute, or $2\pi n$ radians per minute, it does work M (pound × inch) $2\pi n$ per minute, or $\pi n M \div 6$ foot-pounds per minute.

If the twisting moment on a shaft M is proportional to θ radians, the angle through which a certain length of shaft is twisted, then $\frac{1}{2}M\theta$ is the strain energy.

If bending moment M acting on a certain length of a beam between two cross-sections causes an angular change θ, and θ is proportional to M, then $\frac{1}{2}M\theta$ is the strain energy produced by the action of M. These two propositions enable us to calculate the resilience of many springs; that is, the energy which it is possible to store up in them.

The force F, acting through a little distance δx, does work $F \,.\, \delta x$. If F varies, a curve must be drawn showing its value for each value of x, and the work done in any distance is shown as an area or an integral. Thus, if $F = ax$, it is easy to show that the work done from $x = 0$ to any other value is $\frac{1}{2}ax^2$; or from $x = x_1$ to $x = x_2$ the work done is $\frac{1}{2}a\,(x_2^2 - x_1^2)$. Or we may put it that the work done is x multiplied by the *average* value of F, the average value being the half sum of the two extreme values.

Thus if the gradually applied load w on a beam produces the deflection y where y is proportional to w, the energy stored in the loaded beam is $\frac{1}{2}wy$.

But if there is no loss of the energy due to loading, any method of loading which during the deflection y is calculated to do the same work will cause the same deflection. One such method is to suddenly apply half the final load or $\frac{1}{2}w$. Here, as in structures generally, we have assumed that deformation is proportional to load. If the law of yielding is force $= f(y)$ where there is any curious law, and if the integral of $f(y)$ with regard to y is $F(y)$; or if in case the deformation is through an angle θ the moment $M = f(\theta)$, then the energy stored in any configuration is $F(y)$ or $F(\theta)$. If a force or moment which, gradually applied, would only produce a yielding y or θ is suddenly applied of its full amount, the yielding is y^1 or θ^1 where $y^1 f(y) = F(y^1)$ or $\theta^1 f(\theta) = F(\theta^1)$.

Thus if the righting moment M of a ship is a known function of the heeling angle θ (say $M = A \sin. a\theta$), if the moment M due to the wind produces the steady heeling angle θ, and the energy stored in the heel is $-\int_0^\theta A\,\frac{1}{a}\cos. a\theta . d\theta = \frac{A}{a}\,(1-\cos. a\theta)$, if this moment were suddenly applied, and the vessel heeled over under a *gust* of wind from o to θ_1, the work done is $\theta_1 M$ or $\theta_1 A \sin. a\theta$; and as this is all stored as $\frac{A}{a}(1 - \cos. a\theta_1)$, we can calculate θ_1, knowing θ from θ_1 $A \sin. a\theta = \frac{A}{a}(1 - \cos. a\theta_1)$. The work is easily done graphically. If the curve O P R V represents the righting moment (Q P) corresponding to the inclination θ (or T P), then a steady moment P Q produces the heel O S if the rectangular area O T P U S O is equal to the area of O P R S O. If the law is $M = A \sin. 4\theta$, the vessel cannot right herself if θ is more than 45 degrees. But if the student studies the matter, he will see that if a steady moment gets the vessel over to anything approaching 45 degrees, she must heel farther than 45 degrees. We are neglecting friction.

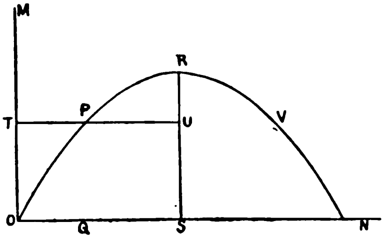


Fig. 167.

There is the same moment at θ_2 as at θ when $\sin. a\theta_2 = \sin. a\theta$; and if $\theta_2 \sin. a\theta = \frac{1}{a}(1 - \cos. a\theta_2)$, the steady wind which would keep the ship at θ will, as a steady gust acting from $\theta = 0$, heel the vessel to θ_2 and beyond. Thus, taking the above example $a = 4$,

$$\sin. 4\theta_2 = \sin. 4\theta, \text{ or } 4\theta_2 = \pi - 4\theta, \text{ or } \theta_2 = \frac{\pi}{4} - \theta,$$

$$\theta_2 \sin. 4\theta = \tfrac{1}{4}(1 - \cos. 4\theta_2),$$

$$\cos. 4\theta_2 = \cos. (\pi - 4\theta) = - \cos. 4\theta,$$

$$\left(\frac{\pi}{4} - \theta\right) \sin. 4\theta = \tfrac{1}{4}(1 + \cos. 4\theta), \qquad \cos. 4\theta = 2\cos.^2 2\theta - 1,$$

$$(\pi - 4\theta) = \frac{1 + \cos. 4\theta}{\sin. 4\theta},$$

$$(\pi - 4\theta) = \frac{\cos. 2\theta}{\sin. 2\theta} = \cot. 2\theta,$$

$$\pi - 4\theta - \cot. 2\theta = 0.$$

Calling this $\phi(\theta) = 0$, I find that $\theta = 11\frac{2}{3}$ degrees. Hence, if a wind is such that it would maintain a steady heel of $11\frac{2}{3}$ degrees; if it caught the vessel suddenly and acted steadily, and we neglect friction, the heel would be as much as $33\frac{1}{3}$ degrees; and if the wind still continued to act with the same moment, the righting moment being now less than the moment due to the wind, the vessel must go on her beam-ends. She will recover from a gust of wind less *strong than this.*

θ deg.	θ	$\phi(\theta)$
18°	·3142	0·5088
16°	·2793	0·4241
12°	·2095	·0576
11°	·1920	− ·1015

208. So long as a constant force acts it produces a uniform acceleration in the direction in which it acts. We may experiment with Attwood's machine, or simply use, as *the body acted on*, a small carriage running very freely on a very smooth level table; and *the force acting*, the pull in a string passing over a pulley on the edge of the table, and having weights in a scale-pan at its end, Fig. 168. Here, however, friction will be greater than in Attwood's machine, We can illustrate the following rule. If a force of 2 lbs. acts on a body whose weight is 50 lbs. at London (the 50 lbs. includes the weight of everything which is set in motion, so that if we use a little weight of 2 lbs. for the purpose of exerting the force, we must remember that this little weight of 2 lbs. is included in the 50 lbs.; we may take 48 lbs. as the body acted upon, and the pull in the string, which is less than 2 lbs., as the force), then the acceleration or increase of the velocity every second is equal to the force divided by the whole mass moved. In this case the mass is 50 ÷ 32·2, or 1·553, so that we have 2 ÷ 1·553, or 1·223 feet per second per second as the acceleration. Thus, if the body started from rest, the velocity would be 1·223 × 5, or 6·115 feet per second at the end of 5 seconds. And now comes the question, how far will the body move from rest in five seconds? Evidently its average velocity during this time is half its terminal velocity, or 3·058, so that 5 × 3·058, or 15·388 feet is the distance. It is evident that to get the space passed over we have multiplied half the acceleration by the square of the time.

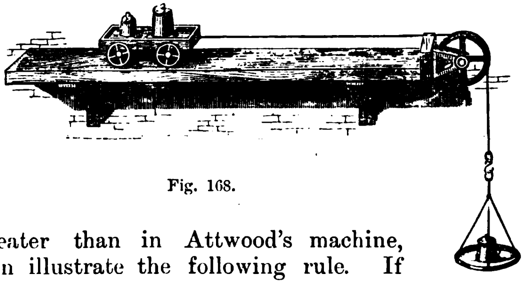


Fig. 168.

I do not suggest that the apparatus of Fig. 168 is the most suitable apparatus for use in the laboratory; there is too much friction, and it seems difficult to measure the velocity. *Attwood's* machine is used for this illustration, and is described in Art. 193. In both pieces of apparatus corrections must be made for the motion of wheels.

When a body falls freely, its own weight is acting on its own mass. For instance, say the weight is 2 lbs., then the mass is $2 \div 32{\cdot}2$, and weight divided by mass is acceleration, which we find to be in every case 32·2 feet per second per second at London. Anywhere else than in London its weight is w lbs., and its mass is, as before, $2 \div 32{\cdot}2$; and consequently its acceleration is $w \div \frac{2}{32{\cdot}2}$. But we call this acceleration g, and hence $w = 2 \times \frac{g}{32{\cdot}2}$. We see that $\frac{w}{g}$ anywhere is the same as anywhere else. Keeping to London, where $g = 32{\cdot}2$ feet per second per second, the velocity at the end of any number of seconds is 32·2 multiplied by this number; and the space fallen through in any number of seconds is half 32·2, or 16·1 multiplied by the square of the number of seconds. You can check these rules by the rules given you for potential and kinetic energy, and you will find them quite consistent with one another.

209. Momentum.—If a body's weight is 2 lbs., its mass is $2 \div 32{\cdot}2$, or ·0621. Now, if the body is moving with a velocity of 20 feet per second, its momentum is $\cdot0621 \times 20$, or 1·242. If this momentum is created or destroyed by a force acting for only one second, the force must be 1·242 lbs.; if it is created or destroyed by a force acting for 5 seconds, the force must be $1{\cdot}242 \div 5$, or ·2485 lb. The mass of a body multiplied by its velocity represents its momentum. Momentum is sometimes defined as the *quantity of motion* possessed by a body. It represents the constant force which, acting for one second, would stop the body. Suppose a certain amount of momentum is produced by a force of 1 lb. acting on a body for one second, the same amount of momentum would be produced by a force of 2 lbs. acting for half a second, or by 1,000 lbs. acting for the thousandth of a second, or by ·001 lb. acting for 1,000 seconds.

Example.—A bullet of 2 ounces, or ·125 lb., at 500 feet per second, directed towards the centre of mass of a body of 200 lbs. at rest, in which it lodges and which is free to move; what is the velocity after the impact? The momentum before impact is $\frac{\cdot125}{32{\cdot}2} \times 500$. The momentum after impact is $\frac{200{\cdot}125}{32{\cdot}2} \times$ the required velocity. Hence, the answer is $\frac{\cdot 125 \times 500}{200\cdot 125}$, or 0·312 feet per second.

Exercise.—**Chipping hammer,** $1\frac{1}{2}$ lbs.; a velocity of 30 feet per second is destroyed in the one five hundredth of a second. What is the average force of the blow?

Ans., 700 lbs. (nearly).

Example.—One hundred and twenty cubic feet of water leave the rim of the wheel **of a centrifugal pump** every minute; its component velocity in the direction of motion of the rim is 25 feet per second. What is the retarding force on the vanes at the rim of the wheel?—*Ans.*, Two cubic feet of water per second have the mass $\frac{2 \times 62\cdot 4}{32\cdot 2}$; this, multiplied by 25, gives the momentum lost by the wheel per second, which is force, and amounts to 97 lbs. If the velocity of the rim is 30 feet per second, the work done per second is 30 × 97 foot-pounds; the work done per minute is 30 × 97 × 60; dividing by 33,000, we find 5·56 horse-power. Assuming that there is no force at the inside of the wheel, the water entering radially and without shock, this is the power given to the water. If we neglect friction inside the wheel and also outside it retarding its motion, this is the total power given to the wheel itself.

What is the work done per pound of water? There are 2 × 62·4 lbs. of water per second, and the work done per second is 30 × 97 foot-pounds, so that the work done per pound of water is $\frac{30 \times 97}{2 \times 62\cdot 4}$, or 23·3 foot-pounds, or energy sufficient to lift the water to a height of 23·3 feet.

210. If the velocity of a body has been produced or destroyed by various forces, each acting for a certain time, multiply each force by the time during which it acted (each of these products is called an **impulse**), and the sum must be equal to the whole momentum generated or destroyed. When we know the *time* during which a certain force has acted on a body giving to it motion, we generally determine the motion by calculating the momentum of the body. When we know the *distance* through which a force has acted on a body giving to it motion, we generally first find the kinetic energy of the body.

211. Knowing the force F acting at any instant on the mass

M, the acceleration a is F ÷ M. Thus, suppose the following values of F to be given; the varying force acting on the mass of a body whose weight is 64·4 lbs. in London. In engineers' units its mass is 64·4 ÷ 32·2, or 2. Suppose that at time 0 the body has a velocity $v=30$ feet per second.

Time, in seconds.	F, in pounds.	a, in feet per second per second.	v, in feet per second.	s, in feet.
0	20	10	30	0
·1	20	10	31	3·05
·2	19	9·5	31·975	6·199
·3	18	9	32·900	9·443
·4	16	8	33·750	12·775
·5	14	7	34·500	16·188
·6	11	5·5	35·125	19·669
·7	8	4	35·600	23·205
·8	5	2·5	35·925	26·781
·9	2	1·0	36·100	30·382
1·0	-1	-0·5	36·125	33·993
1·1	-3	-1·5	36·025	37·601
1·2	-4	-2·0	35·040	41·163

To obtain the numbers in column 4, take an example. Suppose we know that when $t = ·4$ second from the beginning $v = 33·750$ feet per second. Now, in the next 0·1 second, the average acceleration is approximately $\frac{1}{2}$ (8 + 7), or 7·5; and in the time 0·1 the actual increase of velocity is 7·5 × 0·1, or ·75; and this is what we add to 33·750 to get 34·500 the velocity at the end of the little interval.

We warn beginners that there is no easier way than this, of getting several very important, essential ideas, and every number of such a table ought to be calculated. Now, notice that s, the space passed over, is made up by multiplying an interval of time 0·1 by the average velocity during that interval. Thus, at t = ·4, s = 12·775 feet is the distance passed through since the time t = 0. During the interval from t = ·4 to t = ·5 the average velocity is $\frac{1}{2}$ (33·75 + 34·50), or 34·125 feet per second, and the space passed through in this 0·1 second is 34·125 × 0·1, or 3·4125; and this, added to 12·775, gives s=16·1875, or, rejecting the last figure, s=16·188 at the time $t=0\cdot5$ seconds from the beginning. Note that in approximate calculations of this kind we cannot be certain of our last figures in each number. It will assist the student now to illustrate this work by curves. Plot a and t so that the ordinate of BCD is a and the abscissa is t. Anyone who thinks a little must see that the area of BCEO represents (to scale) the total increase of velocity at the time represented by OE. Add this on (taking care about the scale of measurement) to 30, the velocity at O, and we have the true velocity at the time OE. Let the velocities be found and plotted as OGHIJ. In the same way the area of the v curve must give the space s curve; that is, the area of OGHEO represents to some scale the space s passed over since the time O, and we can now show on a curve s at every instant. This is given as OKL.

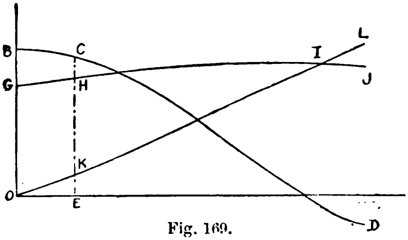


Fig. 169.

To repeat: EC represents to scale the acceleration at the time OE; EH represents to scale the velocity at the time OE; it is the area of OBCEO added to the velocity at the time O, care being taken as to the scale of the diagrams; EK represents s; it is measured as the area of OGHEO.

A student who works such an exercise as this carefully is getting all sorts of **valuable notions**, not merely of mechanics out of practical mathematics. Unfortunately, twenty academic exercises can be worked out without thought or trouble to teacher or student, and by the rules of the game this is sufficient for the passing of examinations. For the present, therefore, my advice will be followed by a few earnest students only—the men who want to know, the men who are not merely in search of examination tips, the men who find academic exercises difficult because they think about what they do. If such exercises as the above ever become obligatory on all examination candidates, of course their academic friends will discover ways of doing such exercises without the trouble of thinking about them.

212. When the resultant force F acting on a body of mass M is *constant*, the acceleration $a=F\div M$ is constant. There is no such case in Nature, but it is commonly studied. When a body falls in a vacuum in an ordinary laboratory, and there are no magnetic or electric or frictional effects, we may for all practical purposes assume that the force, the weight of the body w, is constant. The mass is w $\div g$, and the **constant acceleration** is g, or 32·2 feet per second per second in London. If the student treats this case in a table like that of p. 265, or by means of curves like Fig. 169, he will see that $v = g\,t$, $s = \frac{1}{2}\,g\,t^2$, and hence that $v^2 = 2\,g\,s$. Or if v_0 is the velocity downwards at the time $t = o$ and s is the vertical height through which the body falls from $t = o$ to any other time t, then $v = v_0 + g\,t$, $s = v_0\,t + \frac{1}{2}gt^2$, and $2gs = v^2 - v_0^2$.

If a body w lbs. falls without friction down an inclined plane, making an angle a with the horizontal, the constant force is w sin. a, the constant acceleration is w sin. $a \div \frac{\text{w}}{g}$ o g sin. a.

In any case when the acceleration a is constant, $v = v_0 + a\,t$, $s = v_0\,t + \frac{1}{2}\,a\,t^2$, $2\,a\,s = v^2 - v_0^2$.

213. I have described the units of measurement employed practically, not merely in calculation but in thought, by English-speaking people. In some parts of our work we find it necessary to calculate in a system based upon other units—the centimetre as the unit of length, the inertia or mass of the amount of stuff called one gramme as the unit of mass or inertia, and the second as the unit of time. This is called the C.G.S. system. Its advantages lie in its being used by scientific men of all countries. One of its disadvantages lies in this, that all answers to problems must be multiplied by coefficients to bring them into practical language (see p. 655).

EXERCISES.

1. A bullet takes $2\frac{1}{2}$ seconds to fall to the bottom of a well. What are the depth and the velocity at the bottom? Assume no resistance of the atmosphere.

Ans., Depth $s = \frac{1}{2}g\,(2\frac{1}{2})^2$; and taking $g = 32{\cdot}2$, $s = 100{\cdot}6$ feet. The velocity is $2\frac{1}{2}g$, or 79·5 feet per second.

2. The bullet of (1) leaves the hand with a velocity of 20 feet per second downwards. What are the two answers?

$s = 20 \times 2\frac{1}{2} + \frac{1}{2}g\,(2\frac{1}{2})^2 = 150{\cdot}6$ feet; also $v = 2\frac{1}{2}g + 20 = 99{\cdot}5$ feet per second.

3. The bullet of (1) leaves the hand with a velocity of 20 feet per second upwards. What are the two answers?

$s = -\ 20 \times 2\frac{1}{2} + \frac{1}{2}g\,(2\frac{1}{2})^2 = 50{\cdot}6$ feet; $v = 2\frac{1}{2}g - 20 = 59{\cdot}5$ feet per second.

4. How high did the bullet reach in (3)?

Ans., $v_0^2 = 2gh$, or $20^2 \div 2g = h =$ 6·21 feet.

In the above exercises time and space are measured from leaving the hand.

5. A bullet leaves o, a point on the ground, with an upward velocity of 300 feet per second. Find y, the vertical height of it, at the times $t = 0$, $t =$ ·1, $t =$ ·2, etc., seconds.

6. A bullet leaves o with a horizontal velocity of 400 feet per second, and no force acts upon it to alter its horizontal velocity. Find x, its horizontal distance from o, at the times 0, ·1, ·2, ·3, etc., seconds.

7. If a bullet has both the velocities of (5) and (6) when leaving o, plot its positions on a sheet of squared paper at the times 0, ·1, ·2, etc., and show that the path is parabolic.

8. With different horizontal and vertical components, but the same *total* velocity (500 feet per second), let the bullet of (7) leave o and again plot the path. Do this in several cases. If you knew a little mathematics, you could prove that an angle of elevation of 45 degrees will give the greatest range on a horizontal plane.

9. A force acts on a body of 8 ozs. for 6·9125 minutes, and produces a velocity of 10 feet per second. Find the force. Express it in dynes (see p. 655).

Ans., 0·000372 lb., or 165·6 dynes.

10. How far will a lateral force of 1 oz. move 100 lbs. on a smooth horizontal plane in 5 minutes? *Ans.*, 900 feet.

11. In Attwood's machine, where the weights are 17 ozs. and 16 ozs., find the acceleration and the tension of the cord.

Ans., 0·976 foot per second per second; 1 lb.

12. How long must a force of 14 lbs. act on a body of 1,000 tons to give it a velocity of 1 foot per second? *Ans.*, 5,000 seconds.

13. A rifle 5 feet above a lake discharges a bullet horizontally, which strikes the water 400 feet away. What was the velocity of the bullet?

Ans., 720 feet per second.

14. A man weighing 168 lbs. is standing on the floor of a lift. What force does he exert on it (1) when the lift is stationary? (2) when it is falling freely—that is, with an acceleration of 32·2 feet per second per second? (3) when it is descending with an acceleration of 12 feet per second per second? (4) if it is ascending with the latter acceleration? In each case indicate by a figure what are the forces which act on the man, and give the resultant force in the direction of motion.

Ans., (3) 105 lbs., (4) 231 lbs.

15. A jet of water having a sectional area of 12 square inches, and a velocity of 16 feet per second, impinges normally on a fixed plane surface. What is the mass of the water which comes on the plane per second? What is the momentum of this quantity before impact? What is the force on the plane? If the jet impinges normally on a plane surface which has a velocity of 6 feet per second in the direction of the jet, what is the velocity of the water relatively to the surface? And what is the force exerted on the surface? Find the amount of work per second necessary to maintain the jet, and the work done by it per second; and find the efficiency. *Ans.*, 2·58; 41·28; 41·28 lbs.; 10 feet per second; 16·1 lbs.; 332 ft. lbs.; 96·7 ft. lbs.; ·29.

16. The head of a steam-hammer weighs 50 cwts.; steam is admitted

on the under side for lifting only, and there is a drop of 5 feet. What will be the velocity and momentum of the head the instant before the blow is given if the fall is without resistance? If the time during which the compression of the iron takes place be $\frac{1}{80}$ second, find the average force of the blow. *Ans.*, 17·95 feet per second; 100,500; 804 × 10^4 lbs.

17. A body has its velocity diminished by one-third. By how much are its kinetic energy and momentum diminished? If this diminution was brought about by a certain constant force acting on the body through a distance of 5 feet, through what further distance would this force have to act in order to bring the body to rest? If, on the other hand, the diminution of velocity had taken place in five seconds, what additional time would be required to bring the body to rest, the same constant force still acting? *Ans.*, 4 feet; 10 seconds.

213. Force may be defined as the **space-rate** at which work is done or any form of energy converted into another, or it may be defined as the **time-rate** of transference of momentum.

We would advise students to make two sets of curves from Table IIA., p. 258. The first is given in Fig. 165. The second set

Distance. s	Pressure. p	Acceleration. a	Area = $\frac{v^2}{2}$	v	δt	t
0	100	32·5	0	0		
·5	100	32·5			·2481	
1	100	32·5	32·5	8·06		·2481
1·5	100	32·5			·1027	
2	100	32·5	65·0	11·4		·3508
2·5	80	19·5			·082	
3	66·7	10·9	84·5	13·0		·4328
3·5	57·1	4·7			·0758	
4	50	·1	89·2	13·36		·5086
4·5	44·4	− 3·5			·0756	
5	40	− 6·3	85·7	13·1		·5842
5·5	36·4	− 8·7			·0784	
6	33·3	−10·7	77·0	12·41		·6626
6·5	30·8	−12·3			·0841	
7	28·6	−13·7	64·7	11·38		·7467
7·5	26·7	−14·9			·0936	
8	25	−16·0	49·8	9·98		·8403
8·5	23·5	−17·0			·1106	
9	22·2	−17·8	32·8	8·1		·9509
9·5	21	−18·6			·1490	
10	20	−19·3	14·2	5·33		1·0999
10·25	19·5	−19·6 } −9·8			·1217	
10·5	19	−19·9	4·4	2·97		1·2216
10·75	18·6	−20 } −4·9	−0·5	0	·1700	1·3916
					1·3916	

ought to have s for the horizontal co-ordinates or abscissæ. We have seen that it was a most instructive problem, when given the a, t curve, to find the v, t and s, t curves. Now, suppose we have the a, s curve and we wish to find the v, s and the t, s curves. As a matter of fact, we have already worked out a v, s curve when given an a, s curve in Art. 211. But let us look at it from our new point of view. In the **Bull engine** of Art. 205 the force causing upward motion of W is Ap − W; the mass is W/g, and so the acceleration is $a = (\text{A}p - \text{W}) \div \frac{\text{W}}{g}$, or $a = \left(p - \frac{\text{W}}{\text{A}}\right) \div \frac{\text{W}}{\text{A}g}$. This acceleration is given in Column 3 of the table on p. 269; Column 1 shows s, and from s and a we can draw our s, a curve, which is really shown as GHLIJMK of Fig. 165, a being the ordinate *measured upwards from* KM. Now, $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$, and hence $a\,.\,ds = v\,.\,dv$, or

$\int a\,.\,ds \quad \frac{1}{2}v^2$, or twice the area of the a, s diagram up to any place is the square of the velocity.

We have, in our usual way, worked the integral of $a\,.\,ds$ numerically in the table, and we give the values of v.

Note that as $v = \frac{ds}{dt}$, or $\delta t = \delta s/v$, we get the intervals of time by dividing the intervals of space by the average velocity during the interval. Thus the interval of time from $s = 4$ to $s = 5$ feet is 1 foot $\div \frac{1}{2}$ (13·36 + 13·1), or 0·0756 second; and if we had already determined that $t = {}$·5086 for $s = 4$, we know that $t = {}$·5086 + ·0756 for $s = 5$. In this way the numbers of column 6 were obtained.

If the v, s diagram is given, and we are asked to find the a, s diagram, we notice that

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds} \quad \ldots . \ (1).$$

Therefore a is represented by the sub-normal to the v diagram. We advise no student to use the measurement of the normal of a curve for any useful purpose. It is practically too inaccurate. But from a table of the values of v and s, taken from a curve which will correct errors of observation, values of $\delta v/\delta s$, and therefore of $v\,\delta v/\delta s$ or a, may be calculated with no great error. The values of t may be obtained as in the last example, and the whole motion is known.

Of course if of the variables t, s, v, a any one is known as an algebraic function of the other, it is an exercise in the calculus to find any or all the others in terms of any one; as also the work done, or the kinetic energy, and other things.

Example.—A body has reached the earth at London from space, no other force than the earth's gravitation having acted upon it; what is its velocity? If r_0 feet is the distance of London from the earth's centre, and at any other place reached by the body if the distance is r, then g, the acceleration at r, is $g_0\frac{r_0^2}{r^2}$. Let $2g_0r_0^2$ be called b^2, then

$$g = \tfrac{1}{2}\frac{b^2}{r^2}, \text{ and } \frac{d^2r}{dt^2} \quad -g \quad -\tfrac{1}{2}\frac{b^2}{r^2} \ldots . (1).$$

$$2\frac{dr}{dt}\cdot\frac{d^2r}{dt^2} = -\frac{b^2}{r^2}\cdot\frac{dr}{dt}, \text{ so that } \left(\frac{dr}{dt}\right)^2 \text{ or } v^2 = +\frac{b^2}{r} \ldots . (2).$$

Let $\frac{dr}{dt} = 0$ when $r = \infty$, and we need to add no constant. Hence $\frac{dr}{dt} = -\frac{b}{r^{\frac{1}{2}}}$, and therefore $r^{\frac{1}{2}}dr = -b\,.\,dt$. We take the $-$ sign as we imagine the body falling. Integrating again, we have $\frac{2}{3}r^{\frac{3}{2}} = -bt + c$. Let $t = 0$ when $r = r_0$, so that all our times before reaching the earth will be negative.

$$\tfrac{2}{3}r_0^{\frac{3}{2}} = c \quad \therefore t = \frac{2}{3b}\left(r_0^{\frac{3}{2}} - r^{\frac{3}{2}}\right) \ldots . (3).$$

$$r^{\frac{3}{2}} = r_0^{\frac{3}{2}} - \frac{3bt}{2}, \text{ or } r = \left(r_0^{\frac{3}{2}} - \frac{3bt}{2}\right)^{\frac{2}{3}} \ldots . (4).$$

$$v = \tfrac{2}{3}\left(r_0^{\frac{3}{2}} - \frac{3bt}{2}\right)^{-\frac{1}{3}}\left(-\frac{3b}{2}\right) \ldots . (5).$$

It is (2) that we asked for.

Note that a body of w lbs., when it reaches r from space, has a kinetic energy $\frac{1}{2}\frac{\text{w}}{g}\frac{b^2}{r}$, or w r. We are therefore prompted to study the problem from the energy rather than the momentum point of view. Imagine a body of 1 lb. leaving the earth (say at London).

At the distance r feet from the earth's centre the weight of the body is $\frac{r_0^2}{r^2}$ if r_0 is the distance of the earth's surface from the centre. The potential energy here being v, and being v + δv at $r + \delta r$, δv = weight × $\delta r = \frac{r_0^2}{r^2}\,.\,\delta r$. Hence v $= -\frac{r_0^2}{r} +$ const.

If v$_0$ is the potential energy of 1 lb. (we are really in all this neglecting the fact that the earth moves relatively to the object) at the earth's surface, v$_0 = -\frac{r_0^2}{r_0} +$ const., so that our constant is v$_0 + r_0$. Hence v $= -\frac{r_0^2}{r} +$ v$_0 + r_0$.

The potential energy when $r = \infty$ is v$_0 + r_0$. Hence we have the easily remembered fact:—A body of w lbs. lifted infinitely high from the earth's surface would receive a store of w r_0 foot-pounds of energy if r_0 is the radius of the earth in feet; and if we imagine all this converted into kinetic energy, we see that the velocity of the body coming from rest in space would be $\sqrt{2g_0r_0}$ when $g_0 =$ 32·2. If g is the *gravitational* acceleration at the place r, the body's speed on reaching this place would be $\sqrt{2gr}$; and if we remember that $g = g_0 r_0^2/r^2$, calculation is easy. $v = \sqrt{2g_0 r_0^2/r}$.

If we write $v = -\frac{dr}{dt}$ where t is time, and if $\sqrt{2g_0 r_0^2}$ is called b, then $-r^{\frac{1}{2}} . dr = b . dt$, or $-\frac{2}{3} r^{\frac{3}{2}} = bt + \text{const.}$ Thus, if we count time from the time of reaching the earth's surface, so that it is always negative, let $t = 0$ when $r = r_0$. The constant is $-\frac{2}{3} r_0^{\frac{3}{2}}$, and so $t = \frac{2}{3\,b}(r_0^{\frac{3}{2}} - r^{\frac{3}{2}})$.

Also, as $vr^{\frac{1}{2}} = b$, $r^{\frac{3}{2}} = \frac{b^3}{v^3}$, and hence $t = \frac{2}{3\,b}\left(r_0^{\frac{3}{2}} - \frac{b^3}{v^3}\right)$, or $\frac{b^3}{v^3} = r_0^{\frac{3}{2}} - \frac{3\,bt}{2}$, $v^3 = b^3\left(r_0^{\frac{3}{2}} - \frac{3\,bt}{2}\right)^{-1}$, $v = b\left(r_0^{\frac{3}{2}} - \frac{3\,bt}{2}\right)^{-\frac{1}{3}}$.

125. Force F is rate of change of momentum M. If force F acts for time δt, it increases the momentum of a body by the amount δ M; so we can say either that $F = \frac{dM}{dt}$, or that $\int_0^T F . dt =$ whole gain of momentum if T is the time during which the gain occurs. Of course if F is constant, we have $FT =$ gain in momentum; or the whole momentum gained is the force multiplied by the time. But if F varies, we can only say that the whole gain of momentum, divided by the time, is the *average* force during the time. Here we have a time average.

Space average value of force $= \frac{\text{work done}}{\text{whole distance}}$; **time average** value of force $= \frac{\text{momentum gained}}{\text{whole time}}$.

Continually in dynamics we are considering the two great ideas of energy and momentum. On any system if a force acts from outside bodies it gives energy, and it gives momentum. If no force acts from the outside, the momentum and moment of momentum remain unaltered; and the total energy would remain unaltered were it not that other forms of energy become changed to heat, and a system loses heat by radiation. If the **earth-moon** system were alone in space, we have to consider that its moment of momentum remains constant, whereas its total store of mechanical energy is diminishing. Professor Purser pointed out that this idea gives us the past and future history of the earth-moon system, and Professor Darwin has worked it out for us.

One of the most instructive of laboratory experiments is that in which two bodies, A and B, are suspended so that they may **collide**, their motions before and after collision being measurable. The whole momentum after impact is the same as before impact; and it is very interesting to notice that when A strikes B at rest the extent of swing of the two in combination after impact is not affected by the initial rebounding and chattering and the many little circumstances which cause the *energy* after impact to be quite different from what it was before. Men who do not make experiments of this kind have no clear notions of dynamical phenomena, unless they are very exceptional (that is, men of genius).

When the momentum M, as of a pile-driver, is destroyed in the time T, M/T is the time average **force of the blow**; we have a perfectly definite idea of what is meant. But when we are told that the whole energy of the falling **pile-driver**, divided by the distance through which the pile is forced into the ground, gives us the resistance of the ground to the pile, we get a misleading academic statement. The blow is a complicated phenomenon; and even in the much simpler case of the history of the collision of two billiard-balls we are only now beginning to see how and where the total energy is converted into heat (see Art. 404).

Indicator diagrams of engines are space diagrams of force. Their average heights enable the work done to be calculated.

t	F	$-a$	$-a\delta t =$ $-\delta v$	v	$v\ \delta t$ δs $= v\delta t$	s	F . δs
0	0	0	0	8·8			
·1	49	·141	·028	8·756	1·760		86
·2	97·9	·281	·112	8·772		1·76	
·4	195·8	·562		8·687	3·475		680
·6	293·8	·844	·337			5·24	
·8	391·7	1·125		8·350	3·334		1,310
1·0	489·6	1·406	·562			8·53	
1·2	587·5	1·687		7·787	3·115		1,830
1·4	678·4	1·968	·787			11·69	
1·6	783·4	2·250		7·000	2·800		2,194
1·8	881·3	2·540	1·016			14·49	
2·0	979·2	2·812		5·984	2·393		2,350
2·2	1077·1	3·193	1·277			16·93	
2·4	1175·0	3·374		4·707	1·883		2,220
2·5	1224·0	3·519					
2·6	1175·0	3·374	1·350			18·85	
2·8	1077·1	3·193		3·357	1·353		1,460
3·0	979·2	2·812	1·125			20·22	
3·2	881·3	2·540		2·233	·893		790
3·4	783·4	2·250	·900			21·16	
3·6	678·4	1·968		1·333	·533		362
3·8	587·5	1·687	·675			21·74	
4·0	489·6	1·406		·658	·263		129
4·2	391·7	1·125	·450			22·04	
4·4	293·8	·844		·208	·083		24·5
4·6	195·8	·562	·225	– ·017		22·16	
4·8	97·9	·281	·028	·055	– ·007		– ·7
5·0	0	0	0	– ·045		22·19	
			8·844		21·892		13,635

When a mass is vibrating at the end of a **spiral spring**, the space diagram of the force exerted by the spring upon the body is a straight line. The space average of the force (neglecting the weight of the body) between the end of a swing and the mid position is half what the force was at the end of the swing; whereas the time average of the force in this interval is the fraction $\frac{2}{\pi}$ of the force at the end.

216. *Example.*—A body of 5 tons moving at 6 miles per hour; what are its momentum and kinetic energy? Find the time average of the force which will stop it in 5 seconds.—*Ans.*, The mass is 347·8; momentum, $1·97 \times 10^5$; the kinetic energy is 13466·8; time average of force = 612 lbs.

If the force increases for $2\frac{1}{2}$ seconds, and then diminishes again, in both cases uniformly with time, draw curves showing the velocity with time, and also with distance; also of force with distance. What is the space average of the stopping force?

Draw a curve showing acceleration a and t (a is negative); the integral of this shows v and t, the ordinate at $t = o$ being 6 miles per hour, or 8·8 feet per second. The integral of the v, t curve shows s and t where s is distance from where the force began to act. Now $a \times$ mass represents force. Hence the values of a represent force to some scale. Now plot a new curve showing force and s. The whole area of it is the kinetic energy. The average height of it is the space average of the force.—Ans., 621 lbs.

We have performed the integrations numerically, and have shown the results in the table on p. 273, and we have shown the various curves in Fig. 169A.

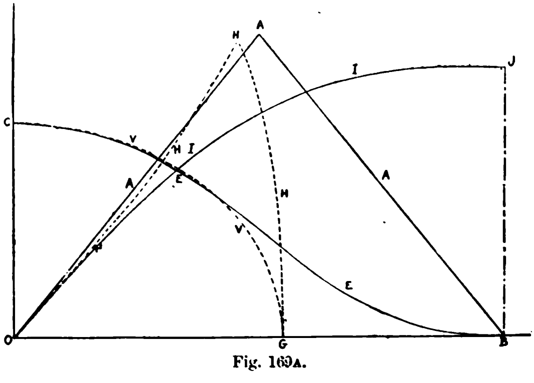


Fig. 169A.

O A A A B shows a and t, t parallel to O D.
O C E E B shows v and t, t parallel to O D.
O I I J B shows s and t, t parallel to O D.
O H H H G shows F and s, s parallel to O D.
O C V V G shows v and s, s parallel to O D.

CHAPTER XII.

MATERIALS USED IN CONSTRUCTION.

217. Mere reading will give to no student a knowledge of the properties of materials. I insist on the necessity for work in a pattern-shop and a fitting-shop and forge; and setting work in machine tools. **Workshops at a college or school** are not intended for the teaching of trades, which can only be done in real shops beside real workmen. A student learns facts about materials which are necessary for his study of mechanics; it is a secondary matter that he also acquires some skill which enables him to learn his trade quickly in a real shop afterwards. Town boys buy their toys and never come in contact with nature; country boys are always **making things** and learning much besides the properties of materials.

218. **Stone.**—The rocks which have once been melted, and have cooled slowly, are usually hard, compact, strong, and durable. They are most easily worked when regard is paid to the fact that they naturally divide up into certain regular shapes. They are all more or less crystalline in texture. **Stratified rocks** are those which have been deposited at the bottom of a sea or river; they are often easily divided in a direction parallel to the layers of which they are built up, but sometimes there are lines of easy cleavage in other directions. These rocks vary very much in appearance, according to the method of their formation, and to the heat and pressure to which they have been subjected, sometimes being very crystalline, strong, and durable, like *marble*. **Slaty rocks** may be hard and durable, or soft and perishable; they are not much used in construction, except as roof covering. **Sandstones** are hardened sand of very different degrees of compactness, porosity, strength, and durability. There are **limestones** whose particles seem to form one continuous mass, and which, when they have been subjected to great heat and pressure, become **marbles**; there are also limestones which are composed of distinct grains cemented together, and which may vary very much in compactness, strength, and durability. Besides these there are *conglomerates*, in which fragments of older rocks are imbedded. A little knowledge of **geology** is necessary in order to understand the properties of *rocks*. **Stones are preserved** by coating them

with some material such as coal-tar, various kinds of oil and paint, or soluble glass, which fills their pores and prevents the entrance of moisture. An **artificial stone**, which can be made in blocks of any required size and shape, is obtained by turning out of moulds and afterwards saturating with a solution of chloride of calcium, a mixture of clean sharp sand and silicate of soda. The chloride of calcium and silicate of soda produce silicate of lime, which cements the sand together, and thus gradually consolidates the whole mass.

219. Bricks.—Bricks are made of tempered (that is, freed from pebbles, saturated with water, and well ground and mixed) clay, moulded, dried gently, then raised to and kept at a white heat in a kiln for some days, and cooled gradually. Sand in the clay prevents too much contraction and helps vitrification. Bricks should have plane parallel surfaces and sharp right-angled edges, should give a clear ringing sound when struck, should be compact, uniform, and somewhat glassy when broken, free from cracks, and able to absorb not more than one-fifteenth of their weight of water. They ought to require at least half a ton per square inch to crush them. The published tests are sometimes much more than a ton per square inch. Probably half the published strengths are the true strengths of bricks or of brickwork. The standard brick is $8\frac{3}{4} \times 4\frac{3}{8} \times 2\frac{3}{4}$ inches. The average weight of brickwork is 116 lbs. per cubic foot. A bricklayer lays from 100 to 200 bricks per hour.

220. Limestone, when burnt in kilns, gives off carbonic acid. If pure it forms **quick-lime**, which combines readily with water, becoming larger in volume. Mixed with clean sand this forms **mortar**, which, in the course of time, hardens by losing its water and combining with carbonic acid from the air. If the burnt limestone were not pure, but contained certain kinds of clayey materials, iron, &c., it would not combine with much water, but when ground up fine, water enables its particles to combine chemically with one another, forming compound silicates with greater or less rapidity, depending on its composition. Such **cement** first *sets*, acquiring a large degree of firmness, and then more slowly and without much expansion becomes as hard as many limestones. These **natural hydraulic limestones** are not much used now. Nearly pure limestone or chalk is mixed with about one-third of its volume of blue clay to produce—when ground and mixed in plenty of water, then drained and *dried, then* burnt to incipient vitrification and ground up again very finely indeed—an artificial cement, which is equal, if not superior, to the natural cement. This is the **Portland cement** now in use. Fineness in the particles is exceedingly important. *Sand* in mortar saves expense, and prevents the cracking of the mortar in drying; coarse sand seems better than fine. Two measures of sand to 1 of slaked lime or 3 to 7 of sand to 1 of cement are the average allowances, but every person who uses mortar ought to test a particular lime or cement to see how much sand it will bear to have mixed with it. **Concrete** is a mixture of 6 of gravel or broken stones and 1 of cement.

From the time of setting the **tensile strength** of cement increases at first rapidly and gradually more slowly. The French standard is 280 lbs. per square inch at the end of seven days, 500 lbs. in 28 days, 640 lbs. in 84 days.

The initial strengths of neat cement, 1 of cement to 2 of sand, 1 of cement to 5 of sand being about $1 : \frac{1}{2} : \frac{2}{9}$; the gains of strength in a year are about as 5 : 4 : 3. Using too much water weakens cement; water about $\frac{1}{5}$ to $\frac{1}{4}$ of the weight of the cement is found to give the best results in testing. The true crushing stress of cement is probably about six times the tensile stress.

221. Timber.—A tree is made up of a great number of little tubes and cells arranged roughly in concentric circles, one circle for each year of growth, because the sap which circulates outside is checked every winter. The process of **seasoning** consists in uniformly drying the timber. As each little portion dries, it contracts and becomes more rigid, and it contracts much more readily in the direction of the circular arrangement of the tubes than it does towards the centre of the tree, and least easily in the direction along the tree. It is obvious, then, that if the tree is dried whole, there will be a tendency to splitting radially. If the tree is cut up before drying we can tell the way in which the planks will warp if we remember the above facts.

Firwoods are easily wrought, and possess straightness in fibre and great resistance to direct pull and transverse load, and are largely used because of their cheapness. They differ greatly in strength, but their weak point is their inability to resist shearing. The best of these is the **red pine** or **Memel** timber from Russia, which can be had in large scantlings, and thus used without trussing. The **white fir** or **Norway spruce** is suitable for planking and light framing, and is imported from Christiania in "deals," "battens," and "planks." **Larch** is a very strong timber, hard to work, and has a tendency to warp in drying, and is therefore not suitable for framing, but is largely used for railway-sleepers and fences, because of its durability when exposed to the weather. **Cedar** lasts long in roofs, but is deficient in strength.

The **English oak** is the strongest and most durable of all woods grown in temperate climates, but is very slow-growing and expensive. Its great durability when exposed to the weather seems to be due to the presence of gallic acid, which, however, in any wood corrodes iron fastenings; trenails or wooden spikes should be used instead. **Teak,** which is grown in the East, is the finest of all woods for the engineer. It is very uniform and compact in texture, and contains an oily matter which contributes greatly to its durability. It is used specially in ship-building and railway carriages. **Mahogany** is unsuitable for exposure to the weather, but it has a fine appearance and is not likely to warp much in drying. It is chiefly used for furniture and ornamental purposes, and to some extent in pattern-making. **Ash** is noted for its toughness and flexibility, and a capability of resisting sudden stresses of all kinds, which make it specially adapted for handles of tools and shafts of carriages. It is very durable when kept dry. It is not obtainable in large scantlings, and is sometimes very difficult to work. **Elm** is valuable for its durability when constantly wet, which makes it useful for piles or foundations under water. It is noted for its toughness, though inferior to oak in this respect, as also in its strength and stiffness. It is very liable to warp. **Beech** is smooth and close in its grain. It is nearly as strong as oak, but is durable only when kept either very dry or constantly wet. It is very tough, but not so stiff as oak. (See also Table VII.)

The **best time for felling timber** is when the tree has reached its maturity, and in autumn when the sap is not circulating. We want to have as little sap in the timber as possible, and in order to harden the sapwood, some foresters are of opinion that the bark should be taken off in the spring before felling. After timber is felled, it is well to square it by taking off the outer slabs.

Timber is for the most part dried by putting it into hot-air chambers, from one to ten weeks according to the thickness. *Even* when kept quite dry, ventilation is necessary to prevent dry rot. The circumstances least favourable to the durability of timber are alternate wetting and drying, as in the case of timber between high and low water mark, whereas good **seasoning and ventilation** are most favourable conditions. The most effective means adopted for **preserving timber** is by saturating it with a black oily liquid called *creosote.* The timber is placed in an air-tight vessel, and the air and moisture extracted from its pores as far as possible. The warm creosote is then forced into these pores at a pressure of 170 lbs. per square inch. In this way timber may be made to absorb from a tenth to a twelfth of its weight of creosote.

I shall give few numbers here for the **strength of timber.** Tests of small specimens are not to be relied upon. The time of felling, the duration of drying, the part of the tree from which the specimen is cut, and many other circumstances affect the strength. A beam will sometimes break with a long continued load only about half of what will fracture it in the ordinary way. The ultimate shearing stress along the grain of ash is about 600, oak 850, pine and spruce 300 lbs. per square inch. For bending, the average value of f in (1) of Art. 341 seems to be for spruce 5,000, yellow pine 7,300, oak 6,000, and white pine 5,000 lbs. per square inch; their Young's moduli being respectively 14×10^5, 17×10^5, 13×10^5, and 11×10^5 lbs. per square inch. The crushing strength of timber may roughly be taken as from 4,000 to 3,000 lbs. per square inch, and its tensile strength as about 10,000 lbs. per square inch.

222. Glass.—Glass is a combined silicate of potassium or sodium, or both, with silicates of calcium, aluminium, iron, lead, and other chemical substances. Certain mixtures of flint and chemicals are melted in crucibles, formed when hot into the required shapes, and cooled as slowly as possible. This may be called the devitrification of glass by slow cooling, giving rise to crystallisation. The more slowly and more uniformly the cooling is effected, the more likely is it that the glass will be without internal strains. When glass is **suddenly cooled,** as when a melted drop falls into water, the outside is suddenly contracted, becomes hard and brittle, and there are such internal strains that if the tapering part be broken or scratched at the point the whole drop crumbles into a state of dust. A blow or scratch on the thick part produces no such effect. Heating and gradual cooling destroy this property. Many peculiarities in the behaviour of metals when heated and cooled seem to be **caricatured in glass**, possibly because they are due to the fact that all the portions of matter which are about to form one crystal must be at the same temperature, and when the substance is a bad conductor of heat there is great variation in temperature. Pure metals are good conductors, but the admixture of small quantities of carbon and of gases hurts their conductivity. **Toughened glass** is the name wrongly given to the hardened glass produced by plunging glass, in a nearly melting state, into a rather hot oily bath. This glass is somewhat in the condition of the glass in a Rupert's drop. It is so hard that it is difficult to cut it with a diamond, but if the diamond cuts too deep the whole mass breaks up into little pieces. Objects made of it may be thrown violently on the floor without breaking.

223. Cast Iron.—Certain chemical changes occur when the ores of iron, generally oxides, are smelted with coke; the iron ceases to be in combination with the oxygen, and appears in metallic form, associated, however, with carbon derived from the fuel. There are usually other impurities besides, derived from the same source or from the ores. When the carbon is all combined with the iron the cast iron is "**white,**" and is very hard and brittle. When only a little is combined, and most of its particles crystallise separately, the cast iron is **grey** in colour; it is weaker and more fusible. Using the common names for the different varieties, No. 1 is darkest in colour, and from No. 4 to No. 1 there is a gradual darkening in colour. In the cupola of the foundry a little purification is effected, and it is found that the composition of a casting is from 97 to 95 per cent. of iron, the remainder being nearly pure carbon, often very largely in the combined form owing to the elimination of one of the impurities—namely, silicon. Nos. 2, 3, 4 are commonly used in the foundry, mixtures being made of them in various proportions according to circumstances. A greater proportion of No. 3 or No. 4 gives greater strength, whereas a greater proportion of No. 1 gives **greater fluidity** and a better power of expanding at the moment when the metal solidifies, so that the sharp corners of the mould are better filled. Higher numbers than 4, as 8, 7, 6, and 5—the white varieties—are seldom used in the foundry, but they may be converted into grey varieties by cooling from a very high temperature at a slow rate, but much more easily and immediately by the addition of certain brands of cast iron containing special impurities such as "siliconeisen" or "glazed pig." A special degree of fluidity and resistance to action of acids is conferred upon cast iron by the presence of a little phosphorus, but this impurity renders iron fragile at low temperatures, just as the presence of much silicon will render it weak and liable to fracture from shock. **To soften a hard casting,** it is heated in a mixture of bone-ash and coal-dust or sand, and allowed to cool there slowly.

The **density** of cast iron varies from 6·8 in dark grey foundry iron to 7·6 in white iron. Of late years cast iron has greatly improved in **strength;** this is due probably to our better knowledge. Contracts are sometimes undertaken to deliver iron of nearly 50 per cent. greater strength than the average number of our table, p. 411. The crushing fracture usually makes an angle of 56 degrees with the axis of a test column. The strengths of **little round columns** of lengths equal to from one to three diameters are much the same, but shorter columns are very much stronger, and longer columns are very much weaker. The reason for this is given in Art. 256. The **specified test** for cast iron is often this—that a bar 3 feet between supports, section 2 inches deep and 1 inch broad, should carry a middle load of 25 to 35 cwt., and will deflect before fracture 0·2 to 0·5 inch. The average ultimate shearing stress is 15 tons per square inch as tested by torsion. **Remelting improves** the strength, but not beyond a certain number of times, a tenacity of 6 tons per square inch in the pig becoming 9 tons after the second melting, and 12 tons after the fifth. This seems connected with the change of the iron from grey to white by increase of the combined carbon and decrease of silicon.

Mr. Turner has arrived at the following percentages as giving the following qualities in the highest degrees :—

	Combined Carbon.	Graphitic Carbon.	Silicon.
Softness...	0·15	3·1	2·5
Hardness	—	—	Under 0·8
General strength ...	0·50	2·8	1·42
Stiffness	—	—	1·0
Tensile strength ...	—	—	1·8
Crushing strength ...	Over 1·0	Under 2·6	About 0·8

224. **Patterns** of objects are usually made in yellow pine (sometimes of metal if many castings are wanted), about one-eighth of an inch per foot in every direction larger than the object is to be, because the iron object **contracts** to this extent in cooling. Gun-metal contracts about one-eleventh of an inch per foot, and brass about twice as much. A thoughtful pattern-maker can often greatly diminish the labour of moulding. **Prints** are excrescences made on the patterns to show in the mould where certain **cores** are to be placed. These cores are made of loam or core-sand in core-boxes, which the pattern-maker supplies; they represent the spaces in the object where the melted metal is not to flow. They are coated with a wash of charcoal dust and clay. Common casting is **green-sand**; there is the more elaborate **dry-sand** for such objects as pipes, and there is the most expensive **loam** moulding, in which the mould is built up without a pattern. You must see for yourself in a foundry what are the usual methods of **preparing a mould**: How the pattern is made so as to draw out easily; how the surface of the sand is blackened; how the moulder arranges his **vents** to let gases escape from the more compact parts of the sand; how he places his **gates** to let the metal run into the mould with just enough rapidity and yet without hurt to the mould. You must also see for yourself, taking sketches in your note-book and making a drawing of the **cupola**, how the pig iron i melted and poured into the moulds; how the moulder stand moving an iron rod up and down in one of the *gates*, producin just so much circulation and eddying motion in the melted iro as is likely to remove bubbles of gas which may otherwise unable to escape from the sides and corners of the mould, well as to prevent the formation of cavities by shrinkage "piping": how in some castings he exposes to the air cert parts which would otherwise cool too slowly for the rest of object; how next morning he *screens* his sand and wets You ought to observe the appearance of the castings bef and after they are cleaned up next morning.

225. **The Cooling of Castings.**—The most important mat in connection with **moulding** is that there shall be the s amount of contraction at the same time in every portion o mass of metal as it cools: otherwise, when finished, there m **internal strains**, which very much weaken the object, and produce fracture. In designing the shape of an object to be cast, care is taken that when a thin portion join

one it shall do so by getting gradually thicker, and not by an abrupt change of size. The thin piece exposes more surface, and cooling is effected through the surface. The **thin rim of a pulley** cools sooner than the arms, and becomes rigid sooner; when the arms cool they contract so much as sometimes to produce fracture near the junction. In a **thick cylindric object** the outer portion becomes rigid first; now when the inner portion contracts it tends to make the outer portion contract too much, and the outer portion prevents the inner from contracting so much as it ought to, so that the outer portion retains a compressive strain, and the inner a tensile strain. When a hollow cylinder is cast, and is required to withstand a great **bursting pressure**—that is, all the metal is required to withstand tensile tresses—it is usual to cool it from the inside by means of a *metal core*, in which cold water circulates. The inside now becomes rigid sooner, the outer portions as they solidify contract, and tend to make the inner portion contract more than it naturally would, and there is a permanent state of compressive strain inside and tensile strain outside in the object, which materially helps it to resist a bursting pressure. This inequality of contraction and production of internal strains in objects cause them to vary in their total bulk as compared with that of their patterns, but it is probable that some of this variation is due to the fact that the contraction of grey cast iron is only 1 per cent. of its linear dimension, whereas white cast iron contracts 2 to 2½ per cent. The fractional difference between **size of pattern** and the finished object varies from one-twenty-fifth of an inch per foot in small, thin objects to one-eighth of an inch per foot in heavy pipe castings and girders. Small castings seem to be considerably stronger than large ones. As there is always great inequality in the rate of cooling of a casting near **a sharp corner**, internal strains may be expected here, and also an inequality in the nature of the cast iron, since the grey variety gets whiter the more rapidly it is cooled. In nearly all bodies a re-entrant corner is a place of weakness (see Art. 303), and is specially to be guarded against in castings. Crystals of cast iron and other metals group themselves along lines of flow of heat. When a plate or wire of iron or steel is rolled or pulled, the crystals become more longitudinal, and the wire or plate becomes stronger, whereas annealing allows the crystals to arrange themselves laterally, and the material is weakened.

It is said that time gradually reduces some internal strains. Castings which have been rapidly cooled by being cast in an iron mould (painted on its inside with loam) are *white*, and very hard in those parts which lie nearest the mould, whereas they are grey and strong inside. These, like the hollow cylinder above mentioned, are called **chilled castings**. When a cleaned casting, preferably of white iron, is put in a box, surrounded with oxide of iron (hematite iron ore or rolling mill scale), and kept at a white heat for a length of time (say a week), its surface, to a depth dependent on the time, loses its carbon and becomes pure or wrought iron, which is much tougher than cast iron. The teeth of wheels are sometimes heated in this way. Such are **malleable castings.** Malleable cast iron seems to be of 50 per cent. greater tensile strength than cast iron, with a contraction of 8 per cent. before fracture; it stands about 60 tons per square inch crushing stress. Melted cast iron possesses the property of dissolving pieces of wrought iron, and is then said to be **toughened cast iron.** When sufficient iron is so added it becomes an inferior variety of cast steel.

226. Wrought Iron.—Cast iron is exposed to the air in a melted state for a long time, and the carbon is burnt out of it. The pig-iron really undergoes two processes, one called *refining*, the other **puddling**. It is then hammered and rolled, when hot, into bars of various shapes. The quality of wrought-iron bars as bought in the market varies greatly. We have *common* iron, used for rails, ships, and bridges; *best*, *double best*, and *treble best* Staffordshire iron, used for boilers and forgings generally; **Lowmoor**, Bowling, and other good irons for the most difficult forgings; and, lastly, **charcoal iron**, which is nearly pure. Iron is softer and more ductile the purer it is. Traces of sulphur make it **red-short,** difficult to work hot. Phosphorus has effects like carbon, but also makes the iron cold-short, or treacherous cold. **Maganese and silicon** seem also to act like carbon, but we are still afraid of them. There is usually $\frac{1}{4}$ to $\frac{1}{2}$ of 1 per cent. of manganese present, and one-third of these amounts of silicon if castings are wanted. Up to the temperatures of ordinary boilers, the tensile strength of iron is not much diminished by heating, but **at a red heat** it is very much less than in the cold state. **Repeated forging** increases the strength of wrought iron up to a certain number of times, after which it diminishes the strength. This is why small rods and small forgings and the outside of large forgings are respectively of stronger material than large rods and large forgings and the inside of large forgings. Cuts of metal in certain directions from heavy forgings seem surprisingly weak. **By rolling and hammering when hot,** iron gets a fibrous texture, and becomes more tenacious. By hammering when cold, or by long continued strains of a vibratory kind, wrought iron changes its fibrous and tough for a crystallised and more brittle condition. This brittle condition may be removed by heating and slowly cooling (annealing). Iron wire is stronger the thinner it is. Bar iron is generally stronger than angle or T-iron, and this again than plate iron. The toughness of an iron bar is best shown by the contraction it undergoes before it breaks. The section of a very tough bar may contract as much as 45 per cent. in area. **Case hardening** of a wrought-iron object is effected by heating it in a box with bone dust and horn shavings. The iron absorbs carbon, and is partially converted into steel.

227. Steel.—Steel contains less carbon and impurities than cast iron, and thus lies intermediate between cast iron and wrought iron. Expensive steel is produced by giving carbon to wrought iron (the best Swedish charcoal iron), keeping the iron heated for some days in contact with powdered charcoal, and then hammering it, whilst hot, till it is homogeneous (shear steel), or else (and this is the most usual practice) casting it when melted into ingots.

Cheap steel is produced by taking only a portion of the carbon from very pure varieties of cast iron by a puddling process such as is employed in the production of wrought iron, or by the **Bessemer** process. In the Bessemer process, air is forced into the melted cast iron for a time, and very pure white cast iron is then added to help in removing bubbles of gas. In Art. 235 I shall speak about the **tempering** of steel. Many varieties of soft steel do not harden when suddenly cooled. Some of it is like pure iron almost, and some of it has half as much carbon as the hardest cast steel. It differs from wrought iron in having been melted, and so there are no **minute streaks of slag** giving heterogeneousness; and so a plate is nearly as strong one way as another. The sudden hardening of steel when rapidly cooled seems greater the more carbon there is, up to a certain limit. It is more fusible than wrought iron, and much success has been met with in the production of **steel castings** in spite of the fact that unless certain precautions are taken, and the addition of silicon, aluminium, etc., there is a tendency to contain cavities. This steel has about twice the strength of cast iron. Annealing is necessary after casting. The strength of crucible cast steel is greater than that of any other material, and is greater as it contains more carbon and is harder. The properties of steel depend so much on so many seemingly small things—small impurities, a little too much heating or variation in the rate of cooling at different places—that great care must be taken in working it. By the Bessemer, Basic, and Siemens processes great quantities of steel are produced cheaply, containing small percentages of carbon. This steel has largely replaced wrought iron in its use in locomotive rails, bridges, and ships. Taking it that crucible steel has elastic limits of stress 26 to 20 tons per square inch, and ultimate stress 52 to 34 tons, and contraction of section 5 to 20 per cent. before fracture, these are for Bessemer steel 17 tons, 34 tons, and 20 to 45 per cent. Steels to be easily welded together ought to be of the same kind.

As steel cools from a welding heat it passes through the "blue heat" stage (about 300° C.), where it is brittle. If hammered or bent in this state it is permanently injured, and if the work was local in a large plate the plate becomes treacherous. Soft steel and iron seem to get a little stronger at **very low temperatures.**

The amount of carbon present in steel does not seem to affect much the **Young's modulus,** which in all kinds of steel and wrought iron seems to be about 3×10^7, as the modulus of rigidity is not very different from 12×10^6. Various numbers are given in Table III., in deference to custom. The carbon produces other effects, shown in the following table :—

% Carbon.	Elastic limit stress.	Breaking stress.	Contraction of area.	Ultimate shearing stress.
·14	18	28	·50	22
·51	22	36	·25	26
·96	31	53	·10	37

Stresses are in tons per square inch. A formula has been given : Strength $= 19{\cdot}5 + 11{\cdot}4\ C^2 + 30^2 + 11{\cdot}4\ Mn + 9{\cdot}5\ P$, *and* elongation per cent. $= 42 - 36\ C. - 5{\cdot}5\ Mn - 6\ Si$, where

C, Mn, P, and Si represent the fractional amounts of carbon, manganese, phosphorus, and silicon in the steel.

The oxide and silicate skin on cast iron is less liable to corrosion than clean iron, but it is advisable to paint or varnish all iron exposed to oxidation. Sometimes, as for water-pipes, the iron is heated to 150° C., and placed in pitch with some oil in it at 100° C. Mr. Barff keeps the iron surface exposed to superheated steam at a high temperature, and this coats it with a film of protecting oxide. Iron is "galvanised" by putting it in a bath of melted zinc. Boilers are sometimes protected inside by the contact of blocks of zinc.

Some **alloys of iron and manganese** are very strong, and so hard as to prevent their being readily tooled. They can, however, be both cast and forged. They are specially important to **electrical engineers**, as they are not magnetic. The power of a small trace of manganese to destroy all the magnetic properties of iron is remarkable. Certain alloys of **iron and nickel** are not quite so hard, but they are extremely tough and strong. Steels containing a little **tungsten** or **chromium** have the special properties that fit them for self-hardening tools or projectiles.

Mitis castings are of wrought iron, to which 0·5 to 1 per cent. of aluminium has been added to lower the fusing point. The tenacity seems to be 20 per cent. greater than that of wrought iron, and the ductility about equal to that of wrought iron.

228. Copper is noted for its malleability and ductility when both hot and cold, so that it is readily hammered into any shape, rolled into plates, and drawn into wires. When cast it usually contains much oxide and many cavities, but when pure it may be worked up by hammering into a state of great strength and toughness, whereas slight traces of carbon, sulphur, and other impurities necessitate its being refined to do away with its brittleness. The brittleness produced by hammering when cold is very different, as it is removable by annealing. Phosphorus is sometimes used to assist casting, and the strength is greater with more phosphorus, whose function seems that of reducing the oxides. Copper is an expensive metal, and is only used now for pipes which require to be bent cold, for bolts and plates in places where iron would be more readily corroded, and for electrical purposes. Its tensile strength is more reduced by heating than that of iron.

Boron seems to affect *copper* as carbon does iron, the wire alloy standing 22 to 27 tons per square inch without loss of conductivity.

229. **Brass** consists of about two parts by weight of copper to one of zinc, with a little tin and lead. The copper is first melted and the zinc added, not long before casting. It is used chiefly on account of its fine appearance and the ease with which it can be worked. Cheap brass things have more zinc usually. The small amount of lead added in melting makes it much softer. **Muntz metal** contains more zinc than ordinary brass—3 copper to 2 zinc, or 2 to 1 with a little lead. Like copper, it is not much weakened by heating up to 260° C.; it can be rolled hot. It is used for sheathing ships and for the tubes of boilers.

230. **Sterro** and **Delta metal** are brasses to which iron has been added, the latter name being given to the metal after it has also received special mechanical treatment, the rods being made by being forced to flow under great pressure through dies. Delta metal can be worked hot or cold, and may be brazed. **Bronze** and **gun-metal** are alloys of copper and tin in varying proportions, more tin giving greater hardness. Twelve of copper to 1 of tin seems best for guns. Five of copper to 1 of tin is the hardest alloy used by the engineer in bearings, but bell metal is 3·3 to 1. A slight addition of zinc helps in casting, and increases the malleability. A great many experiments have been made on bronze. Its strength depends very much upon the care taken in mixing and melting the metals, for it is easily injured by oxidation. Gun-metal is a good material for castings, which, however, should be quickly cooled to give more uniformity, density, strength, and toughness; and hence they are sometimes made in cast-iron moulds. Hard bronze is much used for the bearings of shafts. There are also various **soft alloys** of copper with lead, zinc, tin, and antimony, which are used for this purpose. **Phosphor bronze** is an alloy of copper and tin to which some phosphorus has been added. It bears re-melting better than gun-metal, and its properties may be varied at will. It may be either strong and hard, or weaker but very tough. The phosphorus appears to act by removal of the oxide of tin, and to tend to prevent segregation in cooling; and, indeed, we may say that when care is taken against oxidation the difficulties in the way of re-melting gun-metal vanish. Hard wire has broken with from 100 to 150 tons per square inch, and after annealing has stood half these

amounts. It has been used successfully in **railway axle** and **crank shaft** bearings. It is good in resisting shocks, and has been used instead of steel for chisels in **powder factories.** Gun-metal slowly decreases in strength as it is heated, until at a certain temperature its strength is suddenly halved, and there is almost no ductility. Phosphor bronze is less affected.

231. **Manganese bronze** and **Silicon bronze** are special alloys of copper with manganese and with silicon, the former made by adding ferro-manganese to bronze or brass, and used where unusual strength and power (as for **screw propeller blades**) of resisting sea-water are required. It may be cast and also forged. The manganese seems to act like phosphorus in clearing off the oxide. Some containing zinc can be forged and rolled hot. Silicon bronze has fair electric conductivity, and resists atmospheric corrosion when used as telephone wire, and is of great strength. As tested for tension in the condition of wire for telephonic purposes, it seems to stand from 30 to 50 tons per square inch; the drawn copper wire for the same purpose standing about 29 tons to the square inch; phosphor bronze, 44 to 70; brass, 25; German silver, 30; iron, 57; Martin steel, 86; and crucible steel, 102 tons per square inch. Pianoforte wire sometimes stands 150 tons per square inch.

232. **Aluminium bronze** is formed of 9 parts of copper and 1 of aluminium, and has a tenacity of 43 tons per square inch. The 5 per cent. alloy stands 30 tons per square inch. Copper with only 2 to 3 per cent. of aluminium is stronger than brass. The usual alloy is of the colour of gold, but, like all aluminium alloys, must be prepared from materials free from iron. When advantage is to be taken of the lightness of aluminium, there is an alloy of 32 parts of aluminium and 1 of nickel that can be employed, and to which 1 part of copper may also be added. Specimens have broken with 45 and 50 tons per square inch tensile stress. the first with no elongation and the second with 33 per cent. elongation. The aluminium and silicon bronzes and alloys of silver and other metals with aluminium are produced in an electric furnace, the ores being mixed with retort carbon and an electric current passed.

About 7 per cent. each of **copper and antimony** being added to tin, we have **white metal** or Babbitt's metal, which fuse easily and may be cast inside a bracket, round a journal, as the step of a bearing if not too large.

CHAPTER XIII.

TENSION AND COMPRESSION.

233. THE following chapter on the **behaviour of material** when subjected to tension and to compression has so much to do with the physical properties of matter, that students ought, before reading it, to refresh their memories in regard to the simpler principles of **chemistry** and **physics** and the notions which laboratory work in these subjects gives to us in regard to the probable **molecular condition of matter**, and also of that very much larger coarse-grainedness which we sometimes call heterogeneity. Besides giving us general notions, chemistry gives us useful facts as to the changes which occur in the manufacture of metals, the cause of the rusting of metal, the burning of fuel, etc. A little knowledge of **electricity** enables us to have clear ideas as to the action by which when two metals touch, and are also connected by liquid, one of them rapidly corrodes and the other does not, and how it is that oil preserves a polished metal surface. A little knowledge of **heat** tells us how friction wastes mechanical energy; how heat energy is measured; when a body is heated how much it expands; the laws of expansion of gases; the properties of steam; the laws of flow of heat in conduction and radiation, and other phænomena which continually influence our mere mechanical work.

234. It will be found by students who read what is in the smaller printing in this chapter that the usual statements on which we base our mathematical calculations are very incomplete. The manufacturer depends on many curious properties of materials, many of which are familiar to and helpful to inarticulate **workmen** and unknown in the laboratory as yet. Some rude processes and shop beliefs, which sometimes seem to be no more worthy of attention than superstitions, have suggested scientific experiments and new industries. Some phenomena that seem curious at first sight are really easily explained—the behaviour of **James Thomson's overtwisted shaft**, for example, explains many curious things; and our knowledge of such things as how initial strains are induced in castings by unequal cooling has helped us greatly in systematising our notions.

235. There is one phenomenon which has been known almost

since iron was discovered by man—namely, that **steel hardens** with sudden cooling, and we seem to be almost as far from understanding it as our ancestors were. There has been some advance, for we have no such notions about incantations and the virtues of particular kinds of water for quenching the heat as were held in the Middle Ages. Mild steel is almost like pure iron, and does not harden, but with more carbon (over 0·4 per cent.), such as there is in cast steel, the more rapidly the steel is cooled the harder it gets. So that, for example, thin pieces of steel are apt to be harder than thick pieces. The more carbon a steel contains, the less need it be heated before being suddenly cooled to acquire a particular hardness. **Re-heating diminishes the hardness**, or, as it is called, tempers the steel; and the higher the temperature of re-heating, the softer does the steel become. It may roughly be taken that sudden cooling in oil doubles the proof tensile strength of the material—that is, the stress which would permanently hurt the material.

In every workshop the common method adopted for **tempering a fitter's chisel** is as follows:—Heat the chisel to a dull red colour, put the edge in water to a distance of say half an inch, so that it may become very hard; quickly brighten the edge with pumice or a file; watch it till, as it heats by conduction from the thicker portion, you know that a certain temperature has been reached by seeing a certain colour (purplish-yellow for a chisel) of oxide of iron making its appearance. When this colour appears, plunge the whole chisel into water. Thus the steel is first made extremely hard at its edge, and is then brought back to the required degree of hardness by re-heating up to a certain temperature and then cooling. This simple process is in common use. In tempering other objects sometimes much greater care must be taken, since it is often necessary that every portion of the object shall be of the same hardness, and in such cases the whole may be cooled at first and then re-heated in a bath of oil, mercury, or other melted metal whose temperature is definitely known. The effect is of the same kind, however, whether the process is the rough one which I have described or a more careful one.

There must be no attempt to **make large objects glass-hard**: they would cool very unequally and might fly to pieces or develop flaws; a less rapid cooling in hot oil or melted lead tempers such objects in one process. It is interesting to see that one or other *of the above two* principles is carried out in all sorts of industries, but in a great number of different ways. A certain size of **watchmaker's drill** is stuck when at a red heat into sealing-wax. This gives the right temper. Another smaller drill is merely waved about in the atmosphere to cool it. The **tempering colours of steel**, beginning with the hardest, are:—Straw colour to yellow, (this is about 220°C) for light turning tools, milling cutters, screw-cutting dies and taps, punches, chasers; straw to purplish-yellow, rimers, wood-chisels, plane-irons, twist-drills; light purple to dark blue, augers, chisels for steel, axes, chisels for cast iron, chisels for wrought iron, saws for metal; less dark blue (this is about 320°C), screw-drivers and springs. These colours are supposed to indicate fairly exactly the temperature, irrespective of time; but we cannot say that there is conclusive evidence yet that time produces no effect on the thickness of the film of oxide. If true, it is a very curious phenomenon. But surely it cannot be true!

236. The volume gets greater in hardening. Curiously enough, I have a note saying that steel becomes denser by hardening, but its authority is unknown. Repeated hardening and annealing seem to strengthen the steel. It is usual to explain the hardening by saying that in sudden cooling the particles of iron and carbon have not had time to get into their natural positions when cold, and that they jam one another somehow, getting into positions of instability. As regards the influence of impurities, of gases from the atmosphere which may possibly be suddenly imprisoned among the particles, very little is known. It may help towards an explanation to say that Abel found that in annealed steel the carbon is in the form of a **chemical carbide** Fe_3C mixed in the mass. In hardening, the formation of the carbide is prevented (just as suddenly-cooled gases remain dissociated). At various tempers we have various proportions of the carbide, but it is always the same kind of carbide.

Speculation as to the molecular constitution of iron does not yet seem to have sufficient facts to go upon. It is sometimes assumed that there are two kinds of iron mixed together, the soft α particles and the harder β particles. With slow cooling from a high temperature, when the mass is soft, although the particles may be hard, the β particles (practically all are of the β kind at a high temperature) change to α. That there is some great molecular change even in the purest iron is evidenced by recalescence and other allied phenomena. In sudden cooling most of the β particles have no time to change. Any effect due to the carbon is produced at a much lower temperature than that at which the change from β to α occurs in slow cooling; and although the presence of carbon seems necessary to the hardening of steel, changes in its mode of existence are not of much importance. The α particles are changed to β in the plastic condition of iron in the ordinary testing operations at low temperatures.

237. How a pull is exerted.—How is it that a cord transmits force from my hand to an object when I pull the object by means of a string? If you study this matter, you will see that every particle of the string coheres to the next; and although the refusal of one particle to come away from its neighbour might easily be overcome, there are so many of them to be separated at any particular section of the string that it requires a considerable pull to perform this operation. When a string is pulled it really lengthens a little, and it lengthens more the more force is applied, although it may not break. A string is not so easy to experiment with as a wire of metal, because we find that it differs more in its quality at different sections, and it is affected by dampness and many other circumstances. No doubt it is also difficult to obtain a metal wire which shall just be as willing to break at one place as another—that is, which shall be exactly of the same material everywhere; but metal wire is certainly more uniform than string.

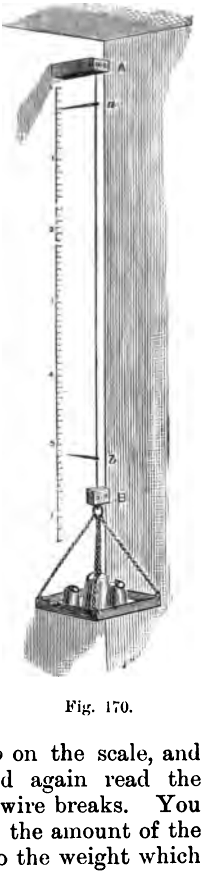


Fig. 170.

238. Strain.—Take, then, a steel wire, A B (Fig. 170), fastened near the ceiling at A, between two pieces of wood, screwed together firmly so that there may be no tendency for the wire to break just at the fastening. Similarly fasten at B a scale-pan arrangement, and first place just so much weight in the pan as keeps the wire taut. Let there be two light little pointers stuck or tied on at *a* and *b*, and let there be a vertical scale on the wall. Now read off the distance between *a* and *b* on the scale, and note the weight. Add more weight, and again read the distance, and continue doing this until the wire breaks. You will prove by means of **squared paper** that the amount of the extension of a wire is nearly proportional to the weight which produces the extension.

239. My students are in the habit of using a more exact method of measurement, a scale hanging from A and two verniers being attached to the wire at *a* and *b*. They also use a method in which a vernier on *a* is brought close to a scale on *b*, the wire being passed over pulleys. In some cases they use micrometer methods of measurement for greater accuracy, and they also experiment with larger specimens, loading them by means of levers or wheels and screws; and advanced students may be allowed to use machines in which loads up to 100 tons are applied, and arrangements may be employed for making automatic records of the load and the extension. If, however, the apparatus is elaborate and imposing as compared with the specimen, a beginner cannot readily pick up the essential idea of an experiment, and hence he had better begin with visible specimens loaded with visible weights. He may proceed to use such a machine as Bailey's wire-testing machine, and afterwards make a few tests with large commercial testing-machines.

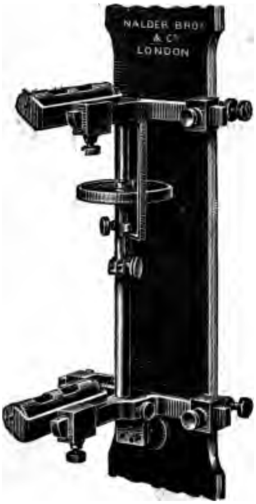


Fig. 170A.

Bailey's latest form of machine is shown in Fig. 171. The specimen, say of ¼-inch wire, is shown at D, being gripped at C and B. By turning the handle, A, we turn a worm driving a worm-wheel, turning a screw whose nut is part of the frame, and so the gripping piece B pulls on the specimen D. The other gripping piece, C, tilts the weight, F, and the amount of tilting which measures the pulling force is indicated by a pointer on the dial, E.

In some English engineering laboratories experimenting with a steam-engine and testing specimens in tension by means of a large testing-machine are supposed to be the only experimental exercises in which students ought to engage. Tests of specimens in compression, bending, and twisting are, however, sometimes made. Consequently a description of all the kinds of large testing-machines which have ever been constructed forms a large part of the college instruction in mechanical engineering. It seems to me, however, somewhat out of place in a text-book, as almost every student has opportunities of examining some such machine for himself. Complete information will be found in a paper by Professor Kennedy read before the Institution of Civil Engineers (Proceedings I.C.E., vol. lxxxviii., 1889).

The machines in most common use apply tensile load to the lower end of a test-piece by means of an hydraulic press. The upper end is pulled by means of a lever (whose fulcrum is a knife-edge), over which a weight may be rolled by machinery into such

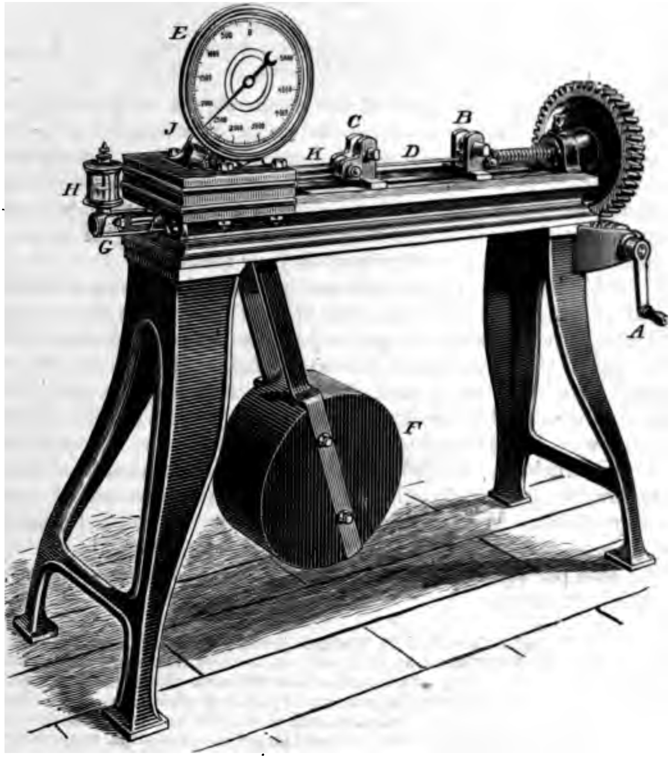


Fig. 171.

positions that the lever is kept horizontal; the position of the weight measures the pull.

Instruments have been designed which register on a sheet of paper (as the pencil of a steam-engine indicator does) the load pulling a rod, and the extension which it produces. A little brass cylinder covered with paper is touched by a pencil on the end of the rod. The amount of rotation of the barrel is regulated so that it is proportional to the load. By this means curves like that of Fig. 172 may rapidly be drawn as the load on the rod is gradually made to increase till the rod breaks (see Art. 244).

When Young's modulus for a material is wanted, my advanced students employ Prof. **Ewing's extensometer.** A half-inch bar of iron being given, even a load of 40 lbs. causes sufficient extension of the distance between two marks about a foot asunder to be measured with sufficient accuracy for the determination of Young's modulus by this beautiful instrument, shown in Fig. 170A.

240. When we speak of the tensile **strain** in the wire, and want to use the term *strain* in an exact sense, we mean *the fraction of itself by which a b lengthens.* Thus, suppose that *a b* was 50 feet and that it lengthens 1 foot, we say that the strain is $\frac{1}{50}$, or ·02, or 2 per cent. I need hardly tell you how important it is to learn the exact meaning of a word like this ; it will give clearness to your ideas.

241. Stress.—If you take another wire of the same material, but of twice the sectional area of this one, you will find that it needs twice as much load to produce the same strain. The reason of this is somehow due to the fact that you have at any section twice as many particles of steel resisting the pull. The pull produced by the load acts at every cross-section in the same way, no matter how long the wire may be ; * but if the wire is thicker at one place than another, then at such a cross-section the pull is distributed over a greater number of pairs of particles. We see, then, that if a wire or rod is transmitting a pull, it is well not to consider the total load, but rather the load per square inch of section. *The load per square inch is called the stress.*

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* One important result of St. Venant's investigation (see Art. 306) is that the **actual distribution** of applied force on a small area is **not important.** At a point not very near, the strain will be the same whatever the distribution of load. Near the gripping-places and places of rapid changes of section in our specimens, the stress cannot be expected to be uniform throughout a cross-section ; hence **test-pieces** are made larger near the grips, as we do not wish to break or study the specimen at a place where the distribution of stress is unknown to us. This is particularly noticeable at a **narrow neck** cut in a round specimen. There is great stress near the ends of the neck, inducing fracture there more readily than elsewhere if the material is hard, but in more ductile material inducing flowing of the metal, which prevents the section of the neck becoming as small before fracture as a long specimen would become, and so increasing the apparent strength of the material. This causes a **strip of boiler-plate** with a drilled or punched hole (the plate must be annealed after punching to destroy local hardness) to seem stronger per square inch of material left at the sides of the hole. If a **specimen is too short,** it does not seem to have freedom to contract as much in section before fracture, and therefore the breaking stress is greater and the fractional elongation is less; for it is to be remembered that breaking stress is calculated as breaking load per square inch of the *original* section. Very **long rods** or wires give too small a breaking stress for a very different reason —namely, because of the greater chance of the existence somewhere of inferior metal. It is reasonable to suppose, and it is found experimentally to be fairly correct, that to obtain the same fractional elongations from the same material, specimens, if of different sizes, ought to be similar.

This is the exact meaning which we give to the word *stress*. Much of the difficulty you may have met with in your reading is due to the fact that you have not made a proper distinction between the meanings of these two words. *Stress* is the load per square inch which produces a fractional alteration of the length of a wire or rod, and this fractional alteration is called the *strain*. Suppose your load to be 6 lbs., and your wire circular in section, with a diameter of 0·05 inch. Then the area of the section is 0·025 × 0·025 × 3·1416, or ·00196 square inch. The stress is 6 ÷ ·00196, or 3,061 lbs. per square inch. You will find that this thin wire gets the same strain with a total load of 6 lbs. as a rod 1 square inch in section would get with a load of 3,061 lbs. If ever you get a problem to work out relating to the lengthening of a wire or rod produced by a load, you must consider not the total lengthening of the wire or rod, but its fractional amount of lengthening, and call this the *strain;* also consider not the total load, but the load per square inch of section, and call this the *stress*, and you will find that for some kinds of wrought iron the tensile stress = the tensile strain × 29,000,000.

242. It is usually assumed that we ought to *expect* the elongation and strength to depend upon load per square inch and not upon the **shape of the section.** To me it seems wonderful that molecules near the surface should not behave differently from molecules remote from the surface. It is possible that the mere shape may have some small effect which is disguised for us by the great differences of material and of physical state in the specimens which we compare. It is only **when we take pains** to obtain homogeneous material that we find the strain very nearly proportional to the stress. Any departure from this rule may be traced to initial strains in the specimens. Unless a casting or cold-hammered or swaged forging is annealed by heating to a bright red heat and a slow cooling, it is heterogeneous. Some parts take a set much more easily than others, because they are of different material or because they are already strained. Instead of Fig. 172, therefore, we ought to expect a figure which is the result of adding together the ordinates of a great number of curves like Fig. 172, the distances O N being very different, in some cases 0. A **cast-iron** beam takes a set for quite small loads, and it is often loaded so as to get a large set before it leaves the foundry. Afterwards its deflection will be practically proportional to load. A man who puts up bells in a house or a telegraph wire "**kills**" the wire beforehand—that is, gives it a permanent set, straining it to its "yield point" indeed, as he finds that after this operation it is harder, will stand great loads without taking a further set, and follows the laws of elasticity better for pulling forces. This is like giving a good set to *a piece of* **riveted work,** which means that the rivets bed better into their holes. When a wire by being **drawn through a die** is reduced to a smaller size, there is a complete alteration in the arrangement of its particles or groups of molecules, and yet the drawn wire has usually greater strength than it had originally. Even hardened steel wire is drawn in this way. **Metals will,** in fact, **flow** (the filling up of the passages in mines shows that rocks also flow) if sufficient stress is applied to them, and at the end of the operation they are as strong or stronger, but less plastic, than before. If they are harder and this quality is not wanted, it may usually be removed by annealing. Plates of iron and steel rolled cold are hardened; rolled hot, they are gradually annealed as they leave the rolls. In constructing a certain magnetic instrument, I find it necessary to anneal a certain piece of iron from so high a temperature that the nearly pure iron is so soft that it almost cannot keep in shape. If this is scratched once by a file, sufficient hardness is induced to make the instrument useless.

243. It is somewhat more difficult to experiment on the **shortening of a strut** or column when it transmits a push, because you cannot use very long struts. A strut tends to bend if it is very long; and when it breaks, unless great care is taken to keep it straight, it breaks more easily the longer it is. The bending action causes the load to act more on one part of the cross-section than another, and the stress—or the pushing force per square inch—is greater at one part of the section than at another. If you experiment, therefore, you must take care to use struts which are prevented from bending. In Chap. XVI. we shall consider the bending of beams, after which you will better understand the present difficulty. It is sufficient for you at present to know that whereas the pull in a tie-bar tends to make it straighter if possible, the push in a strut tends to make it bend. Hence, *in an iron railway-bridge or roof you will see that the tie-bars are thin solid rods usually, and they might be chains or ropes; but the struts must not merely have a proper area of cross-section, this cross-section must also be wide in every direction.* Thus, instead of a solid cast-iron column you always see a hollow one, unless the column is very short. Also, a thin plate of iron suffices for the lower boom or flange of a railway-girder (because it resists a pull), whereas the top boom is a hollow tube, or is U-, or ∩-, or ┌──┐-shaped, because it must resist a push. Long struts, therefore, must be considered in Chap. XXI., after we have investigated the bending of beams.

If, however, **we prevent bending**, the laws for stiffness and strength of long struts are as simple as those of tie-bars. Thus the great struts of the Forth Bridge consist merely of four angle irons arranged as the four parallel edges of a square prism, and they are fastened to one another laterally by very light bracing, whose simple function is to prevent the bending of the angle irons. The strength and stiffness of such a strut are to be calculated from the combined sections of the angle irons.

In a short strut or a long strut prevented from bending (in Art. 373 it will be shown that the lateral constraints are called upon to exert only very small forces to prevent bending), the load per square inch is called the *stress.* The *shortening is a fraction of the whole length of the strut, and this fraction is called the strain.* You will find from your experiments that the strain is proportional to the stress. Thus for wrought-iron struts or columns the compressive stress = the compressive strain × 29,000,000. The multiplying number is found to be the same for the same material, whether it resists a push or a pull. This number is called "Young's Modulus of Elasticity"; it has been measured for various materials, and is given in Table III., p. 302. In using it you must remember that the stress is in pounds per square inch.

Exercise 1.—By how much would a round bar of steel, 120 feet long, whose diameter is 2 inches, lengthen with a pull of 30 tons? Answer:—0·0855 foot.

Exercise 2.—By how much would a column of oak, 7 feet long and 4 inches square, be compressed in supporting a weight of 2 tons? Answer:—0·0013 foot.

244. I have said that if you use **squared paper** after making your experiments, you will find that the strain is proportional to the stress, and the lengthening of a tie bar is proportional to the total pulling force. But you will find that this law is not true when the loads become too great. If your loads are less than about a quarter of the breaking load, you will find on removing them that the wire on which you are experimenting goes back to its original length.* But if your loads much exceed this amount it will be found that the wire has taken a **permanent set**; that is, if you remove the load the wire will not go back to its original length. It remains permanently longer than it originally was, and we say that we have exceeded the *limits of elasticity*. The load which produces this permanent set is said to be the measure of the elastic strength of the wire, for although it does not break the wire it alters it permanently. Now it is only for loads less than this that the law "strain is proportional to stress" is true. Your squared paper for experiments on a steel wire would give a straight line becoming a curve, like Fig. 172.

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* It may not go back **at once** to its old length, but in a few minutes it will be found exactly where it was before you loaded it. Similarly, when the load is put on there is first a sudden lengthening, and after this there is a slight extension going on so long as the load remains, but it practically comes to an end in a few minutes. This after-action, or "creeping," is so slight that I have not till now spoken about it, although we have reason to believe that its investigation would be of great importance. This "creeping," which seems connected with internal friction or viscosity, is absent in the quartz fibres of Professor Boys within a large range of stress, and it is so in soft metals within a small range; and this is curious, because there is much of it in glass; and

When you plot your results, making the distance *m n* represent the extension of the wire for a load represented by the distance *m q*, to any scale you please, you will find that the line passing through your points is straight only from O to M, say, and then it curves. The distance Q M represents the load which produces permanent set. For greater loads than this, the extension is more than proportional to the load, and increases more rapidly until we get at P K a very rapid extension indeed, and the elongations thereafter are so great that they cannot be represented here on the same scale as the part O M. P is called the **yield point**, and the phenomenon is very marked in wrought iron and other ductile materials. It seems always somewhat higher than M, the true elastic limit. A student must obtain results up to the breaking

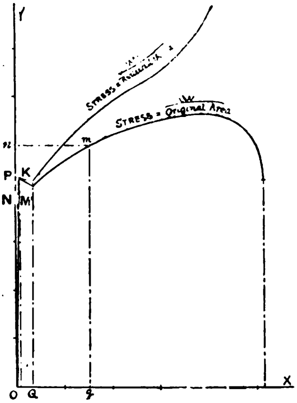

Fig. 172

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certainly within the limits of permanent set there is creeping in brittle substances, such as steel, giving trouble in measuring-instruments. If we say that there is perfect elasticity when the same forces are permanently required to keep a body in any particular shape at the same temperature, a body may be perfectly elastic, and yet it may have viscosity (the term viscosity indicates an elasticity which is affected by the rate of change of strain), and disobey Hooke's law. But there is imperfect elasticity if the forces are dependent not only on the shape of the body, but on its previous shapes, its past history. The only perfectly definite limits of elasticity in nature seem to be those at which, at constant temperature, a vapour becomes a liquid and a liquid becomes a solid, or a vapour becomes a solid.

of a wire for himself, plotting them on squared paper; noting the rapid yield after P, then the much slower but still rapid elongation, becoming again rapid as the specimen gets near breaking. The material is plastic after M. He will note that published diagrams are not of great value unless we know the *rate* at which the load was increased. After P is reached the specimen will continue to increase in length with time, even when the load is not increased, and even when the load is diminished.

245. These great **plastic elongations** are permanent (except for the very small elastic part), and are very different from the great non-permanent compression of cork, or the great extension or shortening with nearly constant volume of indiarubber. When there is continuous yielding under constant forces, the solid body behaves like a fluid, flowing (as lead does easily) without much change of density. It seems from M. Tresca's experiments that when a round punch comes down upon a plate of lead the lead flows rapidly laterally from underneath the punch into the rest of the plate, until the shearing surface is considerably diminished, because the lead is much less thick than the plate from which it is punched. In the squeezing of metals there is local flow wherever the pressure is great.

246. A **fluid** is defined as that which changes its shape under the action of small forces. This is why we call many substances (such as sealing-wax and pitch) fluids. Even their own weights, acting long enough, cause them to flow. Metal statues and brackets several thousands of years old prove that metals are not fluids. Experimental evidence of what stresses infinitely long continued will just cause **metals to flow** is still wanting. Mr. Bottomley found that seemingly similar wires seemed to greatly **increase in their strength** if the increase of load was very gradual indeed. The increase continuing in one case for a month, the strength seemed to be greater by 27 per cent. than on the specimen broken in the ordinary way.

247. When after the yield-point, a load is left upon a specimen, the amount of **hardness** and increase of strength produced are increased by leaving on the load for a longer time. Even when the load is not left on, if the specimen is left unloaded for some time, it seems to become harder and considerably stronger during the interval of rest. In both these cases by hardness I mean a high yield-point. As a matter of fact, the Young's modulus seems to diminish 4 to 9 per cent. as it gets what I now call harder, and annealing increases it again. The material below the yield-point, although seeming very elastic, exhibits **time plasticity**. The nature of the change in the character of these tensile and compressive phenomena when other stresses (such as twisting stresses) are acting is not yet well known; but there may be an apparent alteration in these limits, as deduced from the bending of beams or the twisting of shafts which have initial strains. Thus James Thomson showed that a shaft of ductile material might be twisted until most of the material had taken a set. The shaft, if now examined, would be found to take a set very much more readily with the reversed kind of twisting couple than with one of the kind which was used to give it its set. To understand this properly, a student ought to make a diagram showing the probable stress everywhere in the shaft when left to itself.

248. Moduli of elasticity are slightly altered by **manufacturing processes.** Lord Kelvin found copper and iron wire to alter 5 per cent. in rigidity when subjected to great stretching. Very much greater changes were produced in the Young's moduli by excessive twisting. Lord Kelvin found that good copper wire, annealed by being heated to redness and sudden cooling in cold water, had a modulus of rigidity $6{\cdot}71 \times 10^6$ lbs. per square inch. The same kind of wire heated to redness and cooled slowly, so that it was brittle, had a modulus $5{\cdot}58 \times 10^6$, its density having diminished $2\frac{3}{4}$ per cent.

It is interesting to note that **a watch** goes faster and faster for some time after it is made, but at the end of some months the balance-spring settles down into a state which does not much change afterwards. In this state, then, its elasticity is greater than it was in the beginning. The springs of chronometers are, however, often laid aside as useless after a few years' service, their elastic condition having altered so much since the beginning that they have to be replaced.

Young's modulus sometimes diminishes and sometimes increases with temperature; but more experiments are needed. Wertheim's results are given here :—

TABLE III.

Metal.	Specific Gravity.	Young's Modulus in Millions of lbs. per Square Inch.		
		At 15° C.	At 100° C.	At 200° C.
Lead	11·232	2·46	2·32	...
Gold	18·035	7·9	7·5	7·8
Silver	10·304	10·1	10·3	9·0
Palladium	11·225	13·9	...	...
Copper	8·936	14·9	13·3	11·2
Platinum	21·083	22·0	20·1	18·4
Steel, Drawn, English ...	7·622	24·6	30·3	27·3
Cast Steel	7·919	27·7	27·0	25·4
Iron	7·757	29·6	31·1	25·2

I cannot quite believe these results, because I have never known any specimen of steel to have a less E than 28×10^6. They are reduced from the numbers as selected by Lord Kelvin from Wertheim's original "Memoires"; and I must say that Wertheim's experiments were carefully carried out.

The modulus of rigidity of iron, copper, and brass, according to Kohlrausch, diminishes with temperature. The cubic modulus of water increases 15 per cent. for 53° C. rise of temperature; of alcohol and ether, it diminishes with temperature. The Young's modulus of indiarubber increases with temperature. The conclusions which may be drawn by thermodynamic reasoning from facts like these are interesting.

249. Some metals are believed to become brittle at low temperatures, much as "cold-short" iron or steel containing phosphorus is brittle. In many cases there may be no proper foundation for this belief. There is a slight increase of strength in ordinary iron and steel from low temperatures to near 200° C. Above about 300° C. there is a great lowering of strength produced by rise of temperature. It is believed that iron and mild steel worked at a "blue heat" (lower than red heat) become more deteriorated (brittle, or with a tendency to become brittle afterwards) than if worked cold or at a red heat.

250. Loss of Energy in Change of Strain.—Even in fluids there must be loss of thermodynamic energy in change of volume, because of change of temperature. But the loss of energy in the change of shape of solids depending on the rapidity of change, is much greater than can be accounted for thermodynamically or by motions of the outside air; and it must be put down to internal friction or viscosity. It is very marked in zinc, in indiarubber, and in jellies. The reasons for the viscosity shown in the distortion of gases and liquids are known to us. In twisting and untwisting wires, Lord Kelvin found (1) that the viscosity does not seem to be proportional to the velocity, so that the law is different from that of the viscosity of fluids. (2) Tension in the wire produces a non-permanent increase in the viscosity. (3) The viscosity at any time depends upon the history of the specimen. The viscosity of a wire on Saturday, after a week's experimenting, was greater than when on Monday the experiments were resumed after a Sunday's rest. Young's modulus in iron and copper generally diminishes (in bronze it increases) with repetition of loading if there is little pause. The effect is not so marked if there is a pause.

251. Strength.—Table XXII., p. 658, shows, among other things, the pulling (or tensile) and pushing (or compressive) stress which a material will bear before breaking. Probably if these stresses were allowed to act on the material for some time it would break even if they were not added to. They are obtained from experiments in which the load was increased pretty quickly, and yet quietly—that is, without any jerking or sudden action. The numbers in the table are taken from many sources, and must in general only be regarded as giving rough average values. Ultimate crushing stress is sometimes badly defined, because the materials behave as if gradually becoming plastic. Rolled iron gets a fibrous structure, and may not have the same strengths in all directions. Rolled steel is more uniform. *The smallest stress per square inch which will produce a permanent set in the material is sometimes called the elastic strength. The working stress is usually a fraction of this;* it is the stress which experience tells us to calculate upon for loads acting for a long time on materials, and which we shall be sure are perfectly safe in the case of such materials as are supplied from foundries and forges.

Exercise 1.—How great a pull will a round rod of brass stand before it breaks, if its diameter is 0·3 inch? What pull would produce in it a permanent set, and what is the safe working pull? Answers:—1,237; 484; and 254 lbs.

Exercise 2.—A short hollow cylindric column of cast iron is 8 inches in outer diameter, 5 inches inner diameter. What is the safe load and what load will produce permanent set? Answer:—The area of cross-section is 4 × 4 × 3·1416 minus 2·5 × 2·5 × 3·1416, or 30·63 square inches; 30·63 × 21,000 is 643,230 lbs., or 287 tons; 30·63 × 10,400 is 318,522 lbs., or 142 tons.

252. In making calculations on the stiffness and strength of structures we assume that strains are proportional to stresses. To use in these calculations the results of tests on materials beyond the elastic limit may be unscientific, but this is what we do: employing a **factor of safety,** which is the ultimate strength divided by the working strength. That is, we say that we may use a certain working load on a beam because we find the pushing or pulling forces which it produces in the various parts of the structure; and each of these is, say, one quarter of the pushing or pulling force which would break that particular bar; here the factor of safety is four. We do not mean, or, rather, we ought not to mean, that we might put four times our working load upon the structure before breaking, because we have no theory of what would occur in the structure if the elastic limit of one or more bars were passed. The factor of safety allows also for contingencies; there is the risk of defects in spite of precautions and the possible deterioration with time; sometimes a large factor is used because we suspect inaccuracy in our theory or in our estimated loads, in unforeseen causes of shock and fatigue. It will be evident from this and from what follows that great judgment is needed in fixing a factor of safety, and the following list must be looked upon *merely* as a guide in setting academic problems:—

TABLE IV.

	Steady Load.	Varying Load.	Structures Subjected to Shocks.
Wrought Iron and Steel	3½	5 to 8	10 to 12
Cast Iron	5	6 to 10	15
Timber	7	10 to 15	20
Brickwork and Masonry	20	30	...

Live loads are usually doubled and added to the dead loads. These numbers are sometimes called "factors of our ignorance." The Board of Trade in 1858 adopted the rule that in bridges the stress on wrought iron must not exceed 5 tons per square inch, and later for steel 6½ tons per square inch. We know now from Wöhler's experiments that the *range* of stress is very important. In the Conway bridge, in which the range is not great because the live load is small compared with the dead load, the stresses are as much as 6 tons to the square inch.

253. The load per square inch on a tie or strut is the stress. It is obvious that as the section of a tie-bar becomes smaller, the stress is really the load per square inch of **the diminished section.** The change of area of section in our ordinary constructive materials is so little that we usually consider the stress to be load ÷ original area of section, as this is very convenient. But the section of a bar may alter very greatly when being fractured; and although no experimenter yet seems to have stated it, I am sure that a great deal of useful information is lost because we do not plot tensile strain and actual tensile stress up to the breaking-point. It is somewhat difficult to measure the section with great accuracy, and indeed within the elastic limits the measurement is nearly impossible; but it ought to be attempted above the yield-point, because the section alters sometimes 50 per cent. before fracture. The stress seems to increase very much before fracture.

In indiarubber the alterations in shape with good elasticity are so very great in comparison with what we find in ordinary substances that it is absolutely necessary to measure the real stress for each load and length. Students ought to do this for themselves, as observations even roughly made give instructive results. It is to be remembered that in gases the cubical increase of stress δp produces the cubical strain $-\frac{\delta v}{v}$; and as elasticity = stress ÷ strain, the elasticity of a gas is $-v\frac{dp}{dv}$. The actual volume at the time is to be taken. If we define **Young's modulus**

of the material as $l\frac{dp}{dl}$ where l is length of tie or strut, and p is the load divided by the actual cross-section, we find that for india-rubber it is fairly constant for loads that shorten the specimen 16 per cent. or lengthen it 50 per cent. The most striking thing is the rather rapid increase of the resistance to elongation in the attenuated specimen before it breaks.

254. In some simple structures the value of a material is taken to depend upon the amount of change of shape which it may undergo before it breaks; and **commercial tests** often consist of definite extreme bending or twisting, or the delivery of definite blows to definite specimens.

255. At first the **contraction of section** occurs pretty uniformly along the specimen, but as fracture approaches, the contraction (except in hard metals, such as cast iron and hard steel) becomes localised. Hence this contraction, and not the lengthening of the whole specimen, ought to be taken as a measure of the ductility. When the stretching proceeds slowly, and there are intervals of rest, there is greater uniformity, places of great yielding harden, and the yield takes place afterwards elsewhere. But, again, there have been cases where the loading was so very rapid that almost no local excess of yielding occurred, and the whole specimen yielded fairly evenly, the specimen extending about 50 or 60 per cent., or about double what occurred in a similar specimen tested in the ordinary way.

256. It is interesting to examine the **surface of the fracture,** noting its texture. The fracture surface may be a flat or an oblique cross-section (see Art. 290). In ductile material we often find a combination of the two kinds, sometimes taking the form of conical and flat surfaces. In compression in short specimens the usual way of applying load prevents by friction the lateral enlargement of the ends. The material takes a barrel shape, and can withstand enormous loads, because the internal strain is greatly of the nature of cubical strain, and it is only shear strain that seems to cause fracture (see Art. 291).

257. Art. 290 shows that the shear stress in a tie-bar or strut is greatest on a plane making 45 degrees with the axis. But we do not find that **oblique sections of fracture** are inclined at this angle, possibly for the reason there stated. Tensile stress seems to diminish and compressive stress to increase the resistance to shearing. When, as in blocks of cast iron which are not more than half as high again as the width of base, there is no room for one shearing surface at the angle preferred by the material, the block fractures at a number of surfaces, cones and wedges being formed.

258. Attention ought to be paid to the fact that **local strengthening** and stiffening of a structure may produce general weakness. When a hole is punched in an iron plate there is a local hardening, and probably strength-increase in the material round the hole; but unless the plate is annealed or (as a partial improvement) the hole is rhymered out to remove the hard part, when load is applied the hard part yields less readily than the rest, gets an undue share of load, fractures, and then the softer material fractures.

259. Live Loads.—When a weight is suddenly applied to stretch a wire, it produces greater effects than when slowly and quietly applied. We know the reason. A weight which, slowly applied, would produce an extension of 1 inch, will, when put on and let go, produce an extension of 2 inches. The wire now shortens to its original length, then extends 2 inches, and continues to get shorter and longer, the weight vibrating. As there is friction of some kind among the particles of the wire, and there is also external friction, the lengthenings and shortenings gradually lessen till, in a short time, the wire settles down into the same state as it would have been in if the load had been slowly applied. Now, if we suppose this wire, when stretched 2 inches, to be strained just beyond its elastic strength, it is evident that the **suddenly applied load** does harm; whereas the same load slowly applied would do no harm. The harm is greater if the weight, besides being applied suddenly, is moving before it begins to act on the wire. Take the case of a stone which is being removed by means of a crane. If the stone, happening to fall a little, be brought up by the chain, the increase in the stress on the chain is proportional to the height from which the stone has fallen, and is greater the less the chain is extended. When a wire is lengthened ·1 foot by a weight of 1,000 lbs., which has been increased gradually, we know that the pull on the wire began with 0, and, as the wire gradually extended, the pull became greater, till it is now 1,000 lbs. The average pull was 500 lbs., and 500 × ·1, or 50 foot-pounds, is the total **strain energy stored up** in the wire. If we wish to give more energy to the wire, we must strain it more; and this is just what we do when we let the weight fall suddenly. The extra strains due to loads being live depend upon the mass which we set in motion in applying the load. In some railway bridges it has been found that the increased deflection is 14 per cent. greater, and it is usual to add 50 per cent. to such live loads and treat them as dead.

260. The energy stored up in any strained body may be calculated if we know the stress and the strain. The mainspring of a watch contains a store of energy which is gradually given out by the spring in returning to an unstrained condition. Each strained portion of the spring contains a portion of the store, and if at any place in the body there is too great a store the body will break there.

If w is the *proof load* (or load which will just fall short of producing permanent set) on a tie-bar of length l and cross-section A, the stress being W/A, the strain is W/AE, the elongation is Wl/AE, and the work done as the load is increased from 0 to W is $\frac{1}{2}W^2l/AE$. This is called the resilience of the bar. Now, the volume of the bar is lA, and hence the resilience per unit volume is $\frac{1}{2}W^2/A^2E$. Replacing W/A by f, the proof stress which the material will stand, we have the resilience per cubic inch to be $f^2/2E$. If f is the proof stress in compression, we have the same expression for struts.

When a torque M produces an angular change $\delta\theta$, the work done is $M . \delta\theta$; and when a twisting moment M has been gradually increased from 0 to M, the twist of a shaft increasing from 0 to θ, the work done is $\frac{1}{2}M\theta$.

When there is a shear stress f and a corresponding shear strain f/N, the shear strain energy per unit volume is $f^2/2N$; and if f is the proof shear stress, $f^2/2N$ is called the resilience of the material per unit volume.

The **resilience** is the strain energy which material may store before permanent set takes place. It is evidently $f^2 \div 2E$ per cubic inch in tie-bars and struts, if f is the tensile or compressive stress which would produce permanent set and E is Young's modulus. In shear the resilience is $f^2 \div 2N$, if f is the limiting elastic shear stress and N the modulus of rigidity. When the stress is not uniform, as in beams and shafts, the average resilience per cubic inch is, of course, less. Shocks due to blows, as of falling weights, will often cause the strain energy to exceed the amount which the material will stand, and local set and plastic yielding may take place. Much depends upon the rate at which strain energy is carried off to the rest of the material.

261. Let us consider why a chisel cuts into an iron plate. When I strike the head of a chisel with a hammer I give to the chisel in a very short period of time a certain amount of energy. This energy is transmitted very quickly to the plate through the edge of the chisel. The shorter and **more rigid** the chisel, the **more quickly** is the energy sent through the cutting edge into a portion of the plate. If it is not conveyed away rapidly from the edge, the amount contained in a small portion of material just under the edge is very great, and the material is fractured there. As the energy of strain is proportional to the product of stress and strain or to the square of either stress or strain, the **possibility of fracture** for a material is represented by the **square root of the strain energy** it contains per cubic inch. If a material is brittle, there is a sort of instability which causes fracture at one place to extend to all neighbouring places. And hence, if we deliver with great rapidity to a small portion of such a material a moderate supply of energy, it is sufficient to produce a large fracture. As our material becomes less and less brittle, we must have, over a larger and larger part of the volume in which we want fracture to occur, a sufficient supply of strain energy delivered. Hence, in cutting wood, we use a wooden mallet and a more or less lengthened wooden-headed chisel. The mallet and chisel act as a reservoir for the energy of the blow which is delivered to the wood from the edge of the chisel with comparative slowness and just in sufficient quantity to cause rupture in front of the edge. If the wood, without gaining in strength, became **more rigid** so as to be able to **carry off more rapidly** the energy given to it by the chisel's edge, it would be necessary to make the supply more rapid by using a more rigid chisel and mallet, and as we do this we must take care that the chisel itself near the edge is strong enough to resist fracture. This is one way of considering the effect of a blow. The exact mathematical consideration of what occurs in the impact of elastic bodies is not easy even for spheres and cylinders and other bodies of simple shape (see Art. 404).

262. As we have just seen, the extra stresses due to loads suddenly applied are easy enough to understand. It is not so easy to comprehend why **quietly varying loads** which produce no visible vibrations should produce what we call fatigue. The ordinary kinds of test as to strength under statical load, ductility, or elongation before fracture, applied to old rails, tyres, and other well-used material, show no great difference from what we obtain with new material, and yet the material is really quite different. Sometimes flaws are found, and it may be that fatigue somehow acts in producing minute flaws. The nature of the fracture in Wöhler's experiments is the same as that observed in old tyres and axles; it shows no signs of ductility, and is as if the material were brittle.

263. A piston-rod is subjected to **tensile and compressive stresses, often repeated.** It is found that its breaking strength is not 45,000 lbs. per square inch, which, let us say, it would be for a steady pull or push, but 15,000 lbs. per square inch. If, instead of such an action, we have a tensile stress which varies frequently from 30,000 lbs. per square inch to zero, the rod will break after a time. In the same way, steel which will bear a steady stress of 84,600 lbs. per square inch will only bear 46,500 lbs. per square inch if the stress varies between this and zero, but is always of the same kind; whereas it will only bear 25,000 lbs. per square inch if the stress is sometimes a pull of this amount and is sometimes a push of the same amount.

The above statement, the outcome of Wöhler's experiments begun twenty-five years ago, and Fairbairn's experiments made very much earlier, was made fifteen years ago in the first edition of this book. The experiments made since give results of much the same kind. In the following table we have the most important results arrived at up to the present time, being the stresses in tons which require from five to ten millions or an indefinitely large number of applications of the load to cause fracture:—

TABLE V.

Material.	Ultimate Statical Strength.	Similar Stresses.		One Stress Zero.		Opposite Stresses.		
		Least.	Greatest.	Least.	Greatest.	Least.	Greatest.	
Wrought Iron.	22·8	12·0	20·5	0	15·3	− 8·6	+ 8·6	Wöhler.
Krupp's Axle Steel... ...	52·0	17·5	37·8	0	26·5	−14·1	+14·1	
Untempered Spring Steel.	57·5	12·5	34·8	0	25·5	−13·4	+13·4	
Iron Plate ...	22·8	11·4	19·2	0	13·1	− 7·2	+ 7·2	Bauschinger.
Bar Iron ...	26·6	13·3	22·0	0	14·4	− 7·9	+ 7·9	
Bar Iron ...	26·4	13·2	21·9	0	15·8	− 8·7	+ 8·7	
Bessemer Mild Steel Plate ...	28·6	14·3	23·8	0	15·7	− 8·6	+ 8·6	
Steel Axle ...	40·0	20·0	32·1	0	19·7	−10·5	+10·5	
Steel Rail ...	39·0	19·5	30·9	0	18·4	− 9·7	+ 9·7	
Mild Steel Boiler Plate .	26·6	13·3	22·6	0	15·8	− 8·7	+ 8·7	

This exceedingly great weakening in material due to **fatigue** seems almost as if it had been vaguely known to English engineers from the beginning, and justifies the larger factors of safety which were wisely used in this country fifty years ago in railway bridges. **Professor James Thomson** showed that the two elastic limits ought to be called inferior and superior, as they are not necessarily equal positive and negative, but might even both be on the same side of the zero if the overstraining were great enough, and that variable displacements outside these limits would produce a destructive succession of sets. This theoretical deduction from the consideration of his overtwisted shaft has been completely verified by the experiments of Bauschinger, and may be said to completely explain the phenomena of fatigue. Bauschinger found that the elastic *range* does not alter much, although either end of it may be altered, even, indeed, nearly to the ordinary breaking stress. This change of either limit also is not very permanent, altering with hammering and other violent treatment.

264. We can only refer to a few of the ways in which our principles are applied. When **bolts tightly screwed up** fasten two things—the flanges of a cylinder cover, for example—the bolts get lengthened and other things (let us call them the *cushion)* get shortened. If extra forces are now applied it may be that for a comparatively small extra lengthening of the bolts the cushion is nearly relieved and does not add its force as it did previously; whereas in other cases, when the cushion is more springy, there may be almost no relief, and the old forces due to the cushion may act along with the new forces. Very often this **initial tightening up** of bolts is too great, and we have rule of thumb methods of designing sizes based upon an experience of the carelessness of workmen which it is difficult to express algebraically. One rule based on experience is that the **length of a spanner** shall never much exceed fifteen times the diameter of the bolt. Another, that, in certain kinds of machinery, bolts of less than a certain size shall not be used. Students ought to make careful sketches of the various kinds of bolts and nuts, including **the usual forms of lock nuts** and ways of locking, and also other kinds of fastening. In studying some of the proportions of these important details of machinery our theories are useful in suggestion; in some hands the theory is only a snare. Every true engineer must respect the proportions which have been arrived at by **the fit and try**, the failure and success methods of generations of engineers. When a very novel thing has to be made the good engineer sighs for practical guidance, and he is very cautious in using his theory.

265. It is found then that when a rod is pulled, with however small a force, **not only does it get longer, but its diameter gets less.** When, for example, a rod of glass is pulled so that its length increases by the one-thousandth of itself, it is found that its diameter gets less by the one three-thousandth of itself. When a strut shortens, it also swells laterally. The ratio of the lateral to the axial strains in compression or in tension is called Poisson's ratio. It is of great importance in the theory of structures.

The nature of the strain in a wire which is being extended, or in a column which is being compressed, cannot be said to be simple. If all lines in one direction, and in one direction only, became shorter or longer, the **strain would be called simple,** but it needs rather a complicated system of external compression or extension to produce this effect. No matter how a body is strained, if we consider a small portion of it we shall find that, besides angular changes, any strain simply consists of extensions and compressions in different directions. In fact, imagine a very small spherical portion of the body before it is strained. The effect of strain is to convert the little sphere into a figure called an ellipsoid (that is, a figure every section of which is an ellipse) or a circle. Remember that every section of a sphere is a circle. It may be proved that there were three diameters of the sphere at right angles *to one another,* which remain at right angles to one

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Example.—Thus if M′ N′ (Fig. 173) is part of a long **wire subjected to a pull,** the portion of matter which was enclosed in the very small imaginary spherical surface A B C D before the pull was applied is now enclosed in the ellipsoidal spherical surface A′ C′ B′ D′. The sphere has become an ellipsoid of revolution; A B becomes A′ B′, C D becomes C′ D′. The strain in the direction A B is $\frac{A'B' - AB}{AB}$, and this is equal to the pull in the wire per square inch divided by Young's modulus of elasticity, E. As, however, it is often more convenient to use a multiplier than a divisor, we are in the habit of using the reciprocal of E, and denoting it by the letter α. Thus, if the pull per square inch is p pounds, it produces a strain of the amount $p\alpha$ in the direction A B; the lateral contraction of the material is $\frac{CD - C'D'}{CD}$, and in this case is usually denoted by $p\beta$, the ratio β/α being Poisson's ratio.

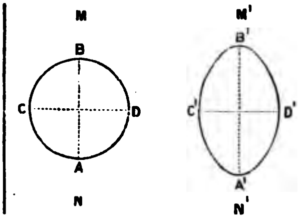

Fig. 173.

266. In the following exercises on struts, the load is supposed to be carefully applied so that there is the same stress at every point of the section :—

1. Find the area of the base of a sandstone column carrying a dead load of 6 tons. Take the ultimate crushing stress as 3,600 lbs. per square inch, and use a factor of safety of 20. *Ans.*, 75 square inches nearly.

2. Find the least safe sectional area of a short cast-iron strut to bear a load of 12 tons. Take the safe stress as 15,000 lbs. per square inch. *Ans.*, 1·8 square inch.

3. The external and internal diameters of a short hollow cast-iron column are 10 inches and 8 inches respectively. If the safe working stress be 15,500 lbs. per square inch, find what load it will safely bear.

If the external diameter had been 7 inches, what ought the thickness to have been to bear a load of 100 tons? *Ans.*, 196 tons; ·73 inch.

4. A tie-rod made from ¾-inch wrought-iron plate has to sustain a load of 15 tons. What should be its width, allowing a working stress of 7,000 lbs. per square inch? *Ans.*, 6·4 inches.

5. The steam pressure in a locomotive boiler is 175 lbs. per square inch; the stay-bolts which connect the flat sides of the firebox with the end plate of the boiler are placed 4 inches from centre to centre, vertically and horizontally. What is the tensile force in each stay-bolt, and what

must be the diameter of each, the metal not being subjected to a greater stress than 5 tons per square inch of the section of the bolts?

Ans., 2,800 lbs.; 0·54 inch.

6. What are the working, proof, and breaking loads of pianoforte wire $\frac{1}{16}$ inch diameter? And if the wire hangs vertically 5 miles, what weight at the end will, together with the weight of the wire, produce the working stress at the top? What is now the average load and the average stress in the wire? By how much will the wire lengthen when subjected to it?

7. A strut 100 feet long is made up of four angle irons of wrought iron $(3\frac{1}{2} + 3\frac{1}{2})\ \frac{1}{2}$, which are prevented from bending. Find the ultimate proof and working loads. How much shortening occurs under the working load? *Ans.*, working load 91,000 lbs.; 0·28 inch.

8. A piston-rod of mild steel 4 inches diameter, 6 feet long, the piston 30 inches diameter; what maximum pressure of steam may be used (1) if the engine is double acting; (2) if the engine is single acting? (See Art. 263.) What lengthening and shortening occur under these pressures? *Ans.*, 53 lbs.; 160 lbs.; (1) 0·08 inch; (2) 0·12 inch.

9. Columns about 8 feet high of brickwork, sandstone, and granite, 14 inches square; what working loads will they carry? If they are 6 feet apart, and carry a wall of brickwork 28 inches thick, to what heights may it be carried? The brick columns being replaced by cast-iron cylinders 6 inches diameter, what ought to be their thickness? How much do they shorten under the load?

Ans., $15\frac{3}{4}$ tons each; 30 ft.; 0·12 inch.

10. The tight side of a gearing-chain taking a pull P, let it act at 5 inches from the centre of the wheel, and transmit 20 horse-power at 100 revolutions per minute; find P. Each of a pair of links takes this pull. If the thickness of a link is one-third of the diameter of the pin, and if its breadth is two and a half times the diameter of the pin, find the section of each link. It is a good exercise to design the chain.

Ans., 5,044 lbs.

11. A single (a little over 0·2 inch thick) leather belt will stand an average pull of 1,000 lbs. per inch of its width. The weakness of the average fastener reduces this to about 300 lbs. per inch of width, and it is usual to take 66 lbs. as a working load. The pull on the tight side of a belt is two and a half times that on the slack side. The pulley is 30 inches diameter, 150 revolutions per minute; find the breadth of a single belt to transmit 5 horse-power. (See also the exercises Art. 185).

Ans., 3·5 inches.

12. Columns of different material, constrained to keep of the same length, of cross-sections A_1, A_2, A_3, etc, of coefficients of expansion α_1, α_2, α_3, etc., unstressed, at a particular temperature, are raised t degrees in temperature. What is the fractional elongation? And what are the stresses in them?

Ans., If unconstrained, their fractional elongations would be $t\alpha_1$, $t\alpha_2$, etc. If x is the real elongation, each has a compressive strain $t\alpha_1 - x$, $t\alpha_2 - x$, etc. Their compressive stresses are $E_1\ (t\alpha_1 - x)$, $E_2\ (t\alpha_2 - x)$, etc., if E_1, E_2, etc., are their Young's moduli, and the total push in each is $A_1 E_1\ (t\alpha_1 - x)$, $A_2 E_2\ (t\alpha_2 - x)$, etc. The sum of these pushes is o (some of them being negative; that is, tensile forces), or $A_1 E_1\ (t\alpha_1 - x) + A_2 E_2\ (t\alpha_2 - x) + \text{etc.} = 0$, or

$x = t \dfrac{A_1 E_1 a_1 + A_2 E_2 a_2 + \text{etc.}}{A_1 E_1 + A_2 E_2 + \text{etc.}}$ .... (1), and the stresses $E_1 (ta_1 - x)$ etc., can be found when x is known.

Thus, for example, suppose there are columns equal in section, two of them of brass, one of cast iron. Then $A_1 = 2 A_2$, $a_1 = 19 \times 10^{-6}$, $a_2 = 11 \times 10^{-6}$ for centigrade degrees; $E_1 = 9{\cdot}2 \times 10^6$, $E_2 = 17 \times 10^6$, $x = t \,.\, \dfrac{2 \times 9{\cdot}2 \times 10^6 \times 19 \times 10^{-6} + 17 \times 10^6 \times 11 \times 10^{-6}}{2 \times 9{\cdot}2 \times 10^6 + 17 \times 10^6} = t \times 15{\cdot}15 \times 10^{-6}$. The compressive stress in the brass is $9{\cdot}2 \times 10^6 (t\, 19 \times 10^{-6} - t\, 15{\cdot}5 \times 10^{-6}) = t \times 30{\cdot}2$ lbs. per square inch. The stress in the cast iron is $17 \times 10^6 (t\, 11 \times 10^{-6} - t\, 15{\cdot}5 \times 10^{-6}) = - t \times 76{\cdot}5$.

Thus, if t is 30 centigrade degrees, there is a compressive stress of 900 lbs. per square inch in the brass, and a tensile stress of 2,300 lbs. per square inch in the iron.

13. A bar of wrought iron 25 feet long and 1·75 inch diameter is heated to 180° C. While in this condition it is made to connect (by means of nuts screwed on the ends) the two side walls of a building which have fallen outwards from the perpendicular. If the walls do not yield against the tendency of the bar to contract, find the pull between them when the bar has cooled down to 80° C. Take the mean coefficient of linear expansion of wrought iron as ·0000124 for 1° C.

Ans., 92,350 lbs.

14. Two bars of copper with a bar of wrought iron between them, all of the same section and length, have their ends rigidly connected together. If the bars are heated from 15° C. to 100° C., find the stresses in the bars, and their fractional elongation, the coefficients of expansion for copper and wrought iron being taken at ·0000172 and ·0000124 respectively.

Ans., copper, 2,940; iron, 6,000 lbs.; 0·00125.

15. A tie-rod 100 feet long and 2 square inches in sectional area carries a load of 32,000 lbs., by which it is stretched ¾ inch. Find the stress, strain, and E.

Ans., 16,000 lbs. per square inch; ·000625; 25,600,000 lbs. per square inch.

16. A wooden tie 40 feet long, 12 inches broad, 7 inches thick, was tested with a pull of 130 tons, which stretched it 1·28 inches. Find the value of E for the timber. *Ans.*, 1,300,000 lbs. per square inch.

17. A vertical wrought-iron tie-rod 200 feet long has to lift a weight of 2 tons. Find the area of the section and the diameter if the greatest strain is ·0005 and E = 30,000,000. Neglect the weight of the rod.

Ans., 0·298 square inch; 0·616 inch.

18. Steam at a pressure of 200 lbs. per square inch is suddenly admitted upon a piston 18 inches in diameter. If the piston-rod be 3 inches in diameter and 7 feet long, what is the compression and strain energy in the rod at maximum compression? E = 30,000,000. Find, also, the maximum stress in the rod.

Ans., ·04 inch; 171 foot-pounds; 14,400 lbs. per square inch.

19. A ship is moored by two cables of 90 feet and 100 feet in length,

respectively. The first cable stretches $2\frac{5}{8}$ inches, and the second stretches 3 inches under the pull of the ship. Find the strain of each cable.

Ans., ·0024 ; ·0025.

The **weight of a rope** in pounds per foot is taken as ag^2 where g is its girth in inches, and **its breaking weight** in pounds is bg^2 where a and b have about the following values:—

	a	b
Tarred Hemp	·04	900
White Hemp	·04	1,300
Iron Wire	·13	4,000
Steel Wire	·13	6,700

20. What lengths of themselves will each of these kinds of rope carry?

21. Compare the weights and strengths of iron- and steel-wire ropes with iron and steel wires of the same circumference.

The Admiralty rule for the proof-load P in tons of the ordinary **close-link chain** of welded iron is $12d^2$, where d is the diameter of the iron in inches; this means nearly 8 tons per square inch in the iron. **For studded chain-cables** it is $P = 18d^2$, which means $11\frac{1}{2}$ tons per square inch in the iron. The working load is from half to one-quarter of this, depending on circumstances. The weight of either chain is about $10d^2$ lbs. per foot. Hemp rope of girth g is taken as being of about the same strength as a chain if $g = 10d$ to $11d$.

22. Two close-link chains, each making an angle of 50 degrees with the vertical, support a load of 10 tons; what is the size of the iron?

Ans., 0·8 inch diameter.

23. A ship of 2,000 tons (take it that one-quarter as much of water moves as the ship moves) is moving at 0·1 knots, and is brought to rest in three seconds, the law of the motion during stoppage being $v = 0{\cdot}1 \cos. kt$, where t is time and k a constant, and v is in knots. The pull comes directly on a studded chain. If the chain gets its proof-load, what is the diameter of the iron?

When three seconds elapse, cos. kt is 0, or $\cos. 3k = \cos. \frac{\pi}{2}$, or $k = \frac{\pi}{6}$. In feet-second units, $v = 0{\cdot}1689 \cos. \frac{\pi}{6}t$, because $1 \text{ knot} = \frac{6{,}080}{60 \times 60}$, or 1·6889 feet per second, and acceleration is $- {\cdot}1689 \times \frac{\pi}{6} \sin. \frac{\pi}{6}t$, being numerically $\cdot 1689 \times \frac{\pi}{6}$, or 0·5305 feet per second per second where greatest. The mass is $2{,}500 \times 2{,}240 \div 32{\cdot}2$, or 173,900 in engineer's units. Hence the greatest force is 92,300 lbs.

24. Find the work which may be stored up in a pound of hard spring

steel when stretched to its elastic limit, taking the modulus of elasticity as 35×10^6 lbs. per square inch; the elastic limit, 100,000 lbs. per square inch; and the weight of one cubic inch, ·29 lb. *Ans.*, 410·5 ft.-lbs.

25. A cylindrical rod of copper ½ inch diameter and 4 feet long, and one of wrought iron ⅝ inch diameter and 3 feet long, are to be stretched the same amount. Compare the forces necessary to do this, the values of E for copper and wrought iron being 17,000,000 lbs. per square inch and 29,000,000 lbs. per square inch respectively. Compare also the amounts of work expended in each case. *Ans.*, 0·28 : 1, 0·28 : 1.

26. What would be the resilience of a steel tie-bar 1 inch in diameter and 4 feet long if the bar becomes permanently stretched under a load of 10 tons, the modulus of elasticity being 32,000,000 lbs. per square inch? *Ans.*, 479 inch-lbs.

267. *Exercise.*—Find every number in the following table. Values of **tensile** ($f_t^2/2\,\mathrm{E}$) or **compressive** ($f_c^2/2\,\mathrm{E}$) **resiliences.** These numbers express the relative values of the following materials for the making of springs in which elongation, compression, or bending occurs. In bending, the smaller of the two values, or possibly an intermediate value, must be taken. The numbers $f_s^2/2\,\mathrm{N}$, or **the shear resiliences** per cubic inch, express the relative values of the following materials for the making of springs, such as spiral springs, in which shearing is most important. The numbers in each case show the amount of energy (in inch-pounds) which it is possible to store in each cubic inch of the material in the most carefully constructed springs. In Art. 518 we give a statement showing how much less these stores usually are in ordinary springs. The work actually done upon ductile materials before fracture is often 1,000 times as great as the resilience, and in hard steel it is 150 times, in cast iron being twenty times the resilience.

	$f_t^2/2\,\mathrm{E}$	$f_c^2/2\,\mathrm{E}$	$f_s^2/2\,\mathrm{N}$
Cast Iron	1	...	...
Wrought Iron	9	...	...
Mild Steel	20	...	...
Best Hard Steel	50	...	...
Copper (Rolled or Drawn) ...	4	...	...
Fir	4	...	...
Oak	6	...	...

268. **The diminution in bulk** of a substance when it is subjected to pressure uniform all round, as, for instance, when it is surrounded by water in an hydraulic press, or sunk in the sea,

has been experimented upon. The lessening in the bulk per cubic inch is called the cubical strain of the substance. The pressure in pounds per square inch all over its surface represents the stress, and it is found that the strain is proportional to the stress. In fact, in any substance the stress is equal to the strain multiplied by a certain number, for which the letter K is usually employed, called the **Modulus of Elasticity of bulk.**

If an increase of pressure δp causes the volume v to become $v + \delta v$ (δv is usually negative), the stress being δp and the strain $-\delta v/v$, the elasticity K is defined by $\delta p = -\mathrm{K}\,\delta v/v$, or $\mathrm{K} = -v\,.\,dp/dv$. In solids it is found that, whether the change takes place quickly (at constant entropy, as it is called in thermodynamics) or slowly (at constant temperature), there is no very great difference; but in gases and liquids it is very important to specify under what circumstances the rate of relative change of p and v is measured. The ratio is in air 1·41; water, 1·004; alcohol, 1·22; ether, 1·58; mercury, 1·38; flint-glass, 1·004; drawn brass, 1·028; iron, 1·019; copper, 1·043.*

The table, page 657, of slow moduli of elasticity of bulk is in pounds per square inch.

269. A cube 1 inch in each edge (Fig. 174), subjected to a uniform compressive force of 1 lb. per square inch on the opposite faces, ADEF and BCLG. Evidently the edges AB, CD, LE, and GF, become $1 - \alpha$ inch in length, α being the reciprocal of Young's modulus used above. Also the edges AD, BC, GL, and FE get the length $1 + \beta$ inch. If now we give to the faces ABCD and EFGL of this cube compressive forces 1 lb. per square inch, it is the edges AF, etc., which shorten, and the edges AB, etc., which lengthen. Again, give the compressive forces to the third pair of opposite faces, ABGF and CDEL, and we have the edges AD, etc., shortening and BG, etc., lengthening. If, now, all three sets of compressive forces act at the same time (that is, the cube gets on every face a pressure of 1 lb. per square inch), as the compressions and extensions are exceedingly small, each edge shortens by the amount α and lengthens by the amount 2β. Hence the edge which used to be 1 inch is now $1 - \alpha + 2\beta$ inch. The cubic contents used to be 1 cubic inch; it is now $1 - 3(\alpha - 2\beta)$ with great exactitude. Hence $3(\alpha - 2\beta)$ is the amount of cubical strain produced by 1 lb. per square inch. That is, the *Modulus of Elasticity of bulk*,

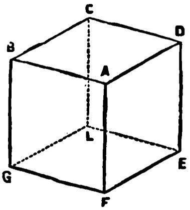

Fig. 174.

$$\mathrm{K} = \frac{1}{3(\alpha - 2\beta)};$$

and if we know α and β it may be calculated.

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* Lord Kelvin (article on elasticity) gives the above, and also the following numbers. The ratios of the quick to the slow Young's moduli are: zinc, 1·008; tin, 1·0036; silver, 1·00315; copper, 1·00325; lead, 1·00310; glass, 1·0006; iron, 1·0026; platinum, 1·0013.

Even very porous bodies, such as cork, have some elasticity of bulk. Fluids and homogeneous solids, such as crystals, are probably perfectly elastic as to bulk even at enormous pressures, and after their removal. Manufactured metals are generally porous, and alter (not necessarily increasing) in density after they have been hammered or drawn.

Experiments on metals with **great negative stress** in all directions are wanting. Liquids do not seem capable of resisting great negative pressures, and the contrast between them and solids in this respect is remarkable.

270. Students now know enough to make calculations on **the stiffness** and strength of ties and struts [only when the struts are kept from bending]. Before going on to other structures, even such simple structures as boilers and pipes, I wish to make a few remarks on the application of theory.

The engineer must have some sort of theory to work upon. I shall give the theory recognised by men like **Rankine**, and also by the most successful practical engineers. It is simple as I shall give it, and fits fairly well a great number of practical conditions. It is founded on the assumption that materials are homogeneous and not loaded beyond elastic limits, and yet it gives us knowledge which the judicious man finds useful beyond elastic limits. It is also founded on the assumption that certain things, too difficult to calculate, are negligible, and hence its mathematical results ought to be **tested by experiment** when this is possible.

To give an example. **Our theory of bending** is founded on the assumption that the plane cross-section of a beam remains plane after bending. Every mathematical result seems to agree with every experiment made on beams, seeming discrepancies being always explainable by the tests not having been confined between the elastic limits of the materials. Again, the more elaborate theory of St. Venant (Art. 311), involving fewer hypotheses, gives results that are practically in agreement with us. Hence we regard this easy theory which I shall give as one which may be depended upon in long beams, in spite of the fact that a plane section of a beam does not remain exactly plane. When it is applied in new cases not yet tested by St. Venant's theory or by experiment, we use it only as a fairly trustworthy guide in our practical work.

Another example. In the great twisting, which might occur without fracture in **indiarubber shafts**, it was known that the plane section of a square shaft did not remain plane. Nevertheless, the warping in ordinary shafts being small, a simple theory was adopted, in which there was the **assumption of no warping**. Results of experiments on round shafts agreed with the simple theory. Results from other shafts did not agree, and St. Venant has shown us why there is a discrepancy. Although his investigation is difficult to follow mathematically, his results are easy enough to comprehend. I find it necessary, therefore, to give not merely the simple theory of the engineer, but an account of St. Venant's results, and also, for advanced students, a short account of St. Venant's theory of beams and shafts. It is to be remembered that our theory assumes perfect elasticity of the material. We shall see that at the bottom edge of a key-way in a shaft we have a place where the stress becomes very great indeed. I have made experiments on such a shaft with a key-way, and I find that it is by no means such a great source of weakness as the theory supposes, and this leads us to consider how the results of the theory are modified by the flow of the material, instead of its fracture, under great stress.

271. In thin-shelled vessels, such as **boilers and pipes**, subjected to fluid pressure p inside, we assume that the tensile stress f is the same throughout the thickness; so that if a is the area of *metal* cut through at any plane section of the boiler, af is the resistance of the metal to the bursting of the boiler at that section. Now the equal and opposite force due to the fluid is $\mathrm{A}p$ if A is the whole area of this plane section of the boiler. Hence $\mathrm{A}p = af$ and $p = af/\mathrm{A}$ . . . (1) gives us the bursting or working pressure if f is the ultimate or working stress. **To prove this**:—

In Fig. 175 let F B C D E be part of a boiler whose separation from the rest by a plane section at F E we are now studying. Arrow-heads are drawn showing the forces with which the fluid everywhere acts normally on the shell. We want to know the resultant of these forces. Imagine a boiler made with the part F B C D E and a rigid flat plate F E closing it. If we neglect the weight of the fluid, all the pressure forces on the shell balance one another. This is Newton's law of motion. The mutual forces of parts of a system cannot affect the motion of the centre of inertia of the whole system. (See Art. 482.) If the above boiler were placed upon a **truck with frictionless wheels**, there will be no more tendency to move on a level road when there is great pressure inside than when there is little. The force due to pressure on any one little portion of the surface balances the forces on all the rest of the surface. Hence it is that if we make a hole there is a want of balance and our truck will move. When we make the hole the pressure everywhere changes because of the motion of the fluid, and hence we can only calculate the unbalanced force by knowing the momentum of the fluid which leaves the vessel per second. In the closed

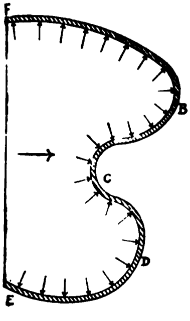

Fig. 175.

vessel of Fig. 175 we know that the resultant pressure on the flat surface F E is $\text{A}\,p$, therefore the equal and opposite force on F B C D E is also $\text{A}\,p$. As an example of this we saw in Art. 122 that the resultant force axially on the ram of an hydraulic press is exactly the same whatever be the shape of the end of it.

Example.—Spherical boiler of diameter D. Any plane section is a circle. If we use the above rule for any such section we find that less pressure will burst the boiler if the section is diametral. The area of such a section is $\text{A} = \frac{\pi}{4} d^2$, and the area of metal laid bare is $a = \pi d t$. Hence (1) becomes $p = \pi d t f \div \frac{\pi}{4} d^2$ or $4\, t f/d$ . . . (2).

272. In a long **cylindric boiler** or pipe it is easy to show that (neglecting the strength of the ends) *the tendency to burst laterally is twice as great as the tendency to burst endwise.* Endwise, the area of a circular cross-section is $\text{A} = \frac{\pi}{4} d^2$, and the metal laid bare is $\pi d t$, so that (1) becomes $p = 4 f t/\text{d}$, just as in a spherical boiler. But laterally, at a plane passing through the axis of the boiler, $\text{A} = l d$ if l is the length, and $a = 2 l t$ if we neglect the metal at the ends; and hence (1) becomes $p = 2 f t/d$ . . . (3). The bursting pressure endwise being twice as great as this, we always take (3) as the formula for the strength of a cylindric pipe or boiler.

Readers will now understand why in cylindric boilers the **longitudinal seams** are always much stronger than the **girth seams.** When the boiler has riveted joints we must, of course, regard the material as weaker than if it could resist tensile stress everywhere like a continuous boiler-plate. The working f for copper ought not to be taken greater than 2,400 lbs. per square inch for steam pipes. **In cast-iron pipes** and in steam-engine cylinders it has to be remembered that the difficulty in getting castings which are of the same thickness everywhere, and the allowance that must be made for tendency to cross-breaking when the pipes are handled, as well as the great allowance that must be made in steam-engine cylinders for stiffness, the difficulty of casting, and boring out, cause such calculations as the above to be somewhat useless. Thus it will usually be found that, whereas a large cast-iron water-pipe is not much *thicker* than the above calculation would lead us to expect, because it is usually carefully moulded in loam, yet a thin cast-iron pipe is often of more than twice such a thickness on the average, and it is our rule never to attempt casting a 9-foot length of pipe of less than $\frac{3}{8}$ inch thick. In these cases the maximum f for cast iron is taken as 1,500 lbs. per square inch, whereas for large pipes we usually take 3,000.

EXERCISES.

1. What must be the thickness of the plates used in the construction of a boiler 10 feet in diameter working under a pressure of 120 lbs. per square inch, taking the efficiency of the joints to be 70 per cent. and the safe stress at 10,000 lbs. per square inch? *Ans.*, 1·03 inch.

2. A copper pipe is 4 inches diameter and $\frac{3}{8}$ inch thick. What is the working pressure? Take $f = 2{,}000$ lbs. per square inch.
Ans., 375 lbs. per sq. in.

3. A vertical cast-iron pipe is 4 inches in internal diameter. The pressure at a certain place is 50 lbs. per square inch. At this place, and at places 100, 200, 300, etc., feet lower in level, find the proper thickness of the metal if the working stress f is taken as 3,000 lbs. per square inch.
Ans., ·033 inch; ·062 inch; ·091 inch; ·149 inch, etc.

4. The rule used for loam-moulded cast-iron water mains is $t = \frac{1}{8} + hd \div 13{,}000$ where h is head of water in feet, d diameter of pipe in inches, t the thickness in inches. A pipe 3 feet diameter, 1 inch thick; find h. Find the corresponding pressure in pounds per square inch. Find what value of f the working stress will cause the ordinary rule for thin cylinders to give the same answer.
Ans., 316 ft., 136·7 lbs. per sq. in., 2,460 lbs. per sq. in.

5. A wrought-iron pipe 2 feet diameter, $\frac{1}{2}$ inch thick; its working stress is 5 tons to the square inch, but strength of plate is diminished 30 per cent. because of riveted joint. What is the working pressure?
Ans., 326 lbs. per sq. in.

6. A cylindrical boiler 12 feet diameter is constructed of $\frac{29}{32}$ inch steel plate. The test pressure applied is 245 lbs. per square inch. Find the stress produced in the plate, and hence deduce the stress in the plate between the rivet holes, the sectional area being there reduced to ·77 of the solid. *Ans.*, 19,500 and 25,300 lbs. per square inch.

273. Storage of Energy in Fluids.—The volume of a cylindric vessel being v and the safe pressure being p, we may take vp as proportional to the energy which may be stored. If the diameter is d and thickness t and length l, the volume is $v = \frac{\pi}{4} d^2 l$. Assume what is known to be true (see Art. 272), that the safe pressure for a long cylindric vessel is (neglecting the strength of the ends) $p = \frac{2tf}{d}$, where f is the safe stress which the material will stand. The weight of the metal, neglecting the ends, is $W = \pi d t l w$, if w is the weight of unit volume of the material. The surface of the vessel is $s = \pi dl$. In all cases we neglect the ends.

Then the storage capacity of energy per unit weight of vessel is $\frac{\pi}{4} d^2 l \frac{2tf}{d} \div \pi dtlw$, or $\frac{f}{2w}$. So we see that it is independent of the diameter.

In boilers, therefore, which must store energy in this way, and where it is of importance that there should be great surface, we must consider surface $\div$ vp. This is $2tf$, or $\frac{2}{tf}$. Hence, the thinner and smaller tubes are, the more surface they have as compared with their storage capacity for energy.

The same considerations cause us to use thin tubes for surface condensation and other purposes, and there is the further consideration that accidents are less likely to be serious.

In cases where energy is stored in hot water and steam, the rate of waste of energy is proportional to the surface, and this requires just the opposite conditions.

EXERCISES.

1. Sixty tubes of wrought iron 4 inches inside diameter, 10 feet long, ¼ inch thick. Find volume, weight, internal area in square inches, and working pressure if working $f = 10{,}000$ lbs. per square inch. Neglect the ends. How many tubes 22 inches diameter, 10 feet long, will have the same volume? And find the thickness suitable for the working pressure. Find also the area and weight.

Ans., 52·36 cub. ft., 6,729 lbs. wt., 12·54 sq. in., 1,250 lbs. per sq. in., 2, 1·375 inches, 380 sq. in., 6,525 lbs.

2. Cylindric boiler of mild steel 5 feet 6 inches in diameter, at elastic limit; pressure 300 lbs. per square inch. What is the thickness? Joints supposed to be of 60 per cent. of strength of unhurt plate. Replace this boiler with tubes 5 inches diameter of same length. How many tubes are needed to make up the same volume? What will be their thickness (no seams)? Their weight? Now replace with 3-inch tubes, finding thickness and weight.

Ans., ·47 inch; 174; ·021 inch; 193 lbs. per foot length; ·013 inch; 199 lbs. per foot.

274. When a **belt or rope** of weight w lbs. per inch of its length is moving with a velocity of v feet per second in a curved path of radius r, the centrifugal force on a small length of it, $r \, . \, \delta\theta$, is $w \, . \, r \, . \, \delta\theta \, \frac{v^2}{rg}$. Now, if a sketch be made showing how this force is balanced by tensile forces T at the ends of the small length $r \, . \, \delta\theta$, it will be seen that the centrifugal force is equal to $T \, . \, \delta\theta$ if $\delta\theta$ is very small; so that $T = wv^2/g$, being independent of the radius. This, then, is the tensile force which acts in a belt or rope when in *motion*—an addition to the tensile force which acts when the rope is at rest—and it must be taken into account in considering the strength of a belt. Notice also that if a is the section of the rim of a pulley of wrought iron, the weight of it is $\cdot 28a$ lb. per inch of its length. Hence the tensile force is $\cdot 28av^2/g$, or $\cdot 28v^2/g$ lb. per square inch is the tensile stress induced in the rim by centrifugal force when it moves at v feet per second. Taking the working tensile stress in wrought iron of a pulley as 6,000 lbs. per square inch, the rim speed of a wrought-iron pulley ought not to exceed 850 feet per second. The usual limiting speed of cast-iron pulley rims is 80 feet per second. Arms, if numerous, serve to diminish this action. If an arm of uniform cross-section moves at n turns per second, the limiting length L of it is $\frac{1}{2\pi n}\sqrt{\frac{1{,}546}{w}f}$, if w is the weight per cubic inch. Thus, if we take $f = 6{,}000$, $w = \cdot 28$, $\text{L} = \frac{916}{n}$. Thus, if $n = 50$ revolutions per second, L is about 18 inches. If such arm has a section a at the distance r from the centre, it is easy to show that if each section has simply to withstand a pull due to the centrifugal force of the part outside it $a = a_0\epsilon^{-br^2}$, where, if r is in inches, $b = \pi^2 w n^2 r^2/193f$.

The condition as to strain of a rotating disc has been investigated by Dr. Chree (see Art. 307).

As the tensile force in a perfectly flexible rope due to its motion is independent of the shape of the path in which a particle is moving, a very rapidly-moving rope, if no external force such as those due to gravity act upon it, has no tendency to any alteration of shape; each particle follows the path of every other, like particles of water in a stream-line in a state of steady motion. We can impulsively alter the shape of such a rope at any place, and then the new shape will be retained. Vibrations will be transmitted by such a rope as if along a naturally straight rope not moving in the direction of its length in which there is the same tension as is here produced by motion. A thinking student will from these facts readily see that there is a quasi-rigidity produced in the rope by the rapid motion.

EXERCISES.

1. Two pulleys, 3 feet 6 inches in diameter, running at 150 revolutions per minute, are connected by a leather belt weighing 0·6 lb. per foot in length. Taking $\mu = \cdot 3$, find the greatest tension in belt when transmitting $7\frac{1}{2}$ horse-power. *Ans.*, 260 lbs.

2. In a travelling crane the driving rope runs at 5,000 feet per minute. Find the tension due to centrifugal action, having given that a rope 1 inch diameter weighs 0·28 lb. per foot of length. *Ans.*, 60·4 lbs.

275. Strength of Thick Cylinders.—Let the inside and outside radii be r_1 and r_0, and the inside and outside fluid pressures be p_1 and p_0. Consider an elementary ring of metal, 1 inch parallel to the axis, inside radius r, outside radius $r + \delta r$. Imagine a compressive stress p inside it and $p + \delta p$ outside (δp is usually negative in our examples), and a compressive stress q in the material (q is usually negative or the stress is tensile) at right angles to the radius. $p \times 2r$ is a force tending to fracture this ring at a diametrical plane; $2\,(p + \delta p)\,(r + \delta r) - 2q\,.\,\delta r$ is the force tending to prevent fracture. Note that there is a possible shear stress on the sides of this strip that we are neglecting, and a careful student will give thought to the matter. Our justification for neglecting it lies in this, that the strength of our cylinder cannot be imagined to depend upon its length; and if we consider a very long cylindric strip, end effects are negligible. Balancing the forces, we have the well-known rule for the strength of a thin cylinder. Divide by the thickness δr, and imagine δr smaller and smaller, and we find $p + r\dfrac{dp}{dr} = q \dots (1)$. As the material is subjected to crushing stresses p and q in two directions at right angles to one another in a cross-sectional plane, the dimensions parallel to the axis of the cylinder elongate by an amount which is proportional to $p + q$. We must imagine this to remain constant if a plane cross-section is to remain a plane, and we make this reasonable assumption. Hence (1) has to be combined with $p + q = 2a \dots (2)$, where $2a$ is a constant. Substituting q from (2), in (1) we get $\dfrac{dp}{dr} = \dfrac{2a}{r} - \dfrac{2p}{r} \dots (3)$, and we find by trial that the solution is $p = a + \dfrac{b}{r^2}$ and $q = a - \dfrac{b}{r^2} \dots (4)$, where the constants a and b are to be found by the conditions of any problem. Thus, in the case of a gun or hydraulic press, let the pressure inside be p_1 where $r = r_1$, and let $p_0 = 0$ where $r = r_0$. If we insert these conditions in (4), we find (q is everywhere negative, and I shall use f, the tensile stress, to replace $- q$)

$$-q = f = p_1 \frac{r_1^2}{r_0^2 - r_1^2} \cdot \frac{r_0^2 + r^2}{r^2} \dots (5).$$

f is greatest at $r = r_1$, and is then $f_1 = p_1 \dfrac{r_0^2 + r_1^2}{r_0^2 - r_1^2} \dots (6)$.

The student is to note that the circular compressive strain at any place is $q\alpha - p\beta$. This is the fractional diminution of the radius r.

A student ought to take an example such as: An hydraulic press has an inside radius $r_1 = 4$ inches; the stress is not to exceed 5,000 lbs. per square inch; find the greatest possible pressure p_1, first, if the thickness is 1 inch, then if the thickness is 2 inches, and so on. Note how little gain there is by increasing the thickness more than a certain amount; and it may be well to write out a list of numbers for various thicknesses, showing among other things the gain in weight.

EXERCISES.

1. In a certain kind of work either one cylindric hydraulic press of 24 inches diameter or four are needed, of same aggregate area and same material, to stand the same pressure. Compare the square area on which the two arrangements will stand. Observe that the ratios of *internal to* external radii will be the same in the small and large presses.

If the student figures it out, he will find that the four will just occupy the same square space as the single press, and they will weigh just the same.

2. **Thin Cylinder.**—Take $r_1 = \frac{1}{2}d$ and $r_0 = \frac{1}{2}d + t$, where d is the inside diameter of a cylindric boiler and t is its thickness, and assume that t is small. Then (6) becomes

$$f = p\,(\tfrac{1}{2}d^2 + dt + t^2)/(dt + t^2) = pd\left(1 + \frac{2t}{d} + \frac{2t^2}{d^2}\right)/2t\left(1 + \frac{t}{d}\right).$$

A first approximation which is generally used, and has been given above, is $f = \dfrac{pd}{2t}$ . . . . (7). A second is $f = \dfrac{pd}{2t} + \frac{1}{2}p$ . . . . (8).

3. A gun of 12 inches internal and 24 inches external diameter is subjected to a maximum internal pressure of 40,000 lbs. per square inch. Find the stress produced at $r = 6, 7\frac{1}{2}, 9, 10\frac{1}{2}$, and 12 inches. Find what was the initial stress everywhere if it was just sufficient to cause the final stress everywhere to be the mean of the stress produced at $r = 6$ and $r = 12$. Now make diagrams showing the state of stress when $p_1 =$ 60,000, 50,000, 40,000, 30,000, 20,000, and 10,000 lbs. per square inch.

Ans., 66,666 ; 47,466 ; 37,037 ; 30,748 ; 4,444 lbs. per square inch.

4. Pipes of a water-pressure supply company are to withstand a possible pressure of 1,000 lbs. per square inch; they are of 6 inches internal diameter. What is the outside diameter, the safe tensile stress of the metal being 3,000 lbs. per square inch? *Ans.*, 8·485 inches.

5. Pipes are to withstand a working pressure of 1,000 lbs. per square inch. If their internal diameters d are 2, 3, 4, 5, and 6 inches in each case, find the thickness. Find in each case the weight w per foot length. Draw a curve showing the relative values of w and d.

Ans., 0·41, 0·62, 0·83, 1·04 inch.

6. A cast-iron water-main is 30 inches internal diameter and 1 inch thick. What is the greatest head of water that it ought to be subjected to? Safe tensile stress, 3,000 lbs. per square inch. Numbers to recollect are: 34 feet of head represent 1 atmosphere, or head in feet ÷ 2·3 = pressure in lbs. per square inch. If the pipes are wrought iron, what ought their thickness to be if safe $f = 10{,}000$ lbs. per square inch, and if the longitudinal seams are of 60 per cent. of the strength of the unhurt plate?

Ans., 460 ft.; ·5 inch.

276. In the above theory we have considered the material initially unstrained; or, rather, the stresses and strains calculated by us are additional to any initial stresses and strains in the material.

The student will see why the outer material of a thick cylinder is comparatively useless if he shows in a curve f for various values of r, calculating from (5), for f decreases as the inverse square of r. If, when $p_0 = p_1 = 0$, there are already strains and stresses in the material, the stresses given in (5) are algebraically added to those already existing at any place. Hence, in casting an hydraulic press, we **chill it internally,** cold water circulating in a metal core painted with loam; and in **making a gun** we build it of tubes, each of which squeezes those inside it. So that there is considerable compressive stress where $r = r_1$ and considerable tensile stress where $r = r_0$ before any pressure comes on inside. We try to create such initial stresses that when there is the maximum pressure p_1 the material has about the same tensile stress in it everywhere. Much knowledge is needed to produce this result in guns.

Exercise.—**Thick spherical shell** subjected to internal fluid pressure. If p is the radial compressive stress at a point at the distance r from the centre, and q is the tangential tensile stress there, show after the manner of Art. 275 that $p = \text{A} + 2\text{B}/r^3$, $q = \text{A} - \text{B}/r^3$, where A and B are constants, which may be found if the internal pressure and the inner and outer radii are stated.

277. The above theory of the strength of thick cylinders seems to agree with our practical experience for such ratios of r_0 and r_1 as we find in the pipes and presses used by engineers. But all the rules given above show that a **flat plate** has no strength. The neglected terms in our theory become important in this case. In truth, the mathematical theory of a shell is so troublesome that we cannot say there is yet a satisfactory treatment of it. The strength of a flat plate has, however, been investigated in a number of cases, and we are led to the following results:—For a circular plate of thickness t and diameter d supported all round its edge, with a normal load of p lbs. per square inch, if f is the greatest stress in the material, $f = 5r^2p/6t^2$. If the plate is *fixed* all round its edge, $f = 2r^2p/3t^2$. A square plate of side s, fixed at the edges, $f = s^2p/4t$. A rectangular plate of length l and breadth b, fixed round the edges, $f = l^4b^2p/2t^2(l^4 + b^4)$. A round plate with a load W in the middle, supported at the edges, $f = \text{W}/\pi t^2$. For stays in square formation, distance asunder being s; each stay has a load ps^2, and the greatest stress in the plate of thickness t is $2s^2p/qt^2$.

278. When the pressure is greatest outside a thin shell, its strength **to resist collapse** ought evidently to follow the law (1), which becomes (3) if the vessel is cylindric; but it is in the very way in which a strut may be relied upon if the slight lateral restraints are provided which prevent bending. So also the thin shell of a boiler flue must be provided with certain restraints against buckling; and just as we have (Art. 372) a theory of laterally unsupported struts, so we have a theory of long boiler flues. Beyond a certain length for a given pressure there is instability, and hence flues are either corrugated or furnished with a number of rings. The most important practical outcome of the theory is that the distance between two rings divided by $\sqrt{dt}$ must not exceed a certain limit. The rules usually followed by boiler-makers are given in the new edition of my book on Steam.

Exercise.—In a **corrugated flue** of 3 feet mean diameter the plate is ½ inch thick, but the corrugations, being longer than the axial length, make it virtually ¾ inch thick. What is the working pressure if the working compressive stress (we allow for corrosion etc.) in the material is 3,000 lbs. per square inch?

Ans., 125 lbs. per square inch.

MORE DIFFICULT EXERCISES ON THICK CYLINDERS.

1. A tube of wrought iron, inside radius 3 inches, outside 4 inches, outside pressure 0. What is the inside pressure P to produce a maximum tensile stress of 15,000 lbs. per square inch? Find the fractional increase in size of the inside radius. Here $p = 0$ where $r = 4$; $p = \mathrm{P}$, and $q = -15{,}000$ where $r = 3$. Inserting these values in (4) Art. 275, we find

$$\begin{array}{l|l} o = a + \dfrac{b}{16} & 15{,}000 = b\left(\dfrac{1}{9} + \dfrac{1}{16}\right) = b\,\dfrac{25}{144} \\ \mathrm{P} = a + \dfrac{b}{9} & b = 86{,}400 \\ q = a - b/r^2 & \\ -15{,}000 = a - \dfrac{b}{9} & a = -5{,}400 \end{array}$$

$$\mathrm{P} = -5{,}400 + 9{,}600 = 4{,}200 \text{ lbs. per square inch.}$$

Let the student calculate $-q$ for several values of r from 3 to 4, and plot his results on squared paper.

r	3	3·5	4
$-q$	15,000	12,453	10,800

The fractional diminution in size of any radius is $q\alpha - p\beta$, or $(a - \beta)\,a - \frac{b}{r^2}(\alpha + \beta)$. Taking $\alpha = \frac{1}{3} \times 10^{-7}$, $\beta = \frac{1}{12} 10^{-7}$, the fractional diminution of the 3 inch radius is

$$\frac{1}{4} 10^{-7} a - \frac{b}{9}\frac{5}{12} 10^{-7} = -5{,}350 \times 10^{-7}.$$

That is, the 3 inch radius increases, becoming $3 \times 5{,}350 \times 10 - 7$, or 0·00161 inch larger.

2. A tube of wrought iron, inside radius 2 inches, outside 3 inches, no pressure inside; pressure $\mathrm{P} = 4{,}200$ lbs. per square inch outside. Find the circular compressive stress everywhere, and also the diminution of the outer radius.

Here in $\qquad p = a + \dfrac{b}{r^2},\ q = a - \dfrac{b}{r^2} \ldots .\ (1).$

$$\left.\begin{array}{c}\text{Fractional diminution} \\ \text{of the 3 inch radius}\end{array}\right\} = (a - \beta)\,a - \frac{b}{9}(\alpha + \beta) \ldots .\ (2)$$

In this case $p = 4{,}200$ where $r = 3$; $p = o$ where $r = 2$.

$$\begin{array}{r|l} 4{,}200 = a + \dfrac{b}{9} & 4{,}200 = b\left(\dfrac{1}{9} - \dfrac{1}{4}\right) = -\dfrac{5}{36}\,b \\ o = a + \dfrac{b}{4} & b = -30{,}240,\ a = 7{,}560 \end{array}$$

Hence $q = 7{,}560 + 30{,}240/r^2$. Thus for the following values of r we have

r	2	2·5	3·0
q	15,120	12,400	10,920

And the 3-inch radius decreases by the fraction of itself

$$\frac{1}{4} \times 7{,}560 \times 10^{-7} + \frac{30{,}240}{9}\,\frac{5}{12}\,10^{-7},$$

or $3{\cdot}29 \times 10^{-4}$, so that the 3 inch radius becomes ·00099 inch smaller.

3. A tube of radius 4 inches outside, and radius 2·9984 inside, is squeezed in some way upon a tube 2 inches inside radius and 3·00098 inches outside radius. Find the compressive circular stress at all points in both tubes.

Ans., It will be seen that I have taken just the sizes necessary to produce the states of Exercises 1 and 2. It is evident that we may take the outside radius of the inner tube as 3 inches, and the inside radius of the outer tube 2·9974 inches. Observe that as the coefficient of expansion of iron is $1{\cdot}2 \times 10^{-5}$, to produce a fractional increase in size ·0026 ÷ 3, the outer tube must be raised in temperature more than $\frac{\cdot 0026}{3} \div (1\ 2 \times 10^{-5})$, or 72 Centigrade degrees, before it will slip over the other.

4. A tube of wrought iron, 2 inches radius inside, 4 inches outside, is subjected to a fluid pressure of 50,000 lb. per square inch inside and *no* pressure outside. Find the tensile stress everywhere in the material or the values of $-q$.

Here in $p = a + \frac{b}{r^2}$, $q = a - \frac{b}{r^2}$, insert $p = 50{,}000$ where $r = 2$, $p = 0$ where $r = 4$, and so find $-q = \frac{266{,}667}{r^2} + 16{,}667$. Of course $-q$ is the tensile stress—

r	2	2½	3	3½	4
$-q$	83,333	59,333	46,300	38,440	33,333

5. To sum up our results. The built-up tube of Exercise 3, with initial tensile stress f or $-q$ of Ex. 3 taken from the tables of Exercises 1 and 2, is subjected to the internal fluid pressure of 50,000 lbs. per square inch of Exercise 4, there being no pressure outside. We have seen that f' or $-q$ of Ex. 4 shows the tensile stress produced by the fluid pressure, and hence f'', which is $f' + f$, is the real tensile stress everywhere in the *compound* tube. Students ought to draw curves showing f, f', and f''.

r	2	$2\frac{1}{2}$	3	3	$3\frac{1}{2}$	4
f	– 15,120	– 12,400	– 10,920	15,000	12,453	10,800
f'	83,333	59,333	46,300	46,300	38,440	33,333
f''	68,213	46,933	35,380	61,300	50,893	44,133

279. These exercises will enable the student to understand the usual calculations of shrinkage which must be made in building up a gun. He will have no great difficulty in working out all the necessary formulæ himself, but a man interested in gun-making ought to refer to three articles published in *Nature* about August, 1890, by Prof. Greenhill. It will be noticed that when a gun is built of tubes there is a sudden change in the tensile stress at the common surface of two tubes. To get a more uniform stress throughout, instead of using many thin tubes we now use an inner tube of steel strong enough to withstand the longitudinal forces, then a thick layer of steel wire wound on with varying tension, and then a covering tube, which is almost unstrained initially. Probably the hydraulic presses of the future will also be built up in this way.

Prof. Greenhill says: "Mr. Longridge's principle of strengthening a tube with wire wound with appropriately varying tension will be found useful in peace and in war. He can claim credit that a gun strengthened on this principle (the 9·2 inch wire gun) was chosen, from its great strength, to test the extreme range of modern artillery in 1888, with what were called the "Jubilee rounds," when, with an elevation of about 40 degrees, a range of 21,000 yards, or 12 miles, was attained, the projectile weighing 380 lbs., and the muzzle velocity being about 2,360 feet per second."

280. The following example will illustrate how calculations may be made:—

Example.—A steel tube 4 inches internal and 5 inches external diameter has steel strip wound on it to the external diameter of 10 inches, under a constant tensile winding stress w lbs. per square inch. Find the radial and circumferential stress anywhere.

Consider the place r in the wire. When the wire has already been wound to an outside radius ρ, let a new layer of thickness $\delta\rho$ be wound on. This produces a pressure $w \,.\, \delta\rho/\rho$. It is easy to see, just as in Example 2, that a radial pressure $w \,.\, \delta\rho/\rho$ at ρ, there being no radial pressure at $r_1 = 2''$, produces at r an increase δp of radial compressive stress, and an increase δq of circumferential compressive stress where

$$\delta p = \frac{w \,.\, \delta\rho/\rho}{\frac{1}{r_1^{\,2}} - \frac{1}{\rho^2}}\left(\frac{1}{r_1^{\,2}} - \frac{1}{r^2}\right) \ldots . \;(1),$$

$$\delta q = \frac{w \,.\, \delta\rho/\rho}{\frac{1}{r^{\,2}} - \frac{1}{\rho^2}}\left(\frac{1}{r_1^{\,2}} + \frac{1}{r^2}\right) \ldots . \;(2).$$

These effects are produced by winding on a new thickness δp of st Integrating these from $\rho = r$ to $\rho = r_0$, we find

$$p = \frac{w}{2}\left(1 - \frac{1}{x}\right)\log.\frac{x_0 - 1}{x - 1} \ldots . \;(3),$$

$$q = \frac{w}{2}\left(1 + \frac{1}{x}\right)\log.\frac{x_0 - 1}{x - 1} \ldots . \;(4),$$

where x stands for $r^2/r_1^{\,2}$ and x_0 for $r_0^{\,2}/r_1^{\,2}$. The tensile stress the strip or wire was w lbs. per square inch, and we see that is now reduced to $w - q = f$ where

$$f = w\left\{1 - \tfrac{1}{2}\left(1 + \frac{1}{x}\right)\log.\frac{x_0 - 1}{x - 1}\right\} \ldots . \;(5).$$

I have used (3) and (5) in calculating the following numbers :—

r	2·5	3	3·5	4	4·5
p	·402 w	·399 w	·315 w	·210 w	·103 w
f	− ·833 w	− ·036 w	·382 w	·65 w	·847 w

At $r = 2\tfrac{1}{2}$ inches, the outside of the steel tube and inside the wire, there is a discontinuity. Calculating from (3) (5), the tensile stress there is $f = -\,0·832\,w$ and $p = 0·402$ Now the metal of the solid tube is subjected to external press ·402 w and no internal pressure, and we find that in it compressive circumferential stress $q = 4·47\,w\left(\tfrac{1}{4} + \frac{1}{r^2}\right) \ldots .$

and the radial stress is $p = 4·47\,w\left(\tfrac{1}{4} - \frac{1}{r^2}\right) \ldots . \;(7).$

I have calculated the values of p and q, or rather of $-\,q$, wh I call f, from (6) and (7) for the solid tube :—

r	2	$2\tfrac{1}{4}$	$2\tfrac{1}{2}$
$-\,q = f$	− 2·235 w	− 2 w	− 1·83 w
p	0	·234 w	·402 w

We know now the internal condition of this coiled structure, which we may call a gun.

Let the gun, considered as homogeneous, $r_0 = 2$, $r_0 = 5$, be subjected to fluid pressure $3\,w$ inside and no pressure outside. It is easy to see that this fluid pressure would produce the tensile stress everywhere $\left(\cdot 57 + \frac{14\cdot 28}{r^2}\right) w$, and as before, leaving out the w, the numbers marked f' in the following table show its value. In this table the initial radial compressive stress p and circular tensile stress f are copied from the previous tables. The actual tensile stress, therefore, in the gun is $f' + f$, shown as f''.

r	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
p	0	·234	·402	·402	·399	·315	·210	·103	0
f	−2·235	− 2	−1·83	−0·833	−·036	+·382	+·65	+·847	1
f'	4·14		2·85	2·85	2·16	1·74	1·46		·14
f''	1·90		1·02	2·02	2·12	2·12	2·11		2·14

It will be observed in the above table that the actual tensile stress in the wire is nearly constant. This is an accidental result.

We usually assume that f'' shall be a fixed constant in the wire; f' is known, so we may calculate the necessary values of f, the initial tensile stress everywhere, in the finished gun. f is evidently of the shape $f = m - h/r^2$. To produce this particular f, it is necessary to alter the winding tensile stress w in the wire as the winding proceeds. Thus in (1) and (2), integrating, we have w a function of ρ; so, remembering that w is to be looked upon either as a function of r or ρ, integrating (2) and writing what is equivalent to (5), we have

$$w + \left(\frac{1}{r_1^2} + \frac{1}{r^2}\right) \int_r^{r_0} \frac{w.\rho.d\rho}{1 - \rho^2/r_1^2} = f$$

Differentiating with regard to r, we see that if θ be written for $w - f$ and if x stands for r^2/r_1^2, then

$$\frac{d\theta}{dx} - \theta.\ \frac{1}{2x}\ \frac{1+x}{1-x} + \frac{f}{2x} = 0$$

It is not difficult to solve this and so find the way in which w, the winding tension, ought to vary.

CHAPTER XIV.

SHEAR AND TWIST.

281. LET C D (Fig. 176) be the top of a firm table, F H a long prism of indiarubber glued to the table, A B a flat piece of wood glued along the upper side of the indiarubber. We try in this way to apply a horizontal force to the whole upper surface of the indiarubber, so that if, for instance, the pull in the cord is 20 lbs., and the upper surface of the indiarubber is 10 square inches in area, there will be a force of 2 lbs. per square inch acting at every part of the surface, and this force will be transmitted through the indiarubber to the table. When the length of the prism is great compared with Z F, we may suppose that the bending in it is very small, and in this case we say that the indiarubber is being subjected to a simple **shear strain**; the force per square inch acting on its surface is also acting from each horizontal layer to the next, and is called the **shear stress.** If you had drawn vertical lines like Y′ X before the cord was pulled, you would now find them sloping like Y X. Thus, making a magnified drawing of Y X in Fig. 176, the point Y′ has gone to Y, and any point like M has gone to N. Points touching the table cannot move, but the farther a point is away from this fixed part the farther it can move. Now suppose that Y′ Y is 0·01 inch, and we know that X Y′ is 2 inches, what is the amount of motion of M if M X is 1·7 inch? Evidently

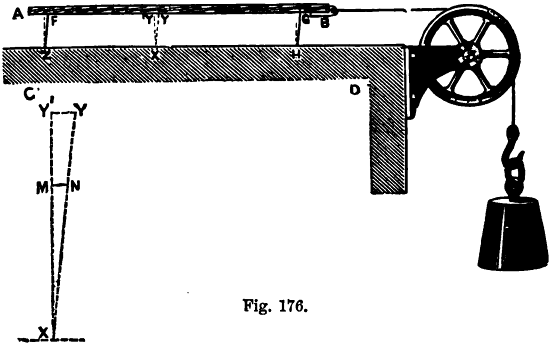

Fig. 176.

Y′ Y is greater than M N just in the proportion of Y′ X to M X, or 2 to 1·7; hence M N is 0·0085 inch. Thus the motion of any point is simply proportional to its distance above the fixed plane, and if we know the amount of motion at, say, a distance of 1 inch, we can calculate what it must be anywhere else. *The amount of motion at one inch above the fixed plane is called the* **shear strain.** The motion is small, and it is evident that the shear strain is the angle Y′ X Y in radians. In this case we have supposed the force on F G to be 2 lbs. per square inch. This is said to be the amount of the **shear stress,** and it produces or is produced by a *shear strain* whose amount is ·005 inch per inch. If the shear stress were 4 lbs. per square inch, you would find the strain to be ·01; if the stress were 8 lbs. per square inch the strain would be ·02. In fact, we find experimentally that the stress and strain are proportional to one another. Thus if, instead of indiarubber, we had a block of tempered steel, we should find that the force in pounds per square inch is equal to 13,000,000 times the strain. This number is called the **modulus of rigidity** for steel; it is given in Table XX. It seems a pity that the name *shearing elasticity* is not given to this number.

282. However long we may make our block of indiarubber in Fig. 176, we shall still have some bending in it; that is, the stress will not be uniformly distributed over each horizontal layer (see Chapter XXI.). To prevent this bending effect, and **to produce a really simple shear strain,** we ought to have force distributed over the ends F Z and G H of the same amount per square inch as we have now acting over F G and Z H. These are shown in Fig. 177. Where P is the pull in the cord of Fig. 176, P′ is the equal and opposite force exerted by the table on the glued underside of the indiarubber, and F and F′ are equal and opposite forces distributed over the ends, such that the couple F F is able to balance the couple P P′. There can now be no bending moment at any place. As F multiplied by the length of the prism is the moment of the couple F F′, and is equal to P multiplied by the vertical dimension, we see that P distributed over the horizontal surface is the same stress per square inch as F distributed over the

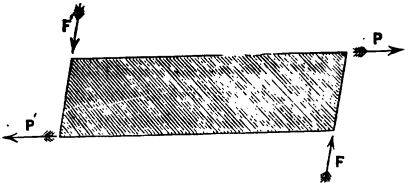

Fig. 177.

ends. From such a material then, if we cut a cubical block A (Fig. 178), its horizontal faces $\mathrm{Y}y$ and $\mathrm{X}x$ are acted upon by equal and opposite tangential forces, and its faces YX yx are acted upon by forces of exactly the same amount. The faces parallel to the paper have no forces acting on them. This will give you the best idea of **simple shear stress.** The material in Fig. 176 near the ends of the block does not get a simple shear; but if the block is very long, then at the middle there is a nearly simple shear acting.

In Fig. 178 the cube $\mathrm{X}\,\mathrm{Y}'\,y'\,x$ has become $\mathrm{X}\,\mathrm{Y}\,y\,x$. Suppose the side of this cube to be 1 inch, then $\mathrm{Y}'\mathrm{Y}$ is the shear strain, which I shall call s. The tangential force distributed over $\mathrm{Y}y$ is p lbs., let us say. Then, if we denote by the letter N the **modulus of the rigidity** of the material, $p = \mathrm{N}s$.

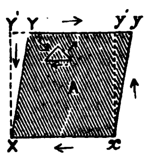

Fig. 178.

283. *Example.*—A beam of steel has one end fixed, and at the other is a weight of 20 tons. The cross section of the beam is 2 square inches in area, and the length of the beam is 5 inches. Besides the deflection of this beam due to bending, there is a certain **deflection due to shearing**; how much is it? Answer: the shear stress is 10 tons, or 22,400 lbs. per square inch. This produces a shear strain of 22,400 ÷ 13,000,000, or ·00172. This is the amount of yielding at 1 inch from the fixed end, and at 5 inches the yielding must be 5 × ·00172, or ·0086 inch.

In a short beam like this, or in one 20 inches long, if we consider, for example, that it is 1 inch broad and 2 inches deep, we may calculate the deflections due to bending and to shear, and reflect upon the fact that in very short beams the yield due to shear is much more important than the yield to bending; whereas in long beams we may, and indeed always do, neglect altogether the deflection due to shear. These reflections are in the main correct, but the actual distribution of shear stress over the sections of a beam not short and not long is unknown to us. Its distribution in the section of a long beam will be given in Art. 369, and it is obvious that the calculation of the deflection due to shear is not so simple as in very short beams.

284. The **shear stress which will produce rupture** is not well known for any substance except cast- and wrought-iron, but the shear stress which will produce permanent set is fairly well known, and we are also agreed as to the ordinary working shear stress of materials. For wrought iron it is usually regarded as less than the working tensile stress; but in a single-riveted lap joint in boiler-plates, as the holes are usually punched (and this weakens the metal), and as rivet iron is usually of a better quality than plate, the cross section of the iron which is left, which is resisting pull, is made to have the same area as the cross sections of all the rivets, which, of course, resist shearing. Besides breaking by either a tensile or a shear stress, a riveted joint may give way by the rivet crushing or being crushed by the side of its hole. Again, in many riveted joints, when the rivets are long, as they tend to contract in cooling and are prevented by the plates, so much tension may remain permanently in them that they are greatly weakened. In bolts there is usually some bending, and consequently a want of perfectly uniform distribution of the shear stress, and they are made larger than rivets in the same positions.

285. In the **punching** of rivet-holes it may be taken that a shearing force V acts on the material; *the area of the curved side of the hole, multiplied by the breaking shear stress of the material per square inch, represents the force with which the punch must be pressed down on the plate.* The punch must be able to resist this force as a compressive stress on its own material. Experiments made on punching-machines show that about 24 tons per square inch is the average shearing force required. This pressure has to be exerted through a very short distance. In **shearing-machines**, if the entire edges of the shears coincided with the plate as soon as they touched anywhere there would be the same sort of effect produced; but by inclining the edges the shearing action does not occur instantaneously at every place, and the rupture being more gradual than in punching, the shearing resistance is usually from 10 to 30 per cent. less. It is very probable that the power lost in punching- and shearing-machines is wasted rather in the friction of the heavy parts of the mechanism than in the almost instantaneous effort of cutting the material. The effort required seems rather that of an impact than of the more gradual action to be found in most existing machines. At the same time we must remember that M. Tresca's experiments indicated a **flowing of the metal**. Machines such as hydraulic bears and shears may be uneconomical as to mere energy, but they produce the higher economy due to convenience and certainty. A man can stop the motion of the hydraulic punch very rapidly if he sees that the plate is not quite right in position. In the fly presses used for hand punching, and used largely in coining, the idea of an impact is already in use; it will come much more into use in large machines when engineers become better acquainted with the distinction between force and energy.

286. Riveted Joints.—Students are supposed to have made sketches of a number of well-proportioned joints. In applying the results of experiment and our imperfect theory to actual structures we must remember that our practical conditions sometimes closely agree with our hypothetical conditions, and then our calculations may be fairly exact; whereas in other cases our theory is only an imperfect sort of guide in our workshop systems of seeking for success and avoiding failure. It is only a very capable engineer who knows exactly what weight ought to be given to his theory in every case.

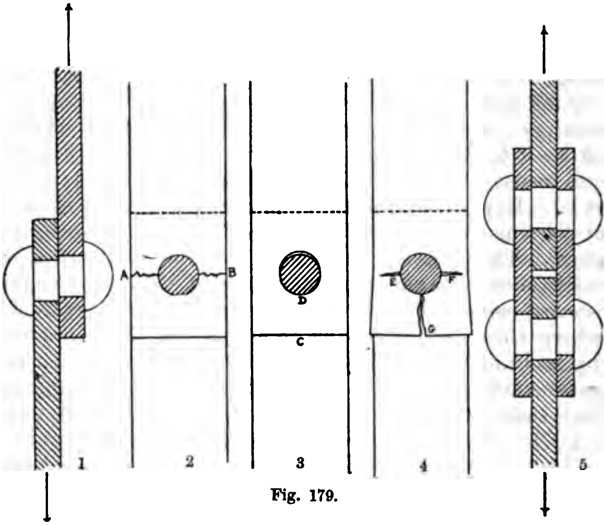

Fig. 179.

If we had a perfect theory we need only consider the various ways in which a riveted joint may fracture. Then we should state the algebraic conditions that the joint shall be equally ready to fracture in these various ways, and we have at once the right proportions.

Thus, take the strip which illustrates the single riveted lap joint of a boiler plate, Fig. 179 (1), or the single riveted butt joint of Fig. 179 (5). A B $= p$, the pitch or distance from centre to centre of rivets; t is thickness of plate; λ is D C, the overlap minus radius of rivet; f_s the shearing stress which the rivet will stand; f_t the tensile stress which the plate will stand.

If fracture occurs as in (1), the tensile force P which the joint will stand is $P = \frac{\pi}{4} d^2 f_s \ldots . (1)$. If in double shear, as in (5), $P = \frac{\pi}{2} d^2 f_s \ldots . (5)$. If, as in (2), $P = (p - d)\, tf_t \ldots . (2)$. If as in (3), when the iron of the plate is crushed by the rivet or the rivet is crushed by the iron, even the most absurd person will hesitate when he puts $P = f_c td \ldots . (3)$, f_c being some kind of surface crushing resistance per square inch, and td being assumed somehow to represent the surface at which crushing may take place. If, as in (4), we assume that the part EGF breaks like a beam fixed at the ends and loaded in the middle, we have a much wilder assumption, giving $P = \lambda^2 tf_t \div \frac{3}{4} d \ldots . (4)$. Now we cannot assume that P has any of the above values. There is always great friction at the joint; the rivet is in a state of unknown tension, and we have no information as to f_s under such circumstances, and in a lap joint there is evidently a bending action due to the plates not being in line. Nevertheless, all the above formulæ and similar formulæ easily made out for all joints may be made use of to *guide* us in obtaining information when we vary proportions and make tests.

For example, tests may be made upon λ and d, and it has been found that λ from the edge of the hole to the edge of the plate must be at least equal to d, and considerations of economy and workmanship guide us in adding about a quarter of an inch when only half-inch rivets are used. Now if we put (4) equal to (3), or $\lambda^2 tf_t \div \frac{3}{4} d = f_c td \ldots . (6)$, we have $\lambda = d \sqrt{\frac{3}{4} \frac{f_c}{f_t}}$. It is dangerous to follow this any further; we have reached a rule (by taking convenient values for the stresses) that agrees so well with the practice of the best makers that we shall be apt to think our theory more valuable than it is. It will be noticed that if write (3) equal to (1) we get a rule which also agrees fairly well with practice; but if we write (4) equal to (5), which we have just as good a right to do, we find that λ ought to be $\sqrt{2}$ times as great when the rivets are in double shear, and this is certainly not in agreement with good practice. If we imagine that instead of (4) we use the idea of a beam breaking by shearing, we again find that λ ought to be proportional to d. But the result of a careful consideration of this and many other points in machine design is that possibly such formulæ as (4) are more likely to be misleading than useful. A complete theory to replace (4) is perfectly possible, but it has not yet had the services of the necessary good mathematician. With (1) and (2) we feel much safer than with the others. Putting them equal, $\frac{\pi}{4} d^2 f_s = (p - d) t f_t \; \ldots \, (7)$. We need some other equation evidently. Now, in all design we try to obtain maximum advantage of some kind. To get maximum economy of material we were obviously right in trying to have a joint equally ready to break in various ways.

But there is a more general kind of economy, not at all easy to express algebraically (see my Calculus for Engineers, Art. 37), which tells us to punch holes in thin plates up to $\frac{3}{4}''$ thick and drill them in thick plates. For thin plates, then, we have such a rule as $d \propto t$, or taking into account cost of riveting, say $d = \frac{3}{4} t + \frac{7}{8}$ if we are not to have too much expense in the fracture of punches [compressive strength of punch more than equal to shearing resistance of plate], whereas we have only one of these things to consider in drilling holes. We may take the rule $d = 1.2 \sqrt{t} \; \ldots \, (8)$ as probably agreeing with the best practice from $t = \frac{1}{4}$ to $t = 1$ inch. If this rule were followed, then, using (7) and (8), we find $\frac{\pi}{4} \, 1{\cdot}44 \, f_s = (p - d) f_t$, or $p - d = {\cdot}36 \pi f_s / f_t = \text{A}$, say. The strength of the joint is the strength of the unhurt plate multiplied by $\frac{p - d}{p}$, or by $\frac{\text{A}}{\text{A} + d} \; \ldots \, (9)$. For a rivet in double shear we use (2), (5), and (8), and find $p - d = {\cdot}72 \pi f_s / f_t$, and we may call this A in the formula. It is just twice the last.

Now for complicated kinds of joint we must make assumptions which are less likely than the above. It is usual to calculate, instead of $p - d$ in (2), a width of strip w equivalent in tensile strength to one rivet in single shear, or $w = \frac{\pi}{4} d^2 f_s / t f_t$. Draw round each rivet a circle of diameter $d + w$, and let lines come to the circles dividing the plate up into strips of the breadth w; thus we allot a strip of plate to each rivet. Students ought to scheme for themselves examples such as are figured in books on "Machine Design." In such books they will also find sketches of many riveted joints which are well worth study in case there are no actual specimens to look at. The result of the calculation is a formula like (9).

The results of actual tests show that instead of A in the numerator of (9), we have a different number $k\,A$, and students who have read Chap. XIII. carefully, and also what I have here given, will know fairly well why k is not equal to unity. We have, then, $d = 1\cdot2\sqrt{t}$, $p = A + d$ . . . . (10). Strength of joint $= \dfrac{k\,A}{A + d} \times$ strength of unhurt plate. . . . (11), where k is given in the following table :—

	Iron Plates.	Steel Plates.
Single Riveted Drilled	·88	1·0
„ „ Punched	·77	0·9
Double „ Drilled	·95	1·06
„ „ Punched	·85	1·0
Treble „ Drilled	...	1·08

And A is given in the following table :—

		Iron Plates and Iron Rivets.		Steel Plates and Steel Rivets.	
		Drilled Holes.	Punched Holes.	Drilled Holes.	Punched Holes.
Lap Joint or Butt Joint with One Covering Plate	Single Riveted	1·20	1·47	0·9	1·08
	Double „	2·22	2·66	1·7	1·93
	Treble „	3·23	...	2·5	...

All these values of A are to be doubled for butt joints with two covering plates.

It is wise to slightly round the outside sharp edges of drilled holes. It is to be remembered that a punched hole is to be called a drilled hole if the plate has been annealed, or if the hole has been rhymered out after punching.

The suitable pressure P lbs. per square inch of a boiler is then $P = t \dfrac{k A}{A + 1{\cdot}2 \sqrt{t}} f \div R$ if t and R are thickness of plate and radius of boiler in inches, and f is the suitable tensile stress in pounds per square inch of the unhurt plate, and where A and k are the numbers given in the above tables for the *longitudinal* seams.

In the following exercises the working f is to be taken as 10,000 lbs. per square inch for wrought iron, and 12,000 lbs. per square inch for mild steel.

EXERCISES.

1. Cylindric boiler 10 feet diameter, $1\frac{1}{4}$-inch steel plates, with steel rivets, drilled holes; butt-joint, with two covering plates; treble riveted. What is the working pressure? What are the pitch and the diameter of the rivets? *Ans* , $P = 213$ lbs. per sq. m. $P = 6{\cdot}34$ inches. $d = 1{\cdot}34$ inch.

2. Cylindric boiler 5 feet diameter, iron plates, iron rivets, double riveted butt-joint, one covering plate, drilled holes; working pressure, 150 lbs. per square inch. Find t. *Ans.*, $t = 0{\cdot}67$ inch.

3. Find the dimensions and strength of a treble riveted butt-joint with double covering plates for plates $\frac{3}{4}$ inch thick. Plates and rivets are of steel. *Ans.*, $d = 1{\cdot}04$ inch. $p = 6{\cdot}04$ inches. Efficiency $= {\cdot}895$.

4. Determine the pitch of the rivets for a single riveted lap-joint of $\frac{1}{2}$ inch plate, so that the joint may be equally strong to resist tearing and shearing. Diameter of rivets is $\frac{7}{8}$ inch. Safe shearing strength 7,800 lbs. per square inch; safe tensile strength is 10,000 lbs. per square inch. *Ans.*, $1{\cdot}813$ inch.

5. In a single riveted lap-joint the plates are $\frac{1}{2}$ inch thick, and the rivets $\frac{7}{8}$ inch diameter; calculate the pitch for the greatest strength of joint, the shearing resistance of the rivet being three-quarters of the tiensle resistance of the plate per square inch of section. *Ans.*, $P = 1{\cdot}78$ inch.

6. The steel plates of a boiler are $\frac{9}{16}$ inch thick, and connected by longitudinal double riveted butt-joints. with covering plates on each side; find suitable proportions for a joint and calculate its efficiency. *Ans.*, $d = 0{\cdot}9$ inch. $p = 4{\cdot}3$ inches. Efficiency $= {\cdot}79$.

7. In a single riveted lap-joint, assume that the shearing strength per unit of area of the rivets is equal to the tearing strength of plates per unit area. If the plates are $\frac{1}{2}$ inch thick, diameter of rivets $\frac{7}{8}$ inch, and the pitch $2\frac{1}{4}$ inches, will the joint yield by tearing or by shearing? Calculate the efficiency of the joints. *Ans.*, Tearing. Efficiency, ·69.

8. Two lengths of mild steel tie-rods, 7 inches × 1 inch, are to be connected with double butt-straps. Determine dimensions and efficiency. *Ans.*, $d = 1{\cdot}3$ inch. Three rivets each side. Straps, $t = 0{\cdot}647$ inch. Efficiency, 0·81.

287. Nature of Shear Strain.—When y' moves to y, Fig. 177, the diagonal $x\,y'$ becomes extended to $x\,y$. Imagine the angle $y'\,x\,y$, as shown, to be an exaggeration of about one hundred times what it ever is in iron or steel. Its original length was $\sqrt{2}$ (the diagonal of a square whose side is 1), and its new length is $\sqrt{2} + \frac{s}{\sqrt{2}}$, as we see very easily. Hence the diagonal x y', *and all lines parallel to this diagonal*, have a tensile strain, whose amount is $\frac{\text{elongation}}{\text{original length}}$ or $\frac{s}{\sqrt{2}} \div \sqrt{2}$, and this is $\frac{s}{2}$. Again, in the same way we find that the diagonal $y' x$, and all lines parallel to it, have a compressive strain whose amount is $\frac{s}{2}$. Thus it has become quite clear that a shear strain simply consists of a **compressive strain** in one direction, accompanied by a **tensile strain** in the perpendicular direction, these strains being each half the shear strain.

We are led to speculate on the behaviour of material subjected to equal compressive and tensile stresses crossing interfaces at right angles to one another. Consider a small right-angled prism of material, shown in Fig. 178, of which M F K (Fig. 180) is a magnified cross-section. Neglect the weight of the prism, and for ease of calculation make M F, say, 1 inch, M K the same, and let the length of the prism at right angles to the paper be also 1 inch. This prism is kept at rest by the matter outside it acting on its three faces. Face M F is pushed by a normal force of p' lb. per square inch, and as its area is just 1 square inch, the resultant push is p' lb. acting at the centre of the square face M F. Similarly the face M K is pulled by a normal force of p' lb. And also the face F K is acted on by tangential forces of p lb. per square inch, and as its area is $\sqrt{2}$ square inch, the total amount of shearing force acting on F K is $p\sqrt{2}$ lb. Now, when three forces keep a body in equilibrium, and two of them are at right angles, the sum of their squares is equal to the square of the third force (this is easily seen if we draw the triangle of forces); hence the square of $p\sqrt{2}$, which is $2p^2$, is equal to $p'^2 + p'^2$ or $2p'^2$. Hence $p = p'$; and we have proved that the **compressive and tensile stresses** which occur in simple shear strain are numerically equal to what we called the shear stress. We have also to recollect that we cannot have shear stress parallel to the paper in planes at right angles to the paper without having equal shear stress parallel to the paper in a set of planes at right angles to the first and also to the paper (see Art. 282).

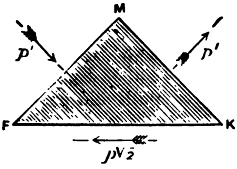

Fig. 180.

Suppose we cut a cube A B C D (Fig. 181) of 1 inch side, from a material subjected to simple shear strain, and let the faces of the cube parallel to the paper have, as before, no stress upon them, the other faces being at right angles to the directions of compression and extension. Shear occurs parallel to the face A C. Let us consider the motion of the point D relative to A C; in fact, regard A C as fixed. Under the sole action of the pushes on A D and B C we know that the side D C shortens by the small amount $p\alpha$ (see Art. 265). Let us set this off from D to M. But when this occurs the side A D lengthens by the amount $p\beta$; set this off from M to D′. Hence the pushing forces on A D and B C cause D to move to D′. Again, the pulling forces on D C and A B further lengthen A D by the distance $p\alpha$, which we set off from D′ to L, and shorten D C by the distance $p\beta$, which we set off from L to D″. Hence the motion of D due to the pulls and pushes acting together is D D″, and we see that this is

$$(DM + LD'')\sqrt{2} \text{ or } (\alpha + \beta)\, p\sqrt{2}.$$

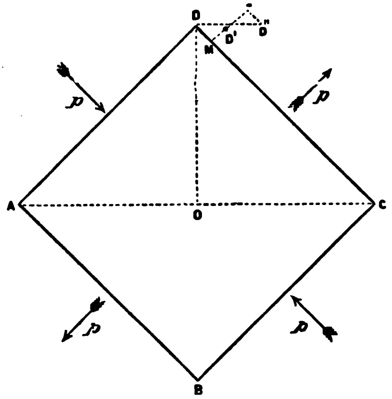

Fig. 181.

But s, the amount of shear, is D D″ ÷ D O, and as $DO = \frac{1}{\sqrt{2}}$ inch, as A D is 1 inch, we have

$$s = (\alpha + \beta)\, p\sqrt{2} \div \frac{1}{\sqrt{2}} \text{ or } 2p\,(\alpha + \beta).$$

That is, *shear strain = shear stress multiplied by* 2 ($\alpha + \beta$). So that *the reciprocal of* 2 ($\alpha + \beta$) *is what we called* N, the **modulus of rigidity** of the material.

289. General Results.—Referring back to Arts. 269 and 288, you will see that we have

Modulus of rigidity ... ... $N = \frac{1}{2\,(\alpha + \beta)}$

Modulus of elasticity of bulk ... $K = \frac{1}{3\,(\alpha - 2\beta)}$

Young's modulus of elasticity ... $E = \frac{1}{\alpha}$;

and you will also see that if we know two of these for any material, we can find the third.

These results are so important that we put them also in the *shapes*:—

$$\alpha = \frac{1}{E} = \frac{1}{3N} + \frac{1}{9K}, \quad \beta = \frac{1}{6N} - \frac{1}{9K}, \quad E = \frac{9NK}{3K+N}, \quad \sigma = \frac{\beta}{\alpha}, \quad \beta = \frac{1}{2N} - \frac{1}{E}.$$

Some French mathematicians have thought that the ratio of β to α (called **Poisson's ratio**, and always denoted by the letter σ), and therefore the ratios of N, K, and E to one another, are constant for isotropic substances, α being always four times β. Experiment has shown that this is not the case, the ratio of α to β being 3 to 2·5 in glass or brass, 3·3 in iron, 4·4 to 2·2 in copper, and in other substances varying from these values very much indeed.

Just as Young's modulus is seldom found from experiments on the extension of wires, but rather from the bending of beams, so the modulus of rigidity is seldom found from experiments like that of Fig. 176, but rather from experiments on the torsion of rods or wires.

In all the cases of simple shear which we have described we had more than simple compression and extension. There was a rotation of the lines in which the compression and extension took place. When there is no rotation, and this is very easy to imagine (let A C not be imagined fixed in Fig. 181, but imagine the lines at 45° to the horizontal to remain fixed in direction), so that the principal directions of strain, as they are called, remain the same in direction, the strain is said to be *pure* (or irrotational), otherwise it is said to be "rotational."

EXERCISES.

1. For mild steel $E = 30 \times 10^6$, $N = 12 \times 10^6$; find α, β, and K.

$$\textit{Ans.,}\ \alpha = \frac{1}{30 \times 10^6}, \quad \beta = \frac{1}{12 \times 10^7}, \quad K = 20 \times 10^6.$$

2. The halves of a flange coupling are bolted together with six bolts at 6 inches from the centre, 60 horse-power transmitted, 100 revolutions per minute; safe shear stress, 8,000 lbs. per square inch. Find the proper diameter of each bolt. *Ans.*, ·41 inches.

3. A 3-inch square steel bar, fixed at one end, loaded with 80 lbs. at its other end; find the deflections due to shearing in the two cases; length 3 inches, length 50 inches; according to Art. 281, and compare with the correct method of Art. 369.

4. Ten cubic inches of wrought iron and 10 cubic inches of water are subjected to fluid pressure of 3 tons per square inch; find the new volumes. If the iron is spherical, what are the old and new diameters?

Ans., Iron, 9·99664 cubic inches; water, 9·9776 cubic inches.
Iron, diameters 2·673 and 2·6727 inches; water, 2·673 and 2·671 inches.

5. For wrought iron $E = 29{,}000{,}000$, $K = 20{,}000{,}000$; find α, β, N, and calculate the value of Poisson's ratio.

$$\textit{Ans.,}\ \frac{1}{29 \times 10^6}, \quad \frac{1}{11 \times 10^7}, \quad 23 \times 10^6;\ 1 \text{ to } 3{\cdot}8.$$

6. A cube of copper 3 inches edge is subjected to a hydrostatic pressure of 4 tons to the square inch; find its new volume. $K = 24{,}000{,}000$. *Ans.*, 26·99 cubic inches.

7. For copper N = 5,600,000, K = 24,000,000; calculate the value of E and Poisson's ratio. *Ans.*, 15·6 × 10^6; 1 to 2·6.

8. A steel punch ¾ inch in diameter is employed to punch a hole in a plate ⅝ inch in thickness: what will be the least pressure necessary to drive a punch through the plate when the shearing strength of the material is 35 tons per square inch? *Ans.*, 51·56 tons.

9. In a fly press for punching holes in iron plates the two balls weigh 30 lbs. each, and are placed at a radius of 30 inches from the axis of the screw, the screw itself having a pitch of 1 inch. What diameter of hole could be punched by such a press in a wrought-iron plate of ¼ inch in thickness, the shearing strength of which is 22·5 tons per square inch? Assume that the balls are revolving at the rate of 60 revolutions per minute when the punch comes into contact with the plate, and that the resistance is overcome in the first sixteenth of an inch of the thickness of the plate.* *Ans.*, 1·136 inch.

290. We have been in the habit of testing material under tensile stress only, or compressive stress only, or shearing stress only. Tests are now greatly required of the strength of material under **combined tensile and compressive stress**, these not being equal in amount as they are in shear.

In an indirect manner we have evidence that if on an interface there is shear stress q and compressive stress p, the strength of the material in resisting fracture *at this section by shearing* is as if a shear stress acted of the amount $q - \mu p$, where μ is a constant for the material. In fact, the compressive stress strengthens the material in its resistance to shearing. It is as if μ were a co-efficient as of friction, preventing sliding. The evidence is the fact that when struts of cast iron, stone, brick, and cement are crushed, fracture usually occurs at a section which makes an angle greater than 45° with the cross-section.

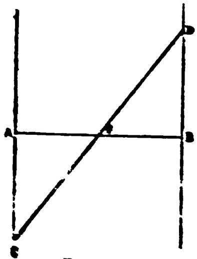

Fig. 182.

In Fig. 182, A B is the cross-section of area A of a tie-bar or strut, and C D is a sloping section making an angle θ with the cross-section: so that the area of C D is A sec. θ. An axial load W produces tensile or compressive stress $p_1 = \frac{W}{A}$ on the cross-section, and $\frac{W}{A \sec. \theta}$ or $p_1 \cos \theta$ in an axial direction on the sloping section. The stress on the sloping section consists, therefore, of a normal tensile or compressive stress $p_1 \cos.^2 \theta$ and a shear stress $p_1 \cos. \theta \ . \sin. \theta$. The material does not tend to break at C D by the normal stress if the material is isotropic, as $p_1 \cos.^2 \theta$ is less than p_1. The shear stress is a maximum, and equal to $\frac{1}{2} p_1$ if θ is 45°. Now, in materials like cast iron, stone, brick, and cement, the ultimate shear stress is less than half the ultimate compressive stress, and it might be expected that the fracture of a strut would be at a section making an angle of 45° with the cross-section. But in every case we find the angle greater than 45°, being nearly constant for the same material. The assumption that there is a resistance to shearing of the nature of friction—that is, proportional to the normal compressive stress—is not only a reasonable-looking hypothesis, but it agrees quite well with the observed phenomena. For a shear stress $p_1 \cos. \theta \sin. \theta$ and a compressive stress $p_1 \cos.^2 \theta$ are by our assumption to be replaced by a shear stress $p_1 (\cos. \theta \sin. \theta - \mu \cos.^2 \theta)$ . . . . (1) ; and this is a maximum on the plane for which $\theta = 45° + \frac{\phi}{2}$ if $\mu = \tan. \phi$.. (2

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* I suppose that we shall always have academic exercises like this and like Ex. 6 of Art. 172. The advanced student will notice (see Art. 486) that it is just as misleading as those absurd pile-driver problems in which force is calculated as the kinetic energy of the pile-driver divided by the distance through which the pile is driven, so that, if the pile is not driven any distance into the ground, the force is infinite. In impact questions like these safety is only to be found in considering force as the time rate of transfer of momentum.

In cast iron, fracture usually occurs on a plane making 54¾° with the cross-section. If our hypothesis were really correct, our co-efficient of internal friction would therefore be ·35 for cast iron.

291. There is much published information on the fracture by **compression of blocks of stone, cement, and bricks.** In almost every case care is taken in loading the usually short specimens that friction at the ends shall prevent the material swelling laterally. When sheet-lead is inserted at the ends it gives a small amount of lateral freedom, and in every case the breaking load is lessened by its use, and therefore it is said to be wrong to use lead. I consider all this published information to be nearly valueless, except that there is some probability that half the usually published ultimate compressive strength for a cube is the true resistance to compression in the material. Hence, when the published strength of stone is given as between 250 and 1,600 tons per square foot, we ought possibly to take it as truly from 125 to 800. In masonry structures the working stress is probably never as great as 50 tons, and generally it does not reach 10 tons per square foot.

292. To see to what extent we can, by means of lateral fluid pressure on a strut, strengthen it, we may provisionally use the theory above given.

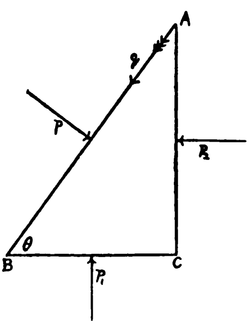

Fig. 183.

If there is compressive stress in a strut, p_1, and a lateral normal pressure p_2, and the material breaks by shearing, let the plane B C be part of the cross-section, with axial compressive stress acting through it; let the plane A C have normal stress p_2 through it; let the interface A B have normal compressive stress p and tangential stress q upon it; consider the equilibrium of a prism 1 inch at right angles to the paper, and let A C = 1 inch. The normal pressure forces on the faces parallel to the paper balance one another independently. Equating the resolved parts of the forces vertically and horizontally, we find

$$p = p_1 . \cos.^2\theta + p_2 \sin.^2\theta, \quad q = (p_1 - p_2) \cos.\theta . \sin.\theta.$$

We shall use the hypothesis that q will not produce fracture until it exceeds f by the amount of the friction μp; that is, we may take it that the value of the shear stress as a producer of rupture is only $q - \mu p$, and it is easy to show that this is greatest when $\theta = 45 + \frac{\phi}{2}$ .... (1). This angle, therefore, is independent of p_1 and p_2. If $q - \mu p$ is put equal to f_s, we see that the least value of p_2 to resist fracture is

$$p_2 = \frac{p_1 (\cos.\theta \sin.\theta - \mu \cos.^2\theta) - f_s}{\cos.\theta \sin.\theta + \mu \sin.^2\theta} \; .... (2),$$

or, indeed, we may say that the principal compressive stresses p_1 and p_2 will produce fracture if

$$p_1 (\cos.\theta \sin.\theta - \mu \cos.^2\theta) - p_2 (\cos.\theta \sin.\theta + \mu \sin.^2\theta) > f_s ...(3),$$

where θ has the value given in (1). p_2 is supposed to be less than p_1. If p_2 is a tensile stress, we have only to give it a negative sign in (3). (3) may be written $mp_1 - np_2 > f_s$ .... (4), where m and n are constants for the material.

Now let p_1, p_2, and p be tensile stresses, and reverse the arrowhead on q. We have the same numerical relations between p, q, p_1, p_2, and fracture occurs by shearing when f_s has the value $q + \mu p$, so that fracture occurs when

$$p_1 (\cos.\theta \sin.\theta + \mu \cos.^2\theta) - p_2 (\cos.\theta \sin.\theta - \mu \sin.^2\theta) > f_s,$$

if θ is taken of such a value as to make $q + \mu p$ a maximum; that is, if $\theta = 45° + \phi/2$ .... (2). In this p_2 is supposed to be less than p_1. If p_2 is a compressive stress, we have only to give it a negative sign in (1). (1) may be written $m^1p_1 - n^1p_2 > f_s$, where m^1 and n^1 are constants for the material. Let $p_2 = 0$. If p_1 is compressive stress, $\theta = 45 + \frac{\phi}{2}$, and it will be found that

$$\frac{2f_s}{p_1} = \sec.\phi - \tan.\phi \; .... (4),$$

$$\text{or } \mu = \tan.\phi = \left\{ 1 - \left(\frac{2f_s}{p_1}\right)^2 \right\} \Big/ \frac{4f}{p_1} \; .... (5).$$

If p_1 is a tensile stress, $\theta = 45 - \frac{\phi}{2}$, and it will be found that

$$\frac{2f_s}{p_1} = \sec.\phi + \tan.\phi \; .... (6),$$

$$\text{or } \mu = \tan.\phi = \left(\frac{4f_s^2}{p_1^2} - 1\right) \Big/ \frac{4f_s}{p_1} \; .... (7).$$

Given compressive stress p_1 and the value of f to find the least value of p_2 for resistance to fracture by shearing; take $\theta = 45^\circ + \frac{\phi}{2}$ as giving the plane on which the tendency to shear is greatest, whatever p_2 may be, and we find

$$p_2 = -\frac{2f_s \,.\, \cos.\phi + p_1\,(1 - \sin.\phi)}{1 + \sin.\phi}.$$

Thus, for example, taking cast iron, let us suppose that $\phi = 28^\circ$ and $\mu = 0{\cdot}53$, as already found, if the ultimate $f_c = 95{,}000$, and $f_s = 28{,}500$, then $\sin.\phi = {\cdot}47$, $p_2 = -\ 34{,}000 + .\ 36\, p_1$ (that is, p_1 increases 27,800 lbs. per square inch for every increase of 10,000 lbs. per square inch in p_2).

It cannot be said that published numbers give any adequate test of the above theory.

If for any material f_c, f_t, f_s are the numerical values (treated all as positive quantities) of the three stresses which the material will stand, $2f_s = f_c\,(\sec.\phi - \tan.\phi) = f_t\,(\sec.\phi + \tan.\phi)$.

Thus, for example, $\frac{fc}{ft} = \frac{1 + \sin.\phi}{1 - \sin.\phi}$.

Now in cast iron the proof-stresses given in Table XXII. are $f_c = 21{,}000$, $f_t = 10{,}500$, $f_s = 8{,}000$, $\frac{fc}{ft} = 2 = \frac{1 + \sin.\phi}{1 - \sin.\phi}$, and hence $\phi = \sin.^{-1}\frac{1}{3} = 19^\circ\frac{1}{2}$, $\tan.\phi = {\cdot}35$, $\theta c = 54^\circ\frac{3}{4}$, $\theta t = 35^\circ\frac{1}{4}$. With this value of ϕ, f_s *ought* to be ${\cdot}356 f_c$, whereas it really is ${\cdot}38 f_c$. This is a discrepancy very allowable, and we may take it as some sort of verification of the theory. These were the first numbers I tried, but I have since found that other published numbers are less satisfactory in their support. For example, f_c ought to be greater than f_t for all materials if the theory is correct. It is evident that special experiments are required as a test.

Tresca and Darwin have propounded the hypothesis that $p_1 - p_2$ is constant; and it ought to be easy, by experiment, to decide whether this is constant, or whether, as by my theory, it increases as p_1 and p_2 increase. For example, I make $p_1 - p_2$ equal to 94,000 when $p_2 = 0$ in cast iron, and equal to 272,000 when $p_2 = 100{,}000$. These are from the ultimate values. As to the permanent set strength of cast iron, I make $p_1 - p_2 = 21{,}000$ when $p_2 = 0$, and 121,000 when $p_2 = 100{,}000$. In fact, instead of making $p_1 - p_2 = 94{,}000$, I make $p_1 - 2{\cdot}78\ p_2 = 94{,}000$.

The hypothesis of Poncelet and St. Venant is that the material is fractured when the strain exceeds a certain amount, and this hypothesis is often used to give ultimate strength conditions when the strains have been investigated mathematically. But in a wrought-iron bar we have sometimes before fracture strains which are several hundreds of times as great as the greatest strains to which the calculations apply, and it seems to me, therefore, that Poncelet's theory has no probability of correctness in its favour.

If we assume that in any kind of earth there is a coefficient of friction μ between layers, but no permanent resistance to true shearing (that is, $f_s = 0$), we get **Rankine's theory of earth pressure** from the above theory. The lateral pressure p_2 necessary to prevent a direct pressure p_1 from causing motion or fracture is given by $\frac{p_2}{p_1} = \frac{1 - \sin.\phi}{1 + \sin.\phi}$. Thus, if the static coefficient of friction in a certain kind of earth is ·9, so that ϕ is 42°, then p_2/p_1 is $\frac{1 - \cdot 6691}{1 + \cdot 6691}$, or, say, $\frac{1}{5}$.

Example.—The weight of a building is 10^7 lbs. The area of the concrete bottom of the foundations is 2,000 square feet. At what depth ought it to be below the level of the soil, if the soil is such that $\phi = 42°$?

Ans., the pressure p_1 lbs. per square foot is 5,000 lbs. To resist this, a horizontal pressure p_2 of 1,000 lbs. is needed. Regarding the horizontal pressure of 1,000 lbs. as a new p_1, it needs a vertical pressure of 200 lbs. per square inch, due to the weight of outside earth, to balance it. If the earth weighs 100 lbs. per cubic foot, the depth of the foundation must be at least 2 feet.

Assuming that the theory is right, the difficulty in carrying out a rule of this kind is that we do not know ϕ, the angle of repose of a particular kind of earth. Rankine assumed that a long mound of earth would, after much weathering and rest, get to have a natural slope ϕ. If the natural surface of earth is horizontal, it is easy to find on the above theory the stress on any interface when motion is about to take place, and particularly on a vertical interface, and so the reason for part of the following rule of Rankine's is known. The study of the stresses when the ground is sloping must be left as an exercise for students.

293. Rankine's Rule for Earth.—Draw an angle X O R, Fig. 184, to represent the static permanent angle of repose of the kind of *earth*. Describe Y R X, a semicircle touching O R. If A B, Fig. 185, is the *vertical face of a wall sustaining a bank* of this *earth*, whose slope is A C, make the angle X O P equal to the inclination of A C to the horizon. Find B D = O Q.A B/O P. Then

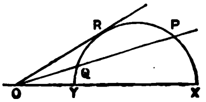

Fig. 184.

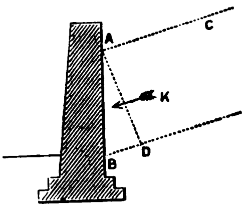

Fig. 185.

A B D is a wedge of earth whose *weight* represents the total *pressure* acting on A B. The pressures act in directions parallel to A C, and the resultant force, representing the total pressure, acts a third of the way up from B to A.

This rule may be compared with the rule of Art. 173 for water pressure against a vertical wall. Rankine neglects friction against the wall face. Students of this subject are directed to a paper by Sir B. Baker, Proc. Inst. C.E., Vol. 65, and its discussion.

When grain is stored in vertical prismatic cylinders, the average pressure in pounds per square foot on the flat bottom is $c\,d\,w$ where d is the diameter in feet, w the weight of a cubic foot, and c is 0·84 for wheat, 0·96 for peas. At greater heights than three times the breadth of section of a bin the pressure on the sides is constant, being about 50 lbs. per square foot for all kinds of grain until we get near the surface.

294. Twisting.—In Fig. 186, AB represents a wire held firmly at A. At B there is a pulley fixed firmly to the wire, and this pulley is acted upon by two cords, which tend to turn it without moving its centre sideways. In fact, they act on the pulley with a turning moment merely. But the pulley can only turn by giving a twist to the wire, and the amount of motion it gets tells us how much the twist is. A little pointer fastened at C moves over a cardboard dial, and tells us accurately how much twist is given to the wire. The angle turned through by the pointer is called the total **angle of twist** at C. If we had a pointer at each of the places G, H, and C, and if A, G, H, and C were one foot apart from one another, we should find that the

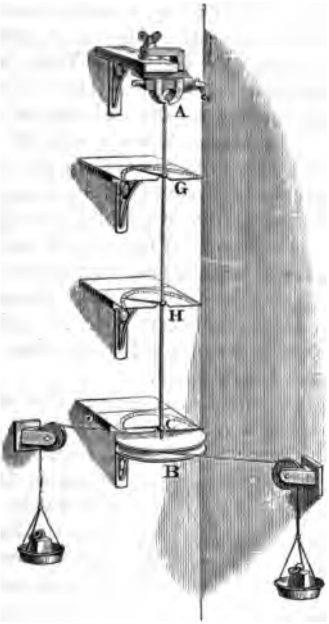

Fig. 186.

angles of twist at G, H, and C are as 1 : 2 : 3 ; in fact, *the angle of twist is proportional to the length of wire twisted.*

You will find that if a twisting moment of 10 pound-feet produces a twist of 4°, then a twisting moment of 20 pound-feet produces a twist of 8°, and, in fact, *the twist is proportional to the twisting moment* which is applied. You will also find that if you try different sizes of wire of the same material, say wire whose diameters are in the proportion of 1, 2, 3, &c., and to each of them you apply the same twisting moment, the amount of twist produced in them will be in the proportion 1, $\frac{1}{16}$, $\frac{1}{81}$, &c.; that is, *inversely as the fourth power of the diameter of the wire.* Lastly, taking wires all of the same diameters and lengths, but of different materials, and applying to them the same twisting moment, *the amount of twist will be inversely proportional to the number which we call the modulus of rigidity* of the material. The exact rules are given in (1) and (2), and the values of N given in Table XX. may be relied upon in such calculations, because they have all been determined from experiments on the twisting of wires and shafts.

Exercise.—A brass wire 20 inches long, 0·1 inch diameter, twists through a total angle of 130 degrees when a twisting moment of 4 pound-inches is applied. Find N for the material. Answer :—$3{\cdot}6 \times 10^6$ lbs. per square inch.

Exercise.—What would be the twist of a shaft of the same material with a twisting moment of 600 pound-inches, 20 feet long, 1·2 inch diameter? Answer :—7·8 degrees.

It is, however, well to notice that the drawn brass will probably have a different value for its N than the brass of a much larger shaft.

It will be seen that the strain is a **shear strain.** Consider M H G (Fig. 187) to be a cross-section of the wire ; then a point which is at H before the twist occurs is found to be at G when there is a twist in the wire, and a point such as P′ moves to P, but a point O in the centre of the wire does not move. Now there is no such motion at the fixed place A, Fig. 186, and in each section there is more of this motion the farther it is away from A ; in fact, the motion is just as it was in the indiarubber of Fig. 176, only that it varies

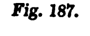

Fig. 187.

in the section, the motion being greatest at the outside of the wire, and nothing at the centre. The material breaks when the shear stress at the surface becomes too great, and the rule found by experiment is that for any material, whatever the length of the wire, the twisting moment which will cause rupture is proportional to the cube of the diameter. It is well known that when a shaft is transmitting power, the horse-power transmitted is proportional to the twisting moment or torque in the shaft multiplied by the number of revolutions made by it per minute. **The rule used by engineers** is this :—$d = 3{\cdot}3 \sqrt[3]{\text{H}/n}$ . . . . (1), as giving the safe diameter of a wroughtiron shaft at n revolutions per minute, if it is only subjected to torsion. Observe that if we double the speed, the shaft is strong enough for double the power. Instead of 3·3 we use 2·9 for a mild steel shaft and 4 for cast iron.

294a. Shafts usually carry pulleys, and are otherwise loaded as beams, as by the pull of belts, and therefore, for reasons given in Art. 379, we take $1\frac{1}{4}$ to $1\frac{1}{3}$ of the above size for mill shafting, and for crank shafts and shafts subjected to shocks we sometimes add 50 per cent. to the diameter as given by (1). We have some explanation in our theory of Art. 263 for this increase; some of it is due to the **variation in stress**, and therefore to fatigue; some of it is due to the fact that in crank shafts the **maximum torque** is often double the average torque. In a long line of shafting, if the power is given off at various places with some irregularity, it may even become evident to the eye that the shaft is perpetually twisting and untwisting, for of course the twist is proportional to the horse-power transmitted if the speed is constant. When this is the case, although the shaft may seem to be strong enough, it is weak because it is not stiff enough. A very long shaft sometimes gets into a state of **torsional vibration** just in the same way that the cage-rope of a coal-mine gets into a state of longitudinal vibration. The nature of this vibration will depend on accidental causes, and should the impulses that give rise to it happen to repeat themselves at proper intervals, the vibration may go on increasing until the torsion at some place may be sufficient to produce rupture. In the same way a number of men walking from side to side of a large ship, just taking as much time in going from one side to the other as the ship takes to make a vibration, may make the rolling dangerous.

It is for this reason that we endeavour to make the period of oscillation of a ship differ greatly from the probable period of waves which she may experience (see Art. 489).

In very small shafting this vibration often occurs, and it is usual to add vaguely $\frac{3}{8}$ to $\frac{5}{8}$ inch to the diameters found by the above rule—a sacrifice to the Goddess of Chance.

295. Consider a little prism, P B (Fig. 188), whose ends lie in two cross-sections of a shaft near together, O being the centre of one of the sections, and O′ the centre of the other. The twisting strain causes B to move to B′, regarding P as fixed. (The motion is, of course, usually very much less than I have here shown it). There must, then, be shearing forces acting on the ends in opposite directions. If α is the **angle of twist** of the shaft per inch of its length, then B O′ B′ is α multiplied by O O′; and if O P or O′ B is r, then B B′ is $r \alpha$ O O′, where α is an angle measured in radians. The shear strain in the little prism is B B′ divided by P B or O O′, so that it is $r\alpha$; hence the shear stress is $N r \alpha$ (see Art. 282). If a is the area of the end of the little prism in square inches, the shear force acting on it is $N r \alpha a$, and as this acts in the direction at right angles to the radius, its moment about O O′ is $N r^2 \alpha a$. But we have a similar moment for every such little area into which the cross-section may be divided, and to find the total torque we must take the sum of all such terms. Now, N and α are the same everywhere, so that in taking such a sum our only difficulty is with the factors $r^2 a$. But the sum of all such terms as $r^2 a$ is called the *moment of inertia* of the section about the axis O O′, and it has been calculated for us. Thus, if D is the diameter of a round shaft, the moment of inertia of its section about an axis through its centre *at right angles to the section* is $\pi D^4 \div 32$, and for a hollow shaft whose outside diameter is D and inside diameter d, the moment of inertia is $\pi (D^4 - d^4) \div 32$; and hence we see that the twisting moment T **necessary to produce a twist** *of α radians per inch in a round shaft of diameter* D *is* $T = \pi N \alpha D^4/32$ . . . . (1), and for a hollow shaft it is $T = \pi N \alpha (D^4 - d^4)/32$ . . . . (2). The torsional rigidity of a shaft is defined as A if $\alpha = T/A$. The values of A will be found in Table XV., Art. 532, for various sections of shaft. The verification of these rules is an excellent laboratory exercise.

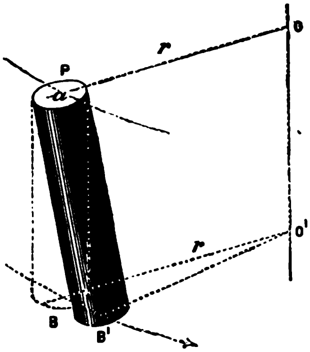

Fig. 188.

296. The strength of a shaft is to be calculated on the assumption *that* rupture occurs when the shear stress $N r \alpha$ mentioned above exceeds the greatest shearing stress to which the material ought to be subjected; and as the stress is greatest when r is the outer radius of the shaft or $\frac{1}{2}$ D, so that $f = \frac{1}{2}\text{N D}a$, and as from equation (1) (Art. 295) we find that $\text{N}a$ is $32\text{T} \div \pi\text{D}^4$, we know that $f = \frac{1}{2}\text{D} \times 32\text{T} \div \pi\text{D}^4$, and this is the condition of strength of a cylindric shaft. It is more compactly put in the form

$$\text{T} = \frac{\pi\text{D}^3 f}{16} \text{ for solid cylindric shafts} \ldots . (1),$$

and in the same way we get

$$\text{T} = \frac{\pi(\text{D}^4 - d^4)f}{16\text{D}} \text{ for hollow cylindric shafts} \ldots . (2),$$

f being the breaking shear stress of the material in pounds per square inch, T the twisting moment in pound-inches which will cause rupture, D the outer diameter, and d the inner diameter (if the shaft is hollow) in inches.

We see, then, that the strength of a solid shaft depends on the cube of its diameter, whereas its stiffness depends on the fourth power of its diameter.

As to the practical rule given in (1) (Art. 294), we saw in Art. 182 that torque in pound-*feet*, multiplied by angular velocity in radians per minute, divided by 33,000, is the horse-power. As we use T in pound-inches, $\text{T} = 12 \times 33{,}000 \times \text{H}/2\pi n$. If we use this in (1) of this article and take $f = 9{,}000$ lbs. per square inch, we find the usual practical rule (1) (Art. 294). Taking $f = 4{,}500$ for cast iron and 13,500 for steel, we find the rules for these materials already given.

297. Shafts Subjected to Twisting and Bending.—It is most convenient here to assume that a student has read the larger print in the next chapter, and knows that if a round shaft of diameter D inches is subjected to a bending moment M pound-inches there are compressive and tensile stresses through the cross-section at the circumference of amount $p = 32\text{M}/\pi\text{D}^3 = 2a\text{M}$, say .... (1) for a solid shaft, and $32\text{M D}/\pi(\text{D}^4 - d^4) = 2b\text{M}$, say .... (2) for a hollow shaft, if d is the internal diameter. We have just found that the shear stress on the same interface, due to a twisting moment T, is $f = a\text{T}$ and $b\text{T}$ .... (3) for the solid and hollow shafts. The other stresses across other interfaces at the point ought now to be studied, but it will be found that it is only necessary to determine the **principal stresses** there—that is, the greatest and least tensile or compressive stresses which act across any interfaces.

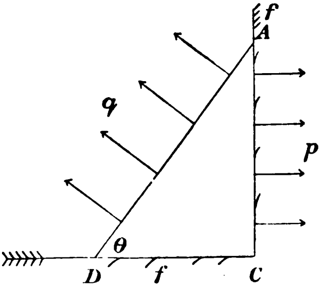

Fig. 189.

298. In Art. 292 we had a simple case of the general rule for stresses given in Art. 290, and this is another case nearly as simple.

Suppose that across an interface A C at right angles to the paper we know that there is a normal stress (either tensile or compressive) p lbs. per square inch and a shear stress f; that on planes parallel to the paper there is no stress; that on the planes at right angles to A C and the paper there is no normal stress. Consider the equilibrium of the prism A C D, and find the traction on A D of q lbs. per square inch. For the sake of ease of calculation we will assume A C to be 1 inch, and that the prism is 1 inch at right angles to the paper. Now, when there is shear stress f across A C there is also an equal shear stress f across D C (Art. 282). What are the forces with which stuff outside acts on the prism? The resultants of the forces on the faces are:—On A C, a horizontal force p and a vertical force f, because A C is 1 square inch in area; on D C, a horizontal force $f \times$ area of D C or $f . \text{D C}$ or $f . \cot . \theta$, as A C is 1; on A D, a force $q \times$ area of A D or $q . \text{cosec}. \theta$, and this force and its direction are what we desire to calculate. Given θ, it would be easy to calculate q and its direction; but our problem is even simpler. It is this: Find on what plane A D there is only a normal stress q with no tangential component, and find q. Resolving forces horizontally, $f \cot . \theta + p = q \text{ cosec}. \theta \times \sin . \theta$, or $f \cot . \theta + p = q \ldots .$ (1). Resolving forces vertically, $f = q . \text{cosec}. \theta . \cos . \theta$, or $f = q . \cot . \theta \ldots .$ (2). Hence, as (2) gives us $\cot . \theta = f/q$, using this in (1), we find $f^2/q + p = q$ or $q^2 - p q = f^2$; so that $q = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + f^2} \ldots .$ (3).

It is easy to find θ. There are two answers, differing by a right angle. **A stress is called a principal stress** if it is normal to the interface, and we see that we have in (3) the two values of the principal stress due to a combination of tensile p and shearing stress f, such as we have supposed. The principal stresses are across interfaces which are at right angles to one another.

Example.—At an interface $p = 6$ tons per square inch and $f = 5$ tons per square inch, then $q = 3 \pm 5{\cdot}83$; so that the principal stresses are 8·83 tons per square inch in tension and 2·83 tons per square inch in compression at right angles to one another.

Exercise.—Wrought iron is not to receive a greater tensile or compressive stress than 5 tons per square inch. There is already a tensile stress of 4 tons per square inch across an interface. What shear stress may also cross that interface?

Ans., $p = 4$ and $q = 5 = 2 + \sqrt{4 + f^2}$ from (3), and hence $f = 2{\cdot}24$ tons per square inch.

Exercise.—A round shaft is in torsion, and the shear stress produced across the section near the circumference is 8,000 lbs. per square inch. At the same section the shaft is subjected to bending, and a compressive stress of 6,000 lbs. per square inch is produced across the same interface. What is the greatest compressive stress in the material there?

Ans., $q = 3{,}000 + \sqrt{9 \times 10^6 + 64 \times 10^6} = 11{,}544$ lbs. per square inch.

Exercise.—If p and f are equal, find them that q may be just 6 tons per square inch.

Ans., Each of them is 3·71 tons per square inch.

299. We see, then, that a **hollow round shaft** subjected to the twisting moment T and the bending moment M is really subjected, at points near the circumference, to the maximum compressive or tensile stress q where $q = b\text{M} + \sqrt{b^2\text{M}^2 + b^2\text{T}^2}$ (see (2) and (3) of Art. 297), or

$$q = \frac{16\,\text{D}}{\pi\,(\text{D}^4 - d^4)}\left\{\text{M} + \sqrt{\text{M}^2 + \text{T}^2}\right\} \quad \ldots . (4),$$

and in solid shafts

$$q = \frac{16}{\pi\,\text{D}^3}\left\{\text{M} + \sqrt{\text{M}^2 + \text{T}^2}\right\} \quad \ldots . (5).$$

Hence the greatest and least compressive or tensile stress existing in a hollow or solid shaft when it is subjected to T and M is exactly the same as the greatest shearing stress in a shaft subjected only to a twisting moment $\text{M} + \sqrt{\text{M}^2 + \text{T}^2}$ .... (6).

300. An overhung crank shaft (Fig. 190) is subjected to a wrench. If the greatest force exerted at the pin A, at right angles to the crank, is F, the twisting moment is F . AC, and we may take the bending moment as F . BC if B is the middle of the journal. Hence the wrench is equivalent to a twisting moment $\text{F}\,(\text{BC} + \sqrt{\text{BC}^2 + \text{AC}^2})$, or F . (BC + AB), a rule which is perhaps a little unexpected.

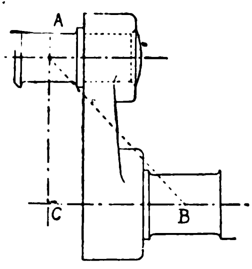

Fig. 190.

301. Ordinary shafts are of wrought iron or mild steel, materials such that their resistances to compression, tension, and shearing are not very different, and therefore when they are subjected to T and M we at once calculate (6), and say that the strength of the shaft is the same as if it were subjected to this twisting moment only. The practical rule (Art. 294) is worked on the idea of a twisting moment only, or $d \propto \sqrt[3]{\text{T}}$. If there is also a bending moment M, which is of the value $k\,\text{T}$, then the rule becomes evidently

$$d = 3{\cdot}3\,\sqrt[3]{\text{M} + \sqrt{\text{M}^2 + \text{T}^2}},\; d = 3{\cdot}3\,\sqrt[3]{\text{T}}\,\sqrt{k + \sqrt{k^2 + 1}}.$$

The extreme values of k for many kinds of shafting have been worked out, and the consequent change in the multiplier of (1), by means of shrewd guessing and calculation; but to my mind it is a better recommendation of the rules given at the end of Art. 294 that they are consistent with theory and with the best practice of engineers.

I have given here the usual theory of shafts subjected to twisting and bending; it assumes that the strength is determined by the maximum stress. A true theory would take account of the considerations of Art. 294a.

If a tensile stress p and a shearing stress f act on the same interface, and if p^1 and p^2 are the principal stresses, we see from (3) of Art. 298 that

$$p_1 = \tfrac{1}{2}p + \sqrt{\tfrac{1}{4}p^2 + f^2},$$
$$p_2 = \tfrac{1}{2}p - \sqrt{\tfrac{1}{4}p^2 + f^2}.$$

The theory of Arts. 290 and 292 tells us that fracture will take place if $m^1p_1 - n^1p_2 > f_s$, where m^1, n^1, and f_s are constants which ought to be known for the material. If p is a compressive stress, we have fracture if $mp_1 - np_2 > f_s$.

The probable values of m, n, m^1, n^1 for cast iron are given in Art. 292. For wrought iron and mild steel it is possible that ϕ of Art. 292 is 0, and hence $m = \frac{1}{2}$, $n = \frac{1}{2}$, $m^1 = \frac{1}{2}$, $n^1 = \frac{1}{2}$. Hence we have fracture either on the compressive or tensile side of a shaft—that is, whether p is compressive or tensile stress—if $\sqrt{\frac{1}{4}p^2 + f^2} > f_s$. Hence, if a shaft is subjected to the twisting moment T and the bending moment M, we calculate an equivalent twisting moment $\sqrt{\mathrm{T}^2 + \mathrm{M}^2}$, and assume that the shaft is subjected to this alone.

For materials in general, and probably even in the case of wrought iron and mild steel, the equivalent twisting moment ought to be calculated from $h\,\mathrm{M} + k\sqrt{\mathrm{T}^2 + \mathrm{M}^2}$, where h and k are constants, depending upon the nature of the material, which are not known at the present time.

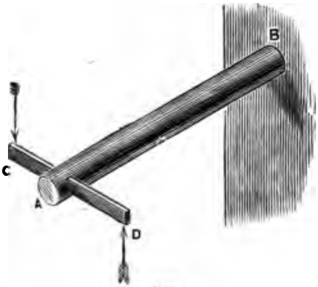

Fig. 191.

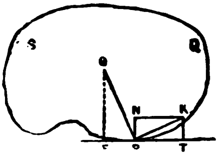

Fig. 192.

302. The demonstration of Art. 296 is found to agree with experiment, but its results must not be applied except to shafts which are **circular in section.** Our assumption, which experience warranted, was that when such a shaft as A B (Fig. 191) is fixed at B, and when

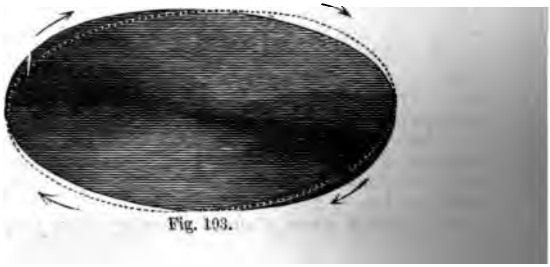

Fig. 193.

interface, and if p^1 and p^2 are the principal stresses, we see (3) of Art. 298 that

$$p_1 = \tfrac{1}{2}p + \sqrt{\tfrac{1}{4}p^2 + f^2},$$
$$p_2 = \tfrac{1}{2}p - \sqrt{\tfrac{1}{4}p^2 + f^2}.$$

The theory of Arts. 290 and 292 tells us that fracture will place if $m^1p_1 - n^1p_2 > f_s$, where m^1, n^1, and f_s are constants w ought to be known for the material. If p is a compressive st we have fracture if $mp_1 - np_2 > f_s$.

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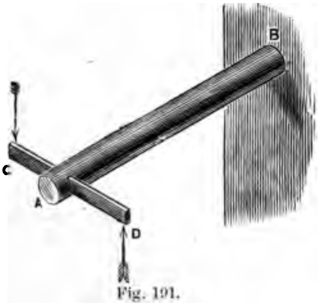
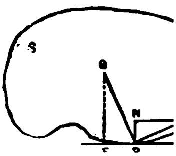


Fig. 191.



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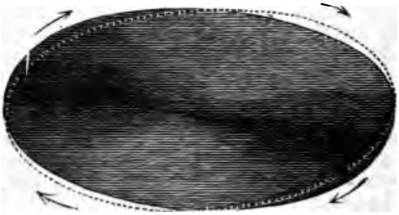

Fig. 193.

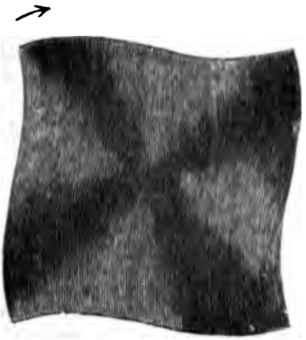

Fig. 194.

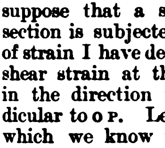

Fig. 195.

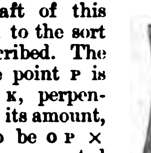

Fig. 196.

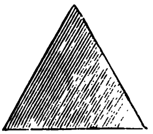

Fig. 197.

to an arm, C D, a twisting couple is applied, every straight line in a section remains straight, and moves through the same angle as every other line. But it can be shown that this is not the case for a shaft of any other than a circular section. Thus, let O (Fig. 192) be the centre of gravity of the section P Q R, and let us suppose that a shaft of this section is subjected to the sort of strain I have described. The shear strain at the point P is in the direction P K, perpendicular to O P. Let its amount, which we know to be O P × angle of twist, be represented by the length of P K. It is easy to show that this is just the same as a shear strain P N in the direction P N, normal to the surface of the shaft at P, together with a shear strain in the direction P T, tangential to the shaft at P. But shear strain in any direction is always accompanied by a similar strain in a plane at right angles to this direction (see Art. 282), so that since we have the shear P N, we must also have a shear parallel to the axis of the prism along the surface at P, and **this cannot be produced merely by a twisting moment.** We must imagine that along with the twisting moment there is a force distributed over the surface of the shaft to produce the above effects. The result of an exact investigation (Art. 313) is that a twisting couple produces a greater twist than might appear from what I have said in Art. 295, and it also produces a **warping** of the naturally plane sections of the shaft. Thus Fig. 193 is the shape assumed by each section of an elliptic shaft, and Fig. 194 of a square shaft. Imagine a section to be distinguishable, say, in a glass shaft by a thin layer of a different colour from the rest. Deeper shading indicates greater distance from the observer who is looking towards the fixed end of the shaft. The arrows show the direction of the twisting moment. In the following three sections, instead of the torque for a twist of one radian being equal to N times the moment of inertia of the cross-section, it is only ·84 times this for a square section (Fig. 195), ·54 times it for the section Fig. 196, and ·6 times it for the section Fig. 197. Indeed, the square section has only ·88 times the torsional rigidity of a cylindric shaft of the same sectional area; Fig. 196 has ·67 times, and Fig. 197 has ·73 times the torsional rigidity of a cylindric shaft of the same sectional area.

The numbers in the column headed W, Table XV., express **the relative strengths to resist twisting** of the various sections there figured.

The torsional rigidity of an elliptic section whose principal semi-diameters are a and b is $\text{N}\,\pi\,a^3b^3/(a^2+b^2)$. If M is the twisting moment, the shear stress at a point x, y (the axis of x being a) is $2\,\text{M}\,\sqrt{b^4x^2+a^4y^2}/\pi a^3b^3$. This is greatest at the end of the minor axis, being $2\,\text{M}/\pi ab^2$.

The torsional rigidity of a rectangle, if its length is two or more times its breadth, is, with some accuracy, the same as that of the inscribed ellipse multiplied by the ratio of their polar moments of inertia. The greatest shear stress occurs at the middle of the longer side of the boundary, and is $3\,\text{M}\,(a^2+b^2)/8\,a^3b^2$, if M is the twisting moment and a is half the longer, b half the shorter, side of the rectangle.

303. A very interesting result of the investigation is that there is always **greatest distortion at that part of the surface** of a shaft where the surface is **nearest the axis.** Thus in an elliptic shaft the substance is most strained at the ends of the shorter diameter of the section. Imagine a very light box to be made so as to contain **frictionless liquid** exactly of the shape of a shaft. If we give a sudden turn to the box about the axis, the liquid will be left behind if the box is circular in section, but it will have motions relatively to the box which can very readily be imagined if the shaft is not circular in section. Now the actual velocity of the liquid at any place **relatively to the box** is in the same direction as, and is proportional to, the shear in a similar shaft when it is twisted. This has been proved by Lord Kelvin. You will see from this that there is very little strain at the projecting ribs of the shaft, whose section is shown in Fig. 196, and just at the projecting angles of Figs. 195 and 197. This reminds me of a general remark which I have to make, and which I must leave without proof. A solid of any elastic substance cannot experience any finite stress or strain in the neighbourhood of a projecting point, unless acted on by outside forces just at the point. In the neighbourhood of an edge it may have strain only in the direction of the edge, and generally there will be exceedingly great strain and stresses at any re-entrant edges or angles. An important application of the last part of the statement is the well-known practical rule that every re-entering edge or angle ought to be rounded to prevent risk of rupture in solid pieces

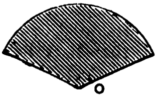

Fig. 198.

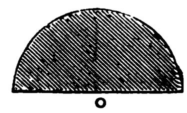

Fig. 199.

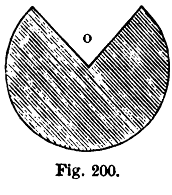

Fig. 200.

designed to bear stress. An illustration of the principle is the stress at the centre of the circular outline in the three sections of shafts (Figs. 198, 199, and 200). In Fig. 198, at o, there is no stress when the shaft is twisted; in Fig. 199 the stress may be calculated; in Fig. 200 the stress is exceedingly great for even the smallest twist (see Art. 302).

EXERCISES.

N.B.—**A most usual error of students is to forget that moments in pound-inches are not numerically the same as in pound-feet. Beginners had better leave the exercises involving bending moment until they have studied bending.**

1. A shaft 1 inch in diameter can safely transmit a torque of 2,400 pound-inches. What diameter of shaft would be required for transmitting 15 H.P. at 200 revolutions per minute? *Ans.*, 1¼ inch.

2. Find the horse-power which may be transmitted by a shaft 4 inches in diameter when running at 150 revolutions per minute, if the stress due to twisting be limited to 9,000 lbs. per square inch. *Ans.*, 273.

3. A line of steel shafting is 80 feet long; if a twisting moment of 4,000 pound-inches be applied at one end, what will be the total angle of twist, the diameter of the shaft being 2½ inches? What horse-power will this transmit at 220 revolutions per minute? *Ans.*, 5·2°; 14 H.P.

4. A solid wrought-iron shaft is to be replaced by a hollow steel shaft of the same diameter. If the material of the latter is 30 per cent. stronger than that of the former, what must be the ratio of internal to external diameter? What is the percentage saving in weight? *Ans.*, 1·44; 46 per cent.

5. The amount of twist in a solid shaft is to be limited to 1° for each 10 feet of length. Find the diameter for a twisting moment of 50 ton-inches, the modulus of torsional rigidity being 10,000,000 lbs. per square inch. *Ans.*, 5·29 inches.

6. A wrought-iron shaft is subjected simultaneously to a bending moment of 8,000 pound-inches, and to a twisting moment of 15,000 pound-inches. Find the twisting moment equivalent to these two and the least safe diameter of the shaft, the safe shear stress being taken at 8,000 lbs. per square inch. *Ans.*, 25,000 pound-inches; 2·52 inches.

7. Find the diameter of a shaft for a winding drum which works under the following conditions: the load lifted is 1½ ton; diameter of drum, 5 feet; width of face of drum, 26 inches; distance from inner face of drum to the middle of the bearing of shaft, 13 inches; maximum stress, 7,000 lbs. per square inch. *Ans.*, 5·44 inches.

8. A wrought-iron shaft 3 inches in diameter, and making 140 revolutions per minute, is supported at wall brackets 16 feet apart. There is a pulley on the shaft, midway between the bearings; if the resultant side pull due to the weight of the pulley and the pull of the belt be 210 lbs., what is the greatest horse-power the shaft will transmit with safety? Safe shear stress, 7,800 lbs. per square inch. *Ans.*, 66.

9. A bar of iron is at the same time under a direct tensile stress of 5,000 lbs. per square inch, and to a shearing stress of 3,500 lbs. per square inch. What would be the resultant equivalent tensile stress on the material? *Ans.*, 6,801 lbs. square inch.

10. Taking the safe tensile stress of wrought-iron to be 10,000 lbs. per square inch, determine whether it would be safe to subject a piece of wrought-iron to a tensile stress of 3½ tons per square inch, together with a shear stress of 3 tons per square inch.

Ans. Unsafe; max. stress = 11,700 lbs. square inch.

11. A wrought-iron shaft is subjected to a twisting moment of 36,000 pound-inches and a bending moment of 18,000 pound-inches; find the diameter when the maximum shear stress is 8,000 lbs. per square inch. Find also the twisting moment which alone would produce a shear stress of the same numerical value.

12. A screw propeller-shaft 10 inches in diameter is subjected to a twisting moment of 35 ton-feet, and to a bending moment of 10 ton-feet, due to the weight of the shaft and the pitching of the ship. What is the maximum compressive stress if the thrust of the screw be 10 tons?

Ans., 2·9 tons.

13. A shaft 12 inches diameter, transmitting a twisting moment of 100 ton-feet, is also subject to a bending moment of 20 ton-feet. Find the maximum stress induced. *Ans.*, 4·3 tons per square inch.

14. Find the diameter of a wrought-iron shaft to transmit 90 horse-power at 130 revolutions per minute. If there is a bending moment equal to the twisting moment, what ought to be the diameter?

Ans., ·27 inches.

15. A steam-engine crank is 12 inches long, and the greatest force which is transmitted through the connecting-rod is 9,000 lbs. Find the diameter of the wrought-iron crank-shaft, taking the safe shear stress at 9,000, the distance of the centre of the crank-pin from the centre of the bearing nearest it being 10 inches, measured horizontally.

Ans., 4·76 inches.

16. A round shaft 3 inches diameter, find the sizes of equivalent shafts of square, elliptic and rectangular sections if the breadth and thickness of each of these latter are as 1 to 2. If these shafts are 20 feet long, and they are transmitting 20 H.P. at 100 revolutions, what is the total twist on each of them. $N = 10{,}500{,}000$.

Ans., Rectangle, 2·15 × 4·3 inches; ellipse, minor axis, 2·38 inches square, 2·73 inches side. Twist on circular shaft = 2·073°, on square shaft, 1·78°; elliptical shaft, 1·05°; rectangular shaft ·93°.

304. [illegible]

306. [illegible] homogeneous strain in general. [illegible] infinitesimal strains, [illegible] this is easy.

Suppose that [illegible] is displaced to $x + u$, $y + v$, $z + w$ where u, v, w are very small, then

$$e=\frac{du}{dx}, f=\frac{dv}{dy}, g=\frac{dw}{dz} \quad \ldots\ldots (1)$$

[illegible] of the stuff in [illegible]. In Fig. [illegible] angles

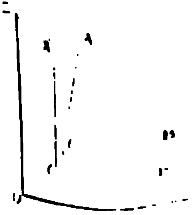

Fig. 203

It is for this reason that we endeavour to make the period of oscillation of a ship differ greatly from the probable period of waves which she may experience (see Art. 489).

In very small shafting this vibration often occurs, and it is usual to add vaguely $\frac{3}{8}$ to $\frac{5}{8}$ inch to the diameters found by the above rule—a sacrifice to the Goddess of Chance.

295. Consider a little prism, P B (Fig. 188), whose ends lie in two cross-sections of a shaft near together, O being the centre of one of the sections, and O′ the centre of the other. The twisting strain causes B to move to B′, regarding P as fixed. (The motion is, of course, usually very much less than I have here shown it). There must, then, be shearing forces acting on the ends in opposite directions. If α is the **angle of twist** of the shaft per inch of its length, then B O′ B′ is α multiplied by O O′; and if O P or O′ B is r, then B B′ is $r \alpha$ O O′, where α is an angle measured in radians. The shear strain in the little prism is B B′ divided by P B or O O′, so that it is $r\alpha$; hence the shear stress is $\text{N} r \alpha$ (see Art. 282). If a is the area of the end of the little prism in square inches, the shear force acting on it is $\text{N} r \alpha a$, and as this acts in the direction at right angles to the radius, its moment about O O′ is $\text{N} r^2 \alpha a$. But we have a similar moment for every such little area into which the cross-section may be divided, and to find the total torque we must take the sum of all such terms. Now, N and α are the same everywhere, so that in taking such a sum our only difficulty is with the factors $r^2 a$. But the sum of all such terms as $r^2 a$ is called the *moment of inertia* of the section about the axis O O′, and it has been calculated for us. Thus, if D is the diameter of a round shaft, the moment of inertia of its section about an axis through its centre *at right angles to the section* is $\pi \text{D}^4 \div 32$, and for a hollow shaft whose outside diameter is D and inside diameter d, the moment of inertia is $\pi (\text{D}^4 - d^4) \div 32$; and hence we see that the twisting moment T **necessary to produce a twist** *of* α *radians per inch in a round shaft of diameter* D *is* $\text{T} = \pi \text{N} \alpha \text{D}^4/32$ . . . . (1), and for a hollow shaft it is $\text{T} = \pi \text{N} \alpha (\text{D}^4 - d^4)/32$ . . . . (2). The torsional rigidity of a shaft is defined as A if $\alpha = \text{T}/\text{A}$. The values of A will be found in Table XV., Art. 532, for various sections of shaft. The verification of these rules is an excellent laboratory exercise.

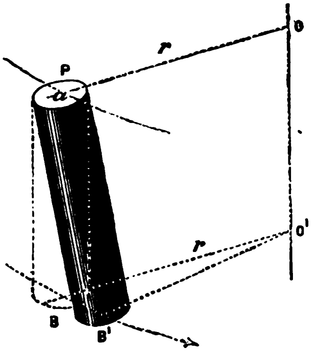

Fig. 188.

296. The strength of a shaft is to be calculated on the assumption that rupture occurs when the shear stress $\text{N} r \alpha$ mentioned above exceeds the greatest shearing stress to which the material ought to be subjected; and as the stress is greatest when r is the outer radius of the shaft or $\frac{1}{2}$ D, so that $f = \frac{1}{2} \mathrm{N\,D}\,a$, and as from equation (1) (Art. 295) we find that $\mathrm{N}\,a$ is $32\,\mathrm{T} \div \pi\,\mathrm{D}^4$, we know that $f = \frac{1}{2}\mathrm{D} \times 32\,\mathrm{T} \div \pi\,\mathrm{D}^4$, and this is the condition of strength of a cylindric shaft. It is more compactly put in the form

$$\mathrm{T} = \frac{\pi\,\mathrm{D}^3 f}{16} \text{ for solid cylindric shafts} \ldots. (1),$$

and in the same way we get

$$\mathrm{T} = \frac{\pi\,(\mathrm{D}^4 - d^4) f}{16\,\mathrm{D}} \text{ for hollow cylindric shafts} \ldots. (2),$$

f being the breaking shear stress of the material in pounds per square inch, T the twisting moment in pound-inches which will cause rupture, D the outer diameter, and d the inner diameter (if the shaft is hollow) in inches.

We see, then, that the strength of a solid shaft depends on the cube of its diameter, whereas its stiffness depends on the fourth power of its diameter.

As to the practical rule given in (1) (Art. 294), we saw in Art. 182 that torque in pound-*feet*, multiplied by angular velocity in radians per minute, divided by 33,000, is the horse-power. As we use T in pound-inches, $\mathrm{T} = 12 \times 33{,}000 \times \mathrm{H}/2\pi n$. If we use this in (1) of this article and take $f = 9{,}000$ lbs. per square inch, we find the usual practical rule (1) (Art. 294). Taking $f = 4{,}500$ for cast iron and 13,500 for steel, we find the rules for these materials already given.

297. Shafts Subjected to Twisting and Bending.—It is most convenient here to assume that a student has read the larger print in the next chapter, and knows that if a round shaft of diameter D inches is subjected to a bending moment M pound-inches there are compressive and tensile stresses through the cross-section at the circumference of amount $p = 32\,\mathrm{M}/\pi\,\mathrm{D}^3 = 2\,a\,\mathrm{M}$, say .... (1) for a solid shaft, and $32\,\mathrm{M\,D}/\pi\,(\mathrm{D}^4 - d^4) = 2\,b\,\mathrm{M}$, say .... (2) for a hollow shaft, if d is the internal diameter. We have just found that the shear stress on the same interface, due to a twisting moment T, is $f = a\,\mathrm{T}$ and $b\,\mathrm{T}$ .... (3) for the solid and hollow shafts. The other stresses across other interfaces at the point ought now to be studied, but it will be found that it is only necessary to determine the **principal stresses** there—that is, the greatest and least tensile or compressive stresses which act across any interfaces.

298. In Art. 292 we had a simple case of the general rule for stresses given in Art. 290, and this is another case nearly as simple.

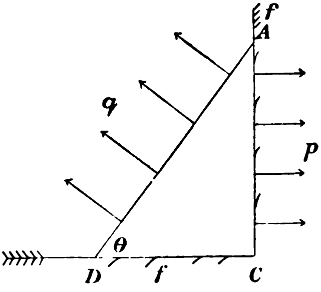

Fig. 189.

Suppose that across an interface A C at right angles to the paper we know that there is a normal stress (either tensile compressive) p lbs. per square inch and a shear stress f; that planes parallel to the paper there is no stress; that on the pla at right angles to A C and the paper there is no normal str Consider the equilibrium of the prism A C D, and find the tract on A D of q lbs. per square inch. For the sake of ease of calculat we will assume A C to be 1 inch, and that the prism is 1 inch right angles to the paper. Now, when there is shear stres across A C there is also an equal shear stress f across D C (Art. 2 What are the forces with which stuff outside acts on the pris The resultants of the forces on the faces are:—On A C, a horizor force p and a vertical force f, because A C is 1 square inch in ar on D C, a horizontal force $f \times$ area of D C or $f \, . \, \text{DC}$ or $f \, . \cot$ as A C is 1; on A D, a force $q \times$ area of A D or $q \, . \, \text{cosec.} \; \theta$, and t force and its direction are what we desire to calculate. Giver it would be easy to calculate q and its direction; but our probl is even simpler. It is this: Find on what plane A D there is o a normal stress q with no tangential component, and find Resolving forces horizontally, $f \cot. \theta + p = q \text{ cosec. } \theta \times \sin$ or $f \cot. \theta + p = q \; \ldots . (1)$. Resolving forces vertica $f = q \, . \, \text{cosec.} \; \theta \, . \cos. \theta$, or $f = q \, . \cot. \theta \; \ldots . (2)$. Hence, (2) gives us $\cot. \theta = f/q$, using this in (1), we find $f^2/q + p = q$ $q^2 - p\,q = f^2$; so that $q = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + f^2} \; \ldots . (3)$.

It is easy to find θ. There are two answers, differing b right angle. **A stress is called a principal stress** if it is normal the interface, and we see that we have in (3) the two values of principal stress due to a combination of tensile p and shear stress f, such as we have supposed. The principal stresses across interfaces which are at right angles to one another.

Example.—At an interface $p = 6$ tons per square inch $f = 5$ tons per square inch, then $q = 3 \pm 5{\cdot}83$; so that principal stresses are 8·83 tons per square inch in tension 2·83 tons per square inch in compression at right angles to another.

Exercise.—Wrought iron is not to receive a greater tensile compressive stress than 5 tons per square inch. There is alread tensile stress of 4 tons per square inch across an interface. W shear stress may also cross that interface?

Ans., $p = 4$ and $q = 5 = 2 + \sqrt{4 + f^2}$ from (3), and he $f = 2{\cdot}24$ tons per square inch.

Exercise.—A round shaft is in torsion, and the shear str produced across the section near the circumference is 8,000 lbs. square inch. At the same section the shaft is subjected to bendi and a compressive stress of 6,000 lbs. per square inch is produ across the same interface. What is the greatest compressive str in the material there?

Ans., $q = 3{,}000 + \sqrt{9 \times 10^6 + 64 \times 10^6} = 11{,}544$ lbs. square inch.

Exercise.—If p and f are equal, find them that q may be j 6 tons per square inch.

Ans., Each of them is 3·71 tons per square inch.

299. We see, then, that a **hollow round shaft** subjected to the twisting moment T and the bending moment M is really subjected, at points near the circumference, to the maximum compressive or tensile stress q where $q = bM + \sqrt{b^2M^2 + b^2T^2}$ (see (2) and (3) of Art. 297), or

$$q = \frac{16D}{\pi(D^4 - d^4)}\left\{M + \sqrt{M^2 + T^2}\right\} \quad \ldots . (4),$$

and in solid shafts

$$q = \frac{16}{\pi D^3}\left\{M + \sqrt{M^2 + T^2}\right\} \quad \ldots . (5).$$

Hence the greatest and least compressive or tensile stress existing in a hollow or solid shaft when it is subjected to T and M is exactly the same as the greatest shearing stress in a shaft subjected only to a twisting moment $M + \sqrt{M^2 + T^2}$ .... (6).

300. An overhung crank shaft (Fig. 190) is subjected to a wrench. If the greatest force exerted at the pin A, at right angles to the crank, is F, the twisting moment is F . AC, and we may take the bending moment as F . BC if B is the middle of the journal. Hence the wrench is equivalent to a twisting moment $F(BC + \sqrt{BC^2 + AC^2})$, or $F . (BC + AB)$, a rule which is perhaps a little unexpected.

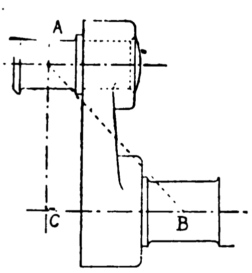

Fig. 190.

301. Ordinary shafts are of wrought iron or mild steel, materials such that their resistances to compression, tension, and shearing are not very different, and therefore when they are subjected to T and M we at once calculate (6), and say that the strength of the shaft is the same as if it were subjected to this twisting moment only. The practical rule (Art. 294) is worked on the idea of a twisting moment only, or $d \propto \sqrt[3]{T}$. If there is also a bending moment M, which is of the value kT, then the rule becomes evidently

$$d = 3{\cdot}3\sqrt[3]{M + \sqrt{M^2 + T^2}},\; d = 3{\cdot}3\sqrt[3]{T}\sqrt{k + \sqrt{k^2 + 1}}.$$

The extreme values of k for many kinds of shafting have been worked out, and the consequent change in the multiplier of (1), by means of shrewd guessing and calculation; but to my mind it is a better recommendation of the rules given at the end of Art. 294 that they are consistent with theory and with the best practice of engineers.

I have given here the usual theory of shafts subjected to twisting and bending; it assumes that the strength is determined by the maximum stress. A true theory would take account of the considerations of Art. 294a.

If a tensile stress p and a shearing stress f act on the same interface, and if p^1 and p^2 are the principal stresses, we see from (3) of Art. 298 that

$$p_1 = \tfrac{1}{2}p + \sqrt{\tfrac{1}{4}p^2 + f^2},$$
$$p_2 = \tfrac{1}{2}p - \sqrt{\tfrac{1}{4}p^2 + f^2}.$$

The theory of Arts. 290 and 292 tells us that fracture will take place if $m^1 p_1 - n^1 p_2 > f_s$, where m^1, n^1, and f_s are constants which ought to be known for the material. If p is a compressive stress, we have fracture if $mp_1 - np_2 > f_s$.

The probable values of m, n, m^1, n^1 for cast iron are given in Art. 292. For wrought iron and mild steel it is possible that ϕ of Art. 292 is 0, and hence $m = \frac{1}{2}$, $n = \frac{1}{2}$, $m^1 = \frac{1}{2}$, $n^1 = \frac{1}{2}$. Hence we have fracture either on the compressive or tensile side of a shaft—that is, whether p is compressive or tensile stress—if $\sqrt{\frac{1}{4}p^2 + f^2} > f_s$. Hence, if a shaft is subjected to the twisting moment T and the bending moment M, we calculate an equivalent twisting moment $\sqrt{\mathrm{T}^2 + \mathrm{M}^2}$, and assume that the shaft is subjected to this alone.

For materials in general, and probably even in the case of wrought iron and mild steel, the equivalent twisting moment ought to be calculated from $h\,\mathrm{M} + k\sqrt{\mathrm{T}^2 + \mathrm{M}^2}$, where h and k are constants, depending upon the nature of the material, which are not known at the present time.

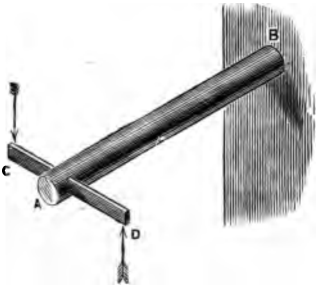

Fig. 191.

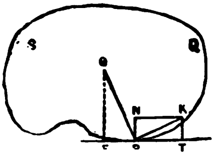

Fig. 192.

302. The demonstration of Art. 296 is found to agree with experiment, but its results must not be applied except to shafts which are **circular in section.** Our assumption, which experience warranted, was that when such a shaft as A B (Fig. 191) is fixed at B, and when

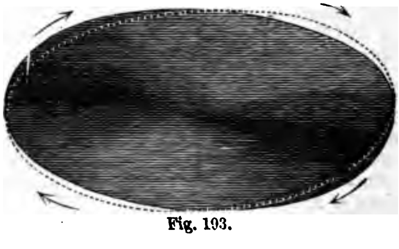

Fig. 193.

to an arm, C D, a twisting couple is applied, every straight line in a section remains straight, and moves through the same angle as every other line. But it can be shown that this is not the case for a shaft of any other than a circular section. Thus, let O (Fig. 192) be the centre of gravity of the section P Q R, and let us suppose that a shaft of this section is subjected to the sort of strain I have described. The shear strain at the point P is in the direction P K, perpendicular to O P. Let its amount, which we know to be O P × angle of twist, be represented by the length of P K. It is easy to show that this is just the same as a shear strain P N in the direction P N, normal to the surface of the shaft at P, together with a shear strain in the direction P T, tangential to the shaft at P. But shear strain in any direction is always accompanied by a similar strain in a plane at right angles to this direction (see Art. 282), so that since we have the shear P N, we must also have a shear parallel to the axis of the prism along the surface at P, and **this cannot be produced merely by a twisting moment.** We must imagine that along with the twisting moment there is a force distributed over the surface of the shaft to produce the above effects. The result of an exact investigation (Art. 313) is that a twisting couple produces a greater twist than might appear from what I have said in Art. 295, and it also produces a **warping** of the naturally plane sections of the shaft. Thus Fig. 193 is the shape assumed by each section of an elliptic shaft, and Fig. 194 of a square shaft. Imagine a section to be distinguishable, say, in a glass shaft by a thin layer of a different colour from the rest. Deeper shading indicates greater distance from the observer who is looking towards the fixed end of the shaft. The arrows show the direction of the twisting moment. In the following three sections, instead of the torque for a twist of one radian being equal to N times the moment of inertia of the cross-section, it is only ·84 times this for a square section (Fig. 195), ·54 times it for the section Fig. 196, and ·6 times it for the section Fig. 197. Indeed,

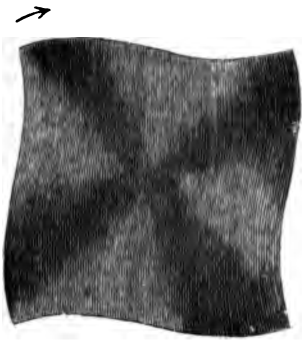
Fig. 194.

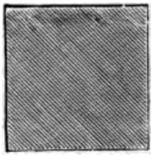
Fig. 195.

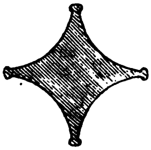
Fig. 196.

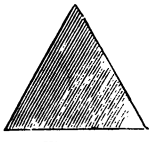
Fig. 197.

the square section has only ·38 times the torsional rigidity of a cylindric shaft of the same sectional area; Fig. 196 has ·67 times, and Fig. 197 has ·73 times the torsional rigidity of a cylindric shaft of the same sectional area.

The numbers in the column headed W, Table XV., express **the relative strengths to resist twisting** of the various sections there figured.

The torsional rigidity of an elliptic section whose principal semi-diameters are a and b is $\text{N}\pi a^3b^3/(a^2 + b^2)$. If M is the twisting moment, the shear stress at a point x, y (the axis of x being a) is $2\,\text{M}\,\sqrt{b^4x^2 + a^4y^2}/\pi a^3b^3$. This is greatest at the end of the minor axis, being $2\,\text{M}/\pi ab^2$.

The torsional rigidity of a rectangle, if its length is two or more times its breadth, is, with some accuracy, the same as that of the inscribed ellipse multiplied by the ratio of their polar moments of inertia. The greatest shear stress occurs at the middle of the longer side of the boundary, and is $3\,\text{M}\,(a^2 + b^2)/8\,a^3b^2$, if M is the twisting moment and a is half the longer, b half the shorter, side of the rectangle.

303. A very interesting result of the investigation is that there is always **greatest distortion at that part of the surface** of a shaft where the surface is **nearest the axis.** Thus in an elliptic shaft the substance is most strained at the ends of the shorter diameter of the section. Imagine a very light box to be made so as to contain **frictionless liquid** exactly of the shape of a shaft. If we give a sudden turn to the box about the axis, the liquid will be left behind if the box is circular in section, but it will have motions relatively to the box which can very readily be imagined if the shaft is not circular in section. Now the actual velocity of the liquid at any place **relatively to the box** is in the same direction as, and is proportional to, the shear in a similar shaft when it is twisted. This has been proved by Lord Kelvin. You will see from this that there is very little strain at the projecting ribs of the shaft, whose section is shown in Fig. 196, and just at the projecting angles of Figs. 195 and 197. This reminds me of a general remark which I have to make, and which I must leave without proof. A solid of any elastic substance cannot experience any finite stress or strain in the neighbourhood of a projecting point, unless acted on by outside forces just at the point. In the neighbourhood of an edge it may have strain only in the direction of the edge, and generally there will be exceedingly great strain and stresses at any re-entrant edges or angles. An important application of the last part of the statement is the well-known practical rule that every re-entering edge or angle ought to be rounded to prevent risk of rupture in solid pieces

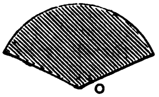

Fig. 198.

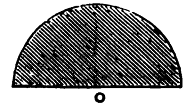

Fig. 199.

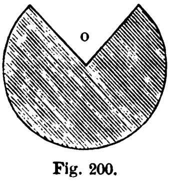

Fig. 200.

designed to bear stress. An illustration of the principle is the stress at the centre of the circular outline in the three sections of shafts (Figs. 198, 199, and 200). In Fig. 198, at o, there is no stress when the shaft is twisted; in Fig. 199 the stress may be calculated; in Fig. 200 the stress is exceedingly great for even the smallest twist (see Art. 302).

EXERCISES.

N.B.—**A most usual error** of students is to forget that moments in pound-inches are not numerically the same as in pound-feet. Beginners had better leave the exercises involving bending moment until they have studied bending.

1. A shaft 1 inch in diameter can safely transmit a torque of 2,400 pound-inches. What diameter of shaft would be required for transmitting 15 H.P. at 200 revolutions per minute? *Ans.*, 1¼ inch.

2. Find the horse-power which may be transmitted by a shaft 4 inches in diameter when running at 150 revolutions per minute, if the stress due to twisting be limited to 9,000 lbs. per square inch. *Ans.*, 273.

3. A line of steel shafting is 80 feet long; if a twisting moment of 4,000 pound-inches be applied at one end, what will be the total angle of twist, the diameter of the shaft being 2½ inches? What horse-power will this transmit at 220 revolutions per minute? *Ans.*, 5·2°; 14 H.P.

4. A solid wrought-iron shaft is to be replaced by a hollow steel shaft of the same diameter. If the material of the latter is 30 per cent. stronger than that of the former, what must be the ratio of internal to external diameter? What is the percentage saving in weight? *Ans.*, 1·44; 46 per cent.

5. The amount of twist in a solid shaft is to be limited to 1° for each 10 feet of length. Find the diameter for a twisting moment of 50 ton-inches, the modulus of torsional rigidity being 10,000,000 lbs. per square inch. *Ans.*, 5·2° inches.

6. A wrought-iron shaft is subjected simultaneously to a bending moment of 8,000 pound-inches, and to a twisting moment of 15,000 pound-inches. Find the twisting moment equivalent to these two and the least safe diameter of the shaft, the safe shear stress being taken at 8,000 lbs. per square inch. *Ans.*, 25,000 pound-inches; 2·52 inches.

7. Find the diameter of a shaft for a winding drum which works under the following conditions: the load lifted is 1½ ton; diameter of drum, 5 feet; width of face of drum, 26 inches; distance from inner face of drum to the middle of the bearing of shaft, 13 inches; maximum stress, 7,000 lbs. per square inch. *Ans.*, 5·44 inches.

8. A wrought-iron shaft 3 inches in diameter, and making 140 revolutions per minute, is supported at wall brackets 16 feet apart. There is a pulley on the shaft, midway between the bearings; if the resultant side pull due to the weight of the pulley and the pull of the belt be 210 lbs., what is the greatest horse-power the shaft will transmit with safety? Safe shear stress, 7,800 lbs. per square inch. *Ans.*, 66.

9. A bar of iron is at the same time under a direct tensile stress of 5,000 lbs. per square inch, and to a shearing stress of 3,500 lbs. per square inch. What would be the resultant equivalent tensile stress on the material? *Ans.*, 6,801 lbs. square inch.

10. Taking the safe tensile stress of wrought-iron to be 10,000 lbs. per square inch, determine whether it would be safe to subject a piece of wrought-iron to a tensile stress of $3\frac{1}{2}$ tons per square inch, together with a shear stress of 3 tons per square inch.

Ans. Unsafe; max. stress = 11,700 lbs. square inch.

11. A wrought-iron shaft is subjected to a twisting moment of 36,000 pound-inches and a bending moment of 18,000 pound-inches; find the diameter when the maximum shear stress is 8,000 lbs. per square inch. Find also the twisting moment which alone would produce a shear stress of the same numerical value.

12. A screw propeller-shaft 10 inches in diameter is subjected to a twisting moment of 35 ton-feet, and to a bending moment of 10 ton-feet, due to the weight of the shaft and the pitching of the ship. What is the maximum compressive stress if the thrust of the screw be 10 tons?

Ans., 2·9 tons.

13. A shaft 12 inches diameter, transmitting a twisting moment of 100 ton-feet, is also subject to a bending moment of 20 ton-feet. Find the maximum stress induced. *Ans.*, 4·3 tons per square inch.

14. Find the diameter of a wrought-iron shaft to transmit 90 horse-power at 130 revolutions per minute. If there is a bending moment equal to the twisting moment, what ought to be the diameter?

Ans., ·27 inches.

15. A steam-engine crank is 12 inches long, and the greatest force which is transmitted through the connecting-rod is 9,000 lbs. Find the diameter of the wrought-iron crank-shaft, taking the safe shear stress at 9,000, the distance of the centre of the crank-pin from the centre of the bearing nearest it being 10 inches, measured horizontally.

Ans., 4·76 inches.

16. A round shaft 3 inches diameter, find the sizes of equivalent shafts of square, elliptic and rectangular sections if the breadth and thickness of each of these latter are as 1 to 2. If these shafts are 20 feet long, and they are transmitting 20 H.P. at 100 revolutions, what is the total twist on each of them. $N = 10{,}500{,}000$.

Ans., Rectangle, 2·15 × 4·3 inches; ellipse, minor axis, 2·38 inches; square, 2·73 inches side. Twist on circular shaft = 2·073°, on square shaft, 1·78°; elliptical shaft, 1·05°; rectangular shaft ·93°.

CHAPTER XV.

MORE DIFFICULT THEORY.

304. MATHEMATICIANS endeavour to help engineers (including in this term all men who apply the principles of natural science) by investigations concerning ideal elastic material shaped like actual beams and shafts. The **mathematical analysis** is exact and difficult, and only a few problems have yet been solved, and it is only by leaving out terms that seem insignificant that we are able to apply the results to actual problems. The **engineer** recognises from the beginning that his problems are too complicated for any exact mathematical investigation. He therefore leaves out his apparently insignificant terms rather at the beginning than the end; but indeed he leaves them out in any part of his investigation if they are likely to give trouble, for he recognises from the beginning that his theory is only to guide him, and that the final appeal must be to experiment. The engineer looks upon the phenomena involved in the loading of the tie-bar as simple because experiment is easy; whereas the mathematician, seeing that a lateral contraction accompanies an axial elongation, regards it as complicated.

The engineer ought to study, and develop, and correct, by experimental observations, his usual method of investigation as described in this book, but he ought also to study the mathematician's treatment of the subject, and let it assist him as he lets experiment assist him. The following very short sketch needs much time for its proper comprehension: it is from the mathematician's point of view. Students may pursue the subject in Mr. Love's treatise on elasticity, or Thomson and Tait's treatise on natural philosophy.

305. The consideration of **homogeneous strain in general** (when any portion of stuff in the shape of a sphere is changed into what is called the strain ellipsoid, any three co-orthogonal diameters of the sphere becoming conjugate diameters of the ellipsoid, planes parallel to one another remain parallel; all parallel lines get the same fractional changes in length) is not so difficult as it is tedious. Unfortunately the authors of books insist on its being studied. For our purposes we have only to deal with **infinitesimal strains**, and this is easy.

Suppose that a point x, y, z is displaced to $x + u$, $y + v$, $z + w$ where u, v, w are very small, then

$$e = \frac{du}{dx}, f = \frac{dv}{dy}, g = \frac{dw}{dz} \; \ldots \ldots (1)$$

are the **tensile** strains of the stuff in directions parallel to x, y, z. In Fig. 201 let the axis of x be at right angles

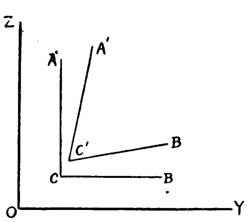

Fig. 201.

to the plane of the paper. Let the traces of two planes, C B and C A, parallel to O Y and O Z, and perpendicular to the paper, become changed to C′ B′, C′ A′. The angle by which A′ C′ B′ is less than the original right angle is evidently a **shear strain.** I shall call the amount of it *a*; and as it is a shear of planes normal to Y, parallel to Z, or a shear of planes normal to Z, parallel to Y, there is no great harm in calling it **a shear** about the axis of X. Now the angle turned through by C A, clockwise, is really the horizontal motion of A minus the horizontal motion of C, divided by A C, since the angle is very small. But this is dv/dz. Let the student be quite sure of this fact. Similarly the angle, anti-clockwise, turned through by C B, is dw/dy, and hence $a = \frac{dw}{dy} + \frac{dv}{dz} \ldots . (2).$

Similarly if b and c are to y and z what a is to x, then

$$b = \frac{du}{dz} + \frac{dw}{dx},\ c = \frac{dv}{dx} + \frac{du}{dy} \ldots . (2).$$

A student may keep these in mind by means of the **mnemonic**

$$\left.\begin{matrix} a & b & c \\ x & y & z \\ u & v & w \end{matrix}\right\} \ldots . (2).$$

Notice that the average rotation of C A and C B, clockwise, in Fig. 201, is half the algebraic sum of the angles turned through by C A and C B, or, calling the average **rotations** of the material $\tilde{\omega}_1$ about the axis of X, $\tilde{\omega}_2$ about the axis of Y, $\tilde{\omega}_3$ about the axis of Z,

$$\tilde{\omega}_1 = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),\ \tilde{\omega}_2 = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right),\ \tilde{\omega}_3 = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right) \ldots . (3).$$

It is evident that when $\tilde{\omega}_1 = \tilde{\omega}_2 = \tilde{\omega}_3 = 0$, or there is *no rotation*, it means that the **strain is pure,** or that the three principal diameters of the strain ellipsoid, called the principal axes of strain, remain parallel to their original directions.

Shear strain involves no change of volume; and if the edges parallel to X, Y, Z of the unit cube become $1 + e$, $1 + f$, $1 + g$, the **volumetric dilatation** or strain is $e + f + g$, since these are small. Or if D is used for this, then

$$\text{D} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \ldots . (4).$$

The conditions $\tilde{\omega}_1 = \tilde{\omega}_2 = \tilde{\omega}_3 = 0$ are evidently the conditions that there is a function ϕ (called the **strain potential**), such that $u = \frac{d\phi}{dx}$, $v = \frac{d\phi}{dy}$, $w = \frac{d\phi}{dz}$; and we see that, in case there is no cubical dilatation, (4) becomes

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0 \ldots . (5).$$

At any point O of a body let there be three **planes** of **reference** meeting at the three mutually perpendicular axes of x, of y, and of z. *Let* **tensile stress** be called positive. Across the plane of yz (usually called the plane x, because the axis of x is normal to it) let the stress be resolved into its three components—X_x, parallel to Ox; Y_x, parallel to Oy; and Z_x, parallel to Oz. Across the planes y and z let the component stresses be X_y, Y_y, Z_y, and X_z, Y_z, Z_z. **To find the stress** components F, G, H in the three directions **across any plane** whose direction cosines (or the direction cosines of its normal) are l, m, n, consider the equilibrium of the tetrahedron formed by the four planes under the action of the tractions acting from material outside it. The area of the sloping face being A, the areas of the other faces are lA, mA, and nA. The resultant force on each area is stress multiplied by each area. We have, then, resolving parallel to x, $FA = X_x lA + X_y mA + X_z nA$ and two other equations. Hence,

$$\begin{aligned} F &= lX_x + mX_y + nX_z \\ G &= lY_x + mY_y + nY_z \\ H &= lZ_x + mZ_y + nZ_z. \end{aligned}$$

In a fluid the static stress is always normal to any interface. Hence F, G, and H are in this case the components of a normal stress P on the new plane, and all the tangential stresses vanish. Hence,

$$\begin{aligned} F &= lP = lX_x \quad \text{or} \quad P = X_x \\ G &= mP = mY_y \quad \text{or} \quad P = Y_y \\ H &= nP = nZ_z \quad \text{or} \quad P = Z_z; \end{aligned}$$

so that the stress is the same across any interface whatsoever.

306. Now consider **in the general case** the equilibrium of a parallelopiped whose corner is at O (Fig. 202), the co-ordinates of which are x, y, z, and let the co-ordinates of the opposite corner O′ be $x + \delta x$, $y + \delta y$, $z + \delta z$, and let us consider the equilibrium of all the surface tractions acting on the faces from the outside. The resultants of the normal forces meet in the centre, and as there are no volumetric forces, they are in equilibrium. They are not shown in Fig. 202.

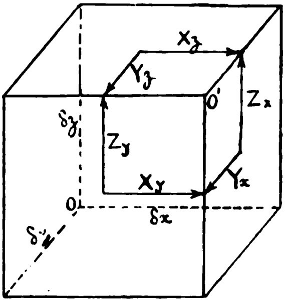

Fig. 202.

We show the tangential forces per unit area with which outside stuff acts on inside on three of the faces. The other three faces have similar forces, the arrows being in opposite senses to these.* Hence their moments are *just* the same as the moments of these about axes through the centre.

---

[*] It is easy to take into account the fact that there may be volumetric forces, and that Z_x on the x plane through O′ is not equal to Z_x on the x plane through O, but is rather $Z_x + \delta x \frac{d}{dx} Z_x$. We do this later, and it is easy to see that these extra terms vanish in the present problem.

Taking moments of the figured forces about the axis through O parallel to x, we find

$$\mathrm{Z}_y \,.\, \delta x \,.\, \delta z \,.\, \tfrac{1}{2}\delta y = \mathrm{Y}_z \delta x \,.\, \delta y \,.\, \tfrac{1}{2}\delta z; \text{ so that } \mathrm{Z}_y = \mathrm{Y}_z.$$

This is **Cauchy's theorem.** We call each of them S. In the same way we have the other two relations here given:—

$$\mathrm{Z}_y = \mathrm{Y}_z = \mathrm{S}; \quad \mathrm{Z}_x = \mathrm{X}_z = \mathrm{T}; \quad \mathrm{X}_y = \mathrm{Y}_x = \mathrm{U}.$$

We also give the names P to X_x, Q to Y_y, R to Z_z.

S, T, U are the shear stresses, and P, Q, and R are the normal tensile stresses in the material. Hence our equation may be written:—

$$\left.\begin{array}{l} \mathrm{F} = l\mathrm{P} + m\mathrm{U} + n\mathrm{T} \\ \mathrm{G} = l\mathrm{U} + m\mathrm{Q} + n\mathrm{S} \\ \mathrm{H} = l\mathrm{T} + m\mathrm{S} + n\mathrm{R} \end{array}\right\} \ldots\ldots (1).$$

Exercise.—Across what interfaces at a point are the stresses normal? That is, **find the principal stresses** and their directions.

If F, G, and H are the components of a force R normal to the plane $l\ m\ n$, then $\mathrm{R}l = \mathrm{F}$, $\mathrm{R}m = \mathrm{G}$, $\mathrm{R}n = \mathrm{H}$; and if we substitute these in (1), remembering that $l^2 + m^2 + n^2 = 1$, we can find $l\,m\,n$ and R. The answer tells us that there are always three directions and amounts of principal stress.

In our most general state of stress, Fig. 203 shows the surface tractions acting on the element $\delta x\ \delta y\ \delta z$ *from the outside.* Imagine equal and opposite forces on the other three faces.

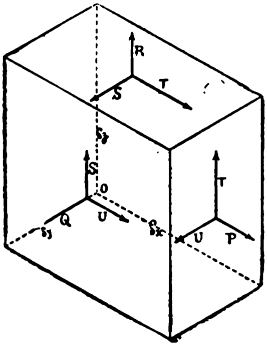

Fig. 203.

Now **let us consider volumetric forces** and the rates of variation of the stress. On the faces meeting at O we have P Q R S T and U. But the forces on the other faces are, as regards normal forces,

$$\mathrm{P} + \delta x \,.\, \frac{d\mathrm{P}}{dx}, \quad \mathrm{Q} + \delta y \,.\, \frac{d\mathrm{Q}}{dy}, \quad \mathrm{R} + \delta z \,.\, \frac{d\mathrm{R}}{dz}.$$

On $\delta y \,.\, \delta z$ we have also $\mathrm{U} + \delta x \dfrac{d\mathrm{U}}{dx}$ and $\mathrm{T} + \delta x \,.\, \dfrac{d\mathrm{T}}{dx}$; on $\delta z \,.\, \delta x$ we have $\mathrm{U} + \delta y \,.\, \dfrac{d\mathrm{U}}{dy}$ and $\mathrm{S} + \delta z \,.\, \dfrac{d\mathrm{S}}{dz}$; on $\delta x \,.\, \delta y$ we have $\mathrm{T} + \delta z \dfrac{d\mathrm{T}}{dz}$ and $\mathrm{S} + \delta z \,.\, \dfrac{d\mathrm{S}}{dz}$. Now we know that the parts of these forces P and P, S and S, etc., balance independently; therefore we need only consider the extra ones as shown in Fig. 204.

If the volumetric force in the direction of increasing x is $\rho \,.\, \mathrm{X} \,.\, \delta x \,.\, \delta y \,.\, \delta z$, then

$$\rho\,\mathrm{X}\,.\,\delta x\,.\,\delta y\,.\,\delta z + \delta y\,.\,\delta z\,.\,\delta x\,\frac{d\,\mathrm{P}}{dx} + \delta y\,.\,\frac{d\,\mathrm{U}}{dy}\,.\,\delta x\,.\,dz$$
$$+ \delta z\,.\,\frac{d\,\mathrm{T}}{dz}\,.\,\delta y\,.\,\delta x = 0.$$

We have similar equations for the other directions, and these reduce to the following equations of equilibrium :—

$$\left.\begin{aligned} \rho\,\mathrm{X} + \frac{d\,\mathrm{P}}{dx} + \frac{d\,\mathrm{U}}{dy} + \frac{d\,\mathrm{T}}{dz} &= 0 \\ \rho\,\mathrm{Y} + \frac{d\,\mathrm{U}}{dx} + \frac{d\,\mathrm{Q}}{dy} + \frac{d\,\mathrm{S}}{dz} &= 0 \\ \rho\,\mathrm{Z} + \frac{d\,\mathrm{T}}{dx} + \frac{d\,\mathrm{S}}{dy} + \frac{d\,\mathrm{R}}{dz} &= 0 \end{aligned}\right\} \ldots . (2).$$

Note that we have assumed the moments due to the forces on the boundary of the element to balance, in proving Cauchy's theorem, or that the bodily forces have no moment. They may have some moment in the matter in which **magnetic stresses** act, but in ordinary stress phenomena we assume the truth of Cauchy's theorem.

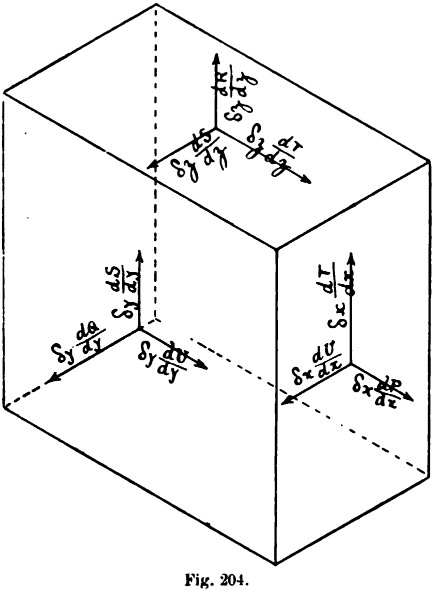

Fig. 204.

If the material is fluid, so that $\mathrm{S} = \mathrm{T} = \mathrm{U} = 0$, $\rho\,\mathrm{X} + \frac{d\,\mathrm{P}}{dx} = 0$, $\rho\,\mathrm{Y} + \frac{d\,\mathrm{Q}}{dy} = 0$, $\rho\,\mathrm{Z} + \frac{d\,\mathrm{R}}{dz} = 0$.

Example.—Let $\rho\,\mathrm{X}$ be called w, the force on unit volume, say the weight in pounds of 1 cubic foot, then P is pressure in pounds per square foot at any depth in a fluid under parallel vertical gravitational force.

If there is **motion in the general case,** we must use $\mathrm{X} - \frac{d^2u}{dt^2}$ instead of X merely, where u is the displacement in the direction of increasing x of the mass $\rho \,.\, \delta x \,\delta y\, \delta z$; and hence we have three equations like $\frac{d\,\mathrm{P}}{dx} + \frac{d\,\mathrm{U}}{dy} + \frac{d\,\mathrm{T}}{dz} + \rho\,\mathrm{X} = \rho\,\frac{d^2u}{dt^2} \;.\;.\;.\;.\; (3).$

At any element of surface of a body ($l\;m\;n$ being the direction cosines of its normal), if there are surface tractions with components F G H, the condition (1) is to be satisfied at the surface.

Now, taking the unit cube, and letting α be $\frac{1}{\mathrm{E}}$, the reciprocal of Young's modulus, and letting β be the lateral contraction in a tie-bar of the material, when α is the axial elongation; under the action of P, Q, and R, the edges become lengthened by the amounts e, f, and g, where $e = \mathrm{P}\,\alpha - (\mathrm{Q} + \mathrm{R})\,\beta$, or

$$\left.\begin{aligned} e &= \mathrm{P}\,\alpha - \mathrm{Q}\,\beta - \mathrm{R}\,\beta \\ f &= -\mathrm{P}\,\beta + \mathrm{Q}\,\alpha - \mathrm{R}\,\beta \\ g &= -\mathrm{P}\,\beta - \mathrm{Q}\,\beta + \mathrm{R}\,\alpha \end{aligned}\right\} \;.\;.\;.\;.\; (4).$$

Solving these equations, we find, writing D for $e + f + g$, and recollecting the fact that $\alpha = \frac{1}{3\mathrm{N}} + \frac{1}{9\mathrm{K}} = \frac{1}{\mathrm{E}}$, $\beta = \frac{1}{6\mathrm{N}} - \frac{1}{9\mathrm{K}}$,

$$\left.\begin{aligned} \mathrm{P} &= \lambda\,\mathrm{D} + 2\,\mathrm{N}\,e \\ \mathrm{Q} &= \lambda\,\mathrm{D} + 2\,\mathrm{N}\,f \\ \mathrm{R} &= \lambda\,\mathrm{D} + 2\,\mathrm{N}\,g \end{aligned}\right\} \;.\;.\;.\;.\; (5),$$

where λ stands for $\mathrm{K} - \frac{2}{3}\mathrm{N}$; K being the modulus of elasticity of bulk, N being the modulus of rigidity.

If S, T, and U are the shear stresses, and a, b, and c are the shear strains (see Art. 305), $\mathrm{S} = \mathrm{N}\,a$, $\mathrm{T} = \mathrm{N}\,b$, $\mathrm{U} = \mathrm{N}\,c$ .... (5). In that Art. we used $\frac{du}{dx}$, etc., instead of e, etc., and so we can write

$$\left.\begin{aligned} \mathrm{P} &= \lambda\,\mathrm{D} + 2\,\mathrm{N}\,\frac{du}{dx} \\ \mathrm{Q} &= \lambda\,\mathrm{D} + 2\,\mathrm{N}\,\frac{dv}{dy} \\ \mathrm{R} &= \lambda\,\mathrm{D} + 2\,\mathrm{N}\,\frac{dw}{dz} \end{aligned}\right\} \;.\;.\;.\;.\; (5),$$

where D stands for $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, and we also have

$$\mathrm{S} = \mathrm{N}\left(\frac{dw}{dy} + \frac{dv}{dz}\right),\quad \mathrm{T} = \mathrm{N}\left(\frac{du}{dz} + \frac{dw}{dx}\right),\quad \mathrm{U} = \mathrm{N}\left(\frac{dv}{dx} + \frac{du}{dy}\right) \ldots (5).$$

$$d\,\mathrm{W} = \mathrm{P}\,.\,de + \mathrm{Q}\,.\,df + \mathrm{R}\,.\,dg + \mathrm{S}\,.\,da + \mathrm{T}\,.\,db + \mathrm{U}\,.\,dc \ldots (6)$$

was proved by **Lord Kelvin** to be a complete differential in two cases—first, at constant temperature; second, when the changes of strain occur so quickly that no heat is gained or lost by the stuff.

Using (5), we find that $d\,\mathrm{W}$ is a complete differential if

$$2\,\mathrm{W} = (\lambda + 2\,\mu)\,(e + f + g)^2 + \mu\,(a^2 + b^2 + c^2 - 4\,fg - 4\,ge - 4\,ef) \;.\;.\;.\;.\; (7):$$

so that $\mathrm{P} = \frac{d\,\mathrm{W}}{dc}$, etc., $\mathrm{U} = \frac{d\,\mathrm{W}}{de}$, etc. .... (7).

Kirchhoff pointed out that W cannot be negative. When W is 0, the whole body moves as a rigid body. He also showed that if the six strains be given at every point in a body it is necessary to impose six independent conditions, such as those of St. Venant given in (7) of Art. 308. Without these there may be "rigid body" displacements as well as the strains. He also showed that if the surface displacements or surface forces on a body are given, there is only one solution possible (Kelvin showed that there was also an unstable solution). Kelvin had shown that one was possible.

St. Venant showed that in calculating the strains due to surface loads it is only in the neighbourhood of the place that the actual distribution of the surface force is of any importance. This is called the principle of **equipollent loads.**

Converting the stresses in (3) into strains, we have

$$(\lambda + \mathrm{N})\frac{d\,\mathrm{D}}{dx} + \mathrm{N}\,.\,\nabla^2 u + \rho\,\mathrm{X} = \rho\frac{d^2u}{dt^2} \;\ldots\ldots\; (8),$$

with two other equations. ∇^2 means $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$. The surface conditions become $\mathrm{F} = l\,(\lambda\,\mathrm{D} + 2\,\mathrm{N}\,e) + m\,\mathrm{N}\,c + n\,\mathrm{N}\,b$, with two others. Translating this, we have the surface conditions (1) becoming

$$\left.\begin{aligned}\mathrm{F} &= l\lambda\,\mathrm{D} + 2\mathrm{N}\left(\frac{du}{dn^1} + m\tilde{\omega}_3 - n\tilde{\omega}_2\right)\\ \mathrm{G} &= m\lambda\,\mathrm{D} + 2\mathrm{N}\left(\frac{dv}{dn^1} + n\tilde{\omega}_1 - l\tilde{\omega}_3\right)\\ \mathrm{H} &= n\lambda\,\mathrm{D} + 2\mathrm{N}\left(\frac{dw}{dn^1} + l\tilde{\omega}_2 - m\tilde{\omega}_1\right)\end{aligned}\right\} \;\ldots\ldots\; (9).$$

Here dn^1 is an element of the normal to the surface.

(8) or (3) may be written

$$(\lambda + 2\mathrm{N})\frac{d\,\mathrm{D}}{dx} - 2\mathrm{N}\frac{d\tilde{\omega}_3}{dy} + 2\mathrm{N}\frac{d\tilde{\omega}_2}{dz} + \rho\,\mathrm{X} = \rho\frac{d^2u}{dt^2} \;\ldots\ldots\; (10),$$

with two other equations.

If we neglect X, Y, Z, the volumetric forces, differentiate (10), etc., with regard to x, etc., and add, we get

$$(\lambda + 2\,\mathrm{N})\,\nabla^2\mathrm{D} = \rho\frac{d^2\mathrm{D}}{dt^2} \;\ldots\ldots\; (11),$$

a well-known equation showing that there is **wave propagation** with velocity $= \sqrt{\frac{\lambda + 2\mathrm{N}}{\rho}}$. Differentiate the third of (8) with respect to y, and the second with regard to z, and subtract, and we get

$$\mathrm{N}\,\nabla^2\tilde{\omega}_1 = \rho\frac{d^2\tilde{\omega}_1}{dt^2} \;\ldots\ldots\; (12),$$

and two other equations. These are equations of **distortional wave propagation** with a velocity $= \sqrt{\frac{N}{\rho}}$.

307. **To illustrate this method of study,** let us consider the problem of the **thick cylinder** of Art. 275. There we considered it from the *stress* point of view. We shall now consider it from the *strain* point of view. At any point in the material at the distance r from the axis there is a radial displacement, which we shall call u, as if the direction r were our old direction x. There is displacement at right angles to u, and its fractional amount—that is, the strain—is evidently u/r, because if a radius r increases to $r + u$, the circumference of the circle increases to $2\pi (r + u)$, so that the fractional increase of the circumference is u/r. Let us imagine any tensile strain parallel to the axis, which we call the direction of z, to be constant. It is evident, also, that the strain is irrotational, and also that the three principal shears are 0. We make these statements to show that we are considering the old practical problem; but the mathematician puts it rather in this way: he assumes certain strains to exist; he works out a mathematical problem, and he takes his chance of the results fitting any practical case.

Let $\frac{du}{dr} = e$, $\frac{u}{r} = f$, $g = g_0$, $a = 0$, $b = 0$, $c = 0$.

1. **No bodily forces,** and the problem a static one—that is, the strains independent of time. Then $D = \frac{du}{dr} + \frac{u}{r}$, and equation (8) or (10) becomes

$$(\lambda + 2N) \frac{d}{dr}\left(\frac{du}{dr} + \frac{u}{r}\right) = 0 \ldots . (1).$$

Hence

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \ldots . (2).$$

And we find on trial that $u = Ar + Br^{-1} \ldots . (3)$ where A and B are arbitrary constants. Knowing the strains, equation (5) (Art. 306) enables us to find the stresses.

$$P = 2(\lambda + N)A - 2NBr^{-2} \ldots . (4),$$
$$Q = 2(\lambda + N)A + 2NBr^{-2} \ldots . (5),$$
$$R = A\lambda + 2Ng_0 \ldots . (6).$$

The first two of these are the values obtained in Art. 275.

Exercise.—Show that D is constant throughout the shell. (2) gives

$$\frac{d^2u}{dr^2} + \frac{r\frac{du}{dr} - u}{r^2} = 0,$$

which is

$$\frac{d^2u}{dr^2} + \frac{d\frac{u}{r}}{dr} = 0.$$

The first integral is

$$\frac{du}{dr} + \frac{u}{r} = \text{constant.}$$

But $D = \frac{du}{dr} + \frac{u}{r}$, and is therefore constant throughout the space for which the differential equation (2) holds.

2. **Rotating Cylinder.**—Volumetric force $\alpha^2 r$ acting radially on unit mass, α being angular velocity. Using this in (8) or (10) of Art. 306, instead of (1), we have

$$(\lambda + 2N)\frac{d}{dr}\left(\frac{du}{dr} + \frac{u}{r}\right) + \rho\alpha^2 r = 0 \ldots . (7).$$

If we try $u = A r^n$, we find that $n = 1$, $n = -1$ will satisfy (7) if A is an arbitrary constant; also $n = 3$ will satisfy (7) if $A = -\rho\alpha^2/8\ (\lambda + 2N)$; and hence

$$u = A r + B r^{-1} - \rho\alpha^2 r^3/8\ (\lambda + 2N) \ldots . (8)$$

where A and B are arbitrary constants. Having the strains, we can calculate the stresses

$$P = \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) + 2N\frac{du}{dr} \ldots . (9),$$

$$Q = \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) + 2N\frac{u}{r} \ldots . (10),$$

$$R = \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) \ldots . (11).$$

If we let $P = 0$ when $r = r_1$, and $P = p$ when $r = r_0$, and we let $\alpha = 0$, we get the answers given in Art. 275 for a thick cylinder.

Taking up this **rotational case by the method in the text** (Art. 275), in the case of a cylindric body rotating with angular velocity α, if ρ is the mass per unit volume, taking into account the centrifugal force on the element whose equilibrium is considered, equation (1) becomes $p \,.\, dr + r \,.\, dp - r^2\rho\alpha^2 \,.\, dr = q \,.\, dr$, and this is $p + r\frac{dp}{dr} - r^2\rho\alpha^2 = q$. The solution now becomes

$$p = C + B r^{-2} + \tfrac{1}{4}\rho\alpha^2 r^2 \ldots . (12).$$

Let us take $p = 0$, both outside and inside.

$$0 = C + B r_1^{-2} + \tfrac{1}{4}\rho\alpha^2 r_1^2,$$
$$0 = C + B r_0^{-2} + \tfrac{1}{4}\rho\alpha^2 r_0^2.$$

From these we find

$$\text{B} = -\tfrac{1}{4}\rho a^2 \frac{r_1^{\,2} - r_0^{\,2}}{r_1^{\,-2} - r_0^{\,-2}},\ \text{C} = \tfrac{1}{4}\rho a^2 \frac{r_1^{\,2} - r_0^{\,2}}{r_1^{\,-2} - r_0^{\,-2}}\, r_0^{\,-2} - \tfrac{1}{4}\rho a^2 r_0^{\,2} =$$

$$\tfrac{1}{4}\rho a^2 \left\{ \frac{r_1^{\,2} - r_0^{\,2}}{r_1^{\,-2} - r_0^{\,-2}}\, r_0^{\,-2} - r_0^{\,2} \right\} = \tfrac{1}{4}\rho a^2 \frac{r_0^{\,-2} r_1^{\,2} - r_1^{\,-2} r_0^{\,2}}{r_1^{\,-2} - r_0^{\,-2}},$$

$$q = \tfrac{1}{4}\rho a^2 \left\{ -r^2 + \frac{r_0^{\,-2} r_1^{\,2} - r_1^{\,-2} r_0^{\,2} + r^{-2}\,(r_1^{\,2} - r_0^{\,2})}{r_1^{\,-2} - r_0^{\,-2}} \right\}.$$

This is greatest where $r = r_0$.

To apply our results to a **rotating solid cylinder**, since we have no statement of the inside stress, the central condition being that there is no radial displacement of the material at the centre, it is necessary to write our equations in terms of the *strains* rather than the stresses. Using, then, the equations of Art. 306, letting $r_1 = 0$,

$u = \text{A}\, r + \text{B}\, r^{-1} - \dfrac{a^2\rho}{8\,(\lambda + 2\text{N})}\, r^3$, and hence $\text{B} = 0$ if the cylinder is complete to the centre. We know, therefore, u everywhere, and we can, if we please, now use the values of p and q.

The solution is only to be regarded correct for a cylinder, not for a thin disc, because the above gives a value for R, the stress parallel to the axis, and there is no surface force on the disc. For the disc we use **Dr. Chree's more correct solution.** Dr. Chree's solution is this:—Let $e = \dfrac{du}{dr}$, $f = \dfrac{u}{r}$, $g = \dfrac{dw}{dz}$, $b = \dfrac{du}{dz} + \dfrac{dw}{dr}$.

We can now easily write out the equations. Try if the solution is

$$u = \frac{a^2\rho}{8\,\text{E}}(1-\sigma)\left\{(3+\sigma)\,a^2 r - (1+\sigma)\,r^3\right\} + \frac{a^2\rho}{6\,\text{E}}\,\sigma\,(1+\sigma)\,r\,(l^2 - 3z^2),$$

$$w = -\frac{a^2\rho}{4\,\text{E}}\,\sigma\left\{(3+\sigma)\,a^2 z - 2\,(1+\sigma)\,r^2 z\right\} - \frac{a^2\rho}{3\,\text{E}}\,\sigma^2\,\frac{1+\sigma}{1-\sigma}\,z\,(l^2 - z^2);$$

it is easy to get the stresses from these values.

In this solution the planes $z = \pm\, l$ are free from stress, and there is no tangential stress on the circumference, and the resultant normal traction per unit length of the circumference is 0 when $r = a$ (that is, $\displaystyle\int_{-l}^{l} \text{P}\,.\,dz = 0$ when $r = a$). By the principle of *equipollent loads*, this means that there is no practical departure from the real condition of things beyond a short distance inside the edge.

308. Bending or Twisting a Prism.—A straight prismatic surface with plane ends bounds isotropic material. Surface tractions are applied to the ends only. The axis of the prism is the axis of z, one end being the origin. When we speak of bending, the deflection will be in the direction of x (that is, bending will take place in the plane xz).

The student will refer back to Fig. 203 to see the exact significations of the stresses P, Q, R, and S, T, U, and the strains $e = \frac{du}{dx}$, $f = \frac{dv}{dy}$, $g = \frac{dw}{dz}$, and $a = \frac{dw}{dy} + \frac{dv}{dz}$, $b = \frac{du}{dz} + \frac{dw}{dx}$, $c = \frac{dv}{dx} + \frac{du}{dy}$: also to the equations of equilibrium (3), and to the equations (5) which express the stresses in terms of the strains, and also to the boundary conditions (1). He had better write out these equations here. We do so, and give them new reference numbers. We assume neither volumetric forces nor dynamical forces in (3), now called (1).

$$\left.\begin{aligned} \frac{d\mathrm{P}}{dx} + \frac{d\mathrm{U}}{dy} + \frac{d\mathrm{T}}{dz} &= 0 \\ \frac{d\mathrm{U}}{dx} + \frac{d\mathrm{Q}}{dy} + \frac{d\mathrm{S}}{dz} &= 0 \\ \frac{d\mathrm{T}}{dx} + \frac{d\mathrm{S}}{dy} + \frac{d\mathrm{R}}{dz} &= 0 \end{aligned}\right\} \ldots . (1).$$

$$\left.\begin{aligned} e &= \mathrm{P}\alpha - \beta(\mathrm{Q} + \mathrm{R}) \\ f &= \mathrm{Q}\alpha - \beta(\mathrm{P} + \mathrm{R}) \\ g &= \mathrm{R}\alpha - \beta(\mathrm{P} + \mathrm{Q}) \end{aligned}\right\} \ldots . (2), \text{ or}$$

$$\left.\begin{aligned} \mathrm{P} &= \lambda\mathrm{D} + 2\mathrm{N}e \\ \mathrm{Q} &= \lambda\mathrm{D} + 2\mathrm{N}f \\ \mathrm{R} &= \lambda\mathrm{D} + 2\mathrm{N}g \\ \mathrm{S} &= \mathrm{N}a,\ \mathrm{T} = \mathrm{N}b,\ \mathrm{U} = \mathrm{N}c \end{aligned}\right\} \ldots . (2).$$

D stands for $e + f + g$. λ is $\mathrm{K} - \frac{2}{3}\mathrm{N}$. Our old

$$\alpha = \frac{1}{3\mathrm{N}} + \frac{1}{9\mathrm{K}} = \frac{1}{\mathrm{E}},\ \beta = \frac{1}{6\mathrm{N}} - \frac{1}{9\mathrm{K}},\ \frac{\beta}{\alpha} = \sigma$$

(Poisson's ratio). K is the modulus of elasticity of bulk: N is the modulus of rigidity, and the surface tractions are

$$\left.\begin{aligned} \mathrm{F} &= l\mathrm{P} + m\mathrm{U} + n\mathrm{T} \\ \mathrm{G} &= l\mathrm{U} + m\mathrm{Q} + n\mathrm{S} \\ \mathrm{H} &= l\mathrm{T} + m\mathrm{S} + n\mathrm{R} \end{aligned}\right\} \ldots . (3).$$

It is to be remembered that the mathematician *assumes* a certain condition of strain, and takes his chance of its fitting some particular practical problem. If it happens to fit the conditions of the problem exactly, it is an exact solution of the problem; and, according to Art. 306, it is the only solution. If it does not exactly fit the conditions (as, for example, Dr. Chree's solution of the rotating disc), we discuss the discrepancy to see whether it is essential or negligible for practical engineering purposes.

St. Venant studies the case of the prism under the following assumptions: P = Q = U = 0 . . . . (4). There is no normal stress, therefore, in the directions x or y (that is, laterally from fibre to fibre); there may be tangential force acting from fibre to fibre in a direction parallel to z. Hence equations (1) give

$$\frac{d\mathrm{T}}{dz} = 0, \quad \frac{d\mathrm{S}}{dz} = 0, \quad \frac{d\mathrm{T}}{dx} + \frac{d\mathrm{S}}{dy} + \frac{d\mathrm{R}}{dz} = 0 \ldots . (5).$$

The surface conditions (3) become

$$l\mathrm{T} + m\mathrm{S} = 0 \ldots . (6).$$

He also assumes that the origin is fixed, and that particles in the directions of the axes of z and y just there are fixed. This means that at the origin

$$u = 0, v = 0, w = 0, \frac{du}{dz} = 0, \frac{dv}{dz} = 0, \frac{du}{dy} = 0 \ldots . (7).$$

The following purely mathematical work is tedious, but quite easy. The student will do it carefully for himself, following our directions. Inserting (4) in (2), so as to calculate the strains, we find that

$$\frac{du}{dx} = \frac{dv}{dy} = -\sigma\frac{dw}{dz} \ldots . (8)$$

where σ is Poisson's ratio, or β/α, or $\frac{1}{2}\lambda/(\lambda + \mathrm{N})$. Also $\mathrm{U} = 0$ means that

$$\frac{du}{dy} + \frac{dv}{dx} = 0 \ldots . (9).$$

$$\left.\begin{aligned} \frac{d\mathrm{T}}{dz} = 0 \text{ means that } \frac{d^2u}{dz^2} + \frac{d^2w}{dz \,.\, dx} = 0, \\ \frac{d\mathrm{S}}{dz} = 0 \text{ means that } \frac{d^2w}{dz \,.\, dy} + \frac{d^2v}{dz^2} = 0 \end{aligned}\right\} \ldots . (10),$$

and the last condition of (5) gives

$$\mathrm{N}\left(\frac{d^2u}{dx \,.\, dz} + \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2v}{dy \,.\, dz}\right) + \lambda\frac{d^2u}{dx \,.\, dz} + \lambda\frac{d^2v}{dy \,.\, dz}$$
$$+ (\lambda + 2\,\mathrm{N})\frac{d^2w}{dz^2} = 0 \ldots . (11).$$

The first and second of (10) and (8) give

$$\frac{d^2u}{dz^2} = \frac{1}{\sigma}\frac{d^2u}{dx^2} \text{ and } \frac{d^2v}{dz^2} = \frac{1}{\sigma}\frac{d^2v}{dy^2} \ldots . (12);$$

(11) and (8) give, after simplification,

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 2\frac{d^2w}{dz^2} = 0 \ldots . (13).$$

Differentiate (13) with respect to z, and (10) with respect to x and y, and use (8) to eliminate u and v, and we get

$$\frac{d^3w}{dz^3} = 0 \ldots . (14).$$

Differentiate (10) with respect to y and x, add and use (9), and we get

$$\frac{d^3w}{dx \,.\, dy \,.\, dz} = 0 \ldots . (15).$$

Differentiate (10) with respect to y and x, and use (8) and (14), and we get

$$\frac{d^3w}{dx^2\,.\,dz} = 0, \quad \frac{d^3w}{dy^2\,.\,dz} = 0 \quad (16).$$

Hence $\frac{dw}{dz}$ **is linear in z, and linear in x and y separately.** Hence

$$\frac{dw}{dz} = \alpha + \alpha_1 x + \alpha_2 y + z\,(\beta + \beta_1 x + \beta_2 y) \;\ldots\ldots\; (17).$$

[α and β are here any constants, and not the coefficients of formula (2)]. Now use (8) with (17), and also (9), and we have

$$u = -\,\sigma\,\left(\alpha x + \tfrac{1}{2}\alpha_1 x^2 + \alpha_2 yz + \beta zx + \beta_2 yzx + \tfrac{1}{2}\beta_1 x^2 z\right) - \\ \tfrac{1}{2}\alpha_1 z^2 - \tfrac{1}{6}\beta z^3 + z u_1 + u_0$$

where u_1 and u_0 are functions of y. Similarly,

$$v = -\,\sigma\,\left(\alpha y + \alpha_1 xy + \tfrac{1}{2}\alpha_2 y^2 + \beta zy + \beta_1 xyz + \tfrac{1}{2}\beta_2 y^2 z\right) - \\ \tfrac{1}{2}\alpha_2 z^2 - \tfrac{1}{6}\beta_2 z^3 + v_0 + v_1 z$$

where v_0 and v_1 are functions of x. Now try these in (9), and find values of u_0, u_1, v_0, v_1, and get

$$w = z\,(\alpha + \alpha_1 x + \alpha_2 y) + \tfrac{1}{2}z^2\,(\beta + \beta_1 x + \beta_2 y) + \phi\,(x, y) = 0.$$

Using this in (13), we see that we must solve

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = -\,(\beta + \beta_1 x + \beta_2 y).$$

To get the complete solution, we find the general solution when the right hand side is zero, and add to it a particular solution. For the particular solution, try

$$\phi = \mathrm{A}\,x^2 + \mathrm{B}\,y^2 + \mathrm{C}\,xy^2 + \mathrm{D}\,x^2 y,$$

and we find it to be

$$-\,\phi = \tfrac{1}{4}\beta\,(x^2 + y^2) + \tfrac{1}{2}\beta_1 xy^2 + \tfrac{1}{2}\beta_2 x^2 y.$$

This, then, is what we must add to the general solution of $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$, taking care that the value of ϕ so found enables the whole value of w to satisfy the boundary conditions. We may now write out w,

$$w = z\,(\alpha + \alpha_1 x + \alpha_2 y) + \tfrac{1}{2}z^2\,(\beta + \beta_1 x + \beta_2 y) - \tfrac{1}{2}\left\{\tfrac{1}{2}\beta\,(x^2 + y^2) \\ + \beta_1 xy^2 + \beta_2 x^2 y\right\} + \phi.$$

309. Boundary Condition.—Use (2), and find from the above values of u, v, w values of S, T, and R. Use them in (6). Integrate both sides of the equation round the *boundary* of a cross-section, and transform the line integrals to surface integrals over the section. This gives us $\beta = 0$. Hence our complete answers are:—

$$\left.\begin{array}{l} u = -\sigma\left(\alpha x + \frac{1}{2}\alpha_1 x^2 + \alpha_2 xy + \frac{1}{2}\beta_1 x^2 z + \beta_2 xyz\right) \\ -\frac{1}{2}\alpha_1 z^2 - \frac{1}{6}\beta_1 z^3 + \frac{1}{2}\sigma\alpha_1 y^2 + z\left(\beta_0 y + \frac{1}{2}\sigma\beta_1 y^2\right) \\ v = -\sigma\left(\alpha y + \alpha_1 xy + \frac{1}{2}\alpha_2 y^2 + \beta_1 xyz + \frac{1}{2}\beta_2 y^2 z\right) \\ -\frac{1}{2}\alpha_2 z^2 - \frac{1}{6}\beta_2 z^3 + \frac{1}{2}\sigma\alpha_2 x^2 - \beta_0 xz + \frac{1}{2}\sigma\beta_2 x^2 z \\ w = z\left(\alpha + \alpha_1 x + \alpha_2 y\right) + \frac{1}{2}z^2\left(\beta_1 x + \beta_2 y\right) + \phi - \\ \frac{1}{2}xy\left(\beta_1 y + \beta_2 x\right) \end{array}\right\} \dots\dots (18)$$

where ϕ satisfies $\dfrac{d^2\phi}{dx^2} + \dfrac{d^2\phi}{dy^2} = 0$ at all points of any cross-section, and the condition

$$l\frac{d\phi}{dx} + m\frac{d\phi}{dy} = \beta_0\left(mx - ly\right) + \beta_1\left[\tfrac{1}{2}lx^2\sigma + \left(\frac{\mathrm{E}}{\mathrm{N}} - \sigma\right)\left(mxy + \tfrac{1}{2}ly^2\right)\right.$$
$$\left. - l\sigma y^2\right] + \beta_2\left[\tfrac{1}{2}m\sigma y^2 + \left(\frac{\mathrm{E}}{\mathrm{N}} - \sigma\right)\left(lxy + \tfrac{1}{2}mx^2\right) - m\sigma x^2\right] \dots\dots (19)$$

at all points of the cylindric boundary. The stresses at any point are $\mathrm{P} = 0$, $\mathrm{Q} = 0$, $\mathrm{U} = 0$.

$$\left.\begin{array}{l} \mathrm{S} = \mathrm{N}\left[\dfrac{d\phi}{dy} - \beta_0 x - \left(\dfrac{\mathrm{E}}{\mathrm{N}} - \sigma\right)\beta_1 xy - \tfrac{1}{2}\beta_2\left\{\sigma y^2 + \right.\right. \\ \left.\left.\left(\dfrac{\mathrm{E}}{\mathrm{N}} - \sigma\right)x^2 - 2\sigma x^2\right\}\right] \\ \mathrm{T} = \mathrm{N}\left[\dfrac{d\phi}{dx} + \beta_0 y - \tfrac{1}{2}\beta_1\left\{\sigma x^2 + \left(\dfrac{\mathrm{E}}{\mathrm{N}} - \sigma\right)y^2 - \right.\right. \\ \left.\left. 2\sigma y^2\right\} - \left(\dfrac{\mathrm{E}}{\mathrm{N}} - \sigma\right)\beta_2 xy\right] \\ \mathrm{R} = \mathrm{E}\left[\alpha + \alpha_1 x + \alpha_2 y + z\left(\beta_1 x + \beta_2 y\right)\right] \end{array}\right\} \dots\dots (20).$$

By giving any values to α, α_1, α_2, β_0, β_1, β_2 **we get all the possible solutions**. It must be remembered that a point $x\,y\,z$ has the new position $x + u$, $y + v$, $w + z$.

Notice that if β_0, β_1, β_2 are 0, then $\phi = 0$, because $\mathrm{S} = \mathrm{N}\dfrac{d\phi}{dy}$ and $\mathrm{T} = \mathrm{N}\dfrac{d\phi}{dx}$, and each of these is 0 at the boundary, and the only conditions to be satisfied by ϕ are that $l\dfrac{d\phi}{dx} + m\dfrac{d\phi}{dy} = 0$ at the boundary. Hence ϕ may be a constant, or rather a function of z; but it is 0 where $z = 0$, and hence $\phi = 0$ everywhere.

310. *Example* 1.—Let all the arbitrary constants vanish except α. We must remember that α is *any* arbitrary constant, and is not the α of Art. 289. Then $u = -\sigma\alpha x$, $v = -\sigma\alpha y$, $w = \alpha z$, $\mathrm{R} = \mathrm{E}\alpha$, and the other stresses vanish. We have, therefore, **a tie bar**.

311. *Example* 2.—**Bending.**—Let all vanish but α_1, then $u = -\frac{1}{2}\alpha_1\left(z^2 + \sigma x^2 - \sigma y^2\right)$, $v = -\alpha_1\sigma xy$, $w = \alpha_1 xz$. Hence $\mathrm{P} = 0$, $\mathrm{Q} = 0$, $\mathrm{R} = \mathrm{E}\alpha_1 x$, $\mathrm{S} = 0$, $\mathrm{T} = 0$, $\mathrm{U} = 0$. The shape of the centre line is obtained by putting $x = 0$, $y = 0$, and we have $u = -\frac{1}{2}\alpha_1 z^2$, $v = 0$,

$w = 0$. So that it becomes the arc of a parabola, or nearly a circle of curvature α_1, lying in the plane x, z. The total force parallel to z across the section $= \iint \text{E}\,\alpha_1 x \,.\, dx \,.\, dy = 0$, because x is measured from the centre of area. Hence the tractions at a section are a torque merely, called here a **bending moment,** whose amount is evidently $- \text{E}\,\alpha_1\,\text{I}_1$ where I_1 is the moment of inertia of the section about the axis of y in the section. Since $w = \alpha_1 xz$, and we make $z = z_0$ a constant, we have the displacement at all points in a section. The section remains plane, and turns through the angle $z_0\alpha_1$.

Change of Shape of Section.—Take a portion of the section of rectangular shape. The boundaries, having been $x = \pm\, a$, $y = \pm\, b$, have become $x = \pm\, a - \frac{1}{2}\alpha_1\sigma\,(a^2 - y^2)$, $y = \pm\, b - \sigma\alpha_1 bx$. So that the boundary consists now of two straight lines and two arcs of parabolas. We note that the upper and lower surfaces of the beam are anticlastic surfaces, the principal curvatures being in the ratio σ. Note (Art. 330) the result of the ordinary theory. It is then evident that—at all events, when the bending moment is constant—the hypothesis on which our ordinary theory is based, namely, that a plane cross-section remains plane, agrees with the more fundamentally correct theory. But we have now to notice that this case of the St. Venant theory only contemplates small displacements, and may not be applied to cases where the displacements are large. The ordinary theory is more useful, therefore, for it makes the shape of the centre line to be circular, as it obviously must be; and no attention need be paid to the fact that this case of the St. Venant theory makes it the arc of a parabola.

312. *Example* 3.—Let all the constants except β_0 vanish, and let β_0 be called $- \tau$. Then $u = - \tau zy$, $v = \tau zx$, $w = \phi\,(x, y)$, where τ is a constant and w is a function of x and y only. We find that

$$\text{P} = \text{Q} = \text{R} = 0,\ \text{S} = \text{N}\left(\frac{d\phi}{dy} + \tau x\right),\ \text{T} = \text{N}\left(\frac{d\phi}{dx} - \tau y\right),\ \text{U} = 0, \text{ and}$$

at the boundary $l\dfrac{d\phi}{dx} + m\dfrac{d\phi}{dy} = \tau\,(ly - mx)$. Everywhere in the section $\dfrac{d^2\phi}{dx^2} + \dfrac{d^2\phi}{dy^2} = 0$. The total tangential force parallel to x is $\iint \text{T} \,.\, dx \,.\, dy$. On writing this out, it will be found transformable

$$\text{into N} \iint \left[\frac{d}{dx}\left\{ x\left(\frac{d\phi}{dx} - \tau y\right)\right\} + \frac{d}{dy}\left\{ x\left(\frac{d\phi}{dy} + \tau x\right)\right\}\right] dx \,.\, dy,$$

$$\text{and into N} \int \left\{ lx\left(\frac{d\phi}{dx} - \tau y\right) + mx\left(\frac{d\phi}{dy} + \tau x\right)\right\} ds \text{ round the}$$

boundary, and this is 0 by the boundary condition Hence the resultant force parallel to x is 0, and the resultant force parallel to y is 0. So that the **forces over the section are a torque** $\mathrm{N}\iint \left\{ x\left(\frac{d\phi}{dy} + \tau x\right) - y\left(\frac{d\phi}{dx} - \tau y\right) \right\} dx\,.\,dy$, which is equal to $\tau \mathrm{N} \mathrm{I} + \mathrm{N}\iint \left(x\frac{d\phi}{dy} - y\frac{d\phi}{dx}\right) dx\,.\,dy = q\,\mathrm{N}\tau\mathrm{I}$, say, where I is the moment of inertia of the cross-section about its centre.

This case, then, is that of a **prism subjected merely to a twisting moment** $q\,\mathrm{N}\tau\mathrm{I}$. When the section is circular, q is 1, and τ is the angle of twist per unit length of the prism. When the section is not circular, q is not 1, and is called St. Venant's torsion factor. To solve the problem, then, when the shape of section is given, we must find ϕ such that $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$ all over the section, with the boundary condition $l\frac{d\phi}{dx} + m\frac{d\phi}{dy} = \tau\,(ly - mx)$. To do this, it is well, as in many other such problems, to find, first, a conjugate function ψ, such that $\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0$, and $\frac{d\phi}{dx} = \frac{d\psi}{dy}$, and $\frac{d\phi}{dy} = -\frac{d\psi}{dx}$. So that the boundary condition becomes

$$l\frac{d\psi}{dy} - m\frac{d\psi}{dx} = \tau\,(ly - mx).$$

Now we see by trial that if $\psi = \frac{1}{2}\tau\,(x^2 + y^2) + c$, the boundary condition is satisfied for $\frac{d\psi}{dy} = \tau y$, and $\frac{d\psi}{dx} = \tau x$. It is evident, then, that if we can find a solution of $\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0$, subject to the condition $\psi = \frac{1}{2}\tau\,(x^2 + y^2) + c$ at all points of the boundary, we have the correct answer.

Example.—**Prism of elliptic section,** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, giving the shape of the boundary. It is found by guessing that $\psi = \mathrm{A}\,(x^2 - y^2)$ is a solution, if A is properly evaluated to satisfy the boundary condition. So that $\mathrm{A}\,(x^2 - y^2) = \frac{1}{2}\tau\,(x^2 + y^2) +$ const. at the boundary—that is, where $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$. We substitute this, and find that A is $\frac{1}{2}\tau\,(a^2 - b^2)/(a^2 + b^2)$. So that $\psi = \mathrm{A}\,(x^2 - y^2)$, and $\phi = -2\,\mathrm{A}\,xy$.

Evidently the curves in the section along which ϕ is constant are rectangular hyperbolas, with the principal diameters as axes. **The total twisting couple is**

$$\mathrm{M} = \tau\mathrm{N}\mathrm{I} + 2\,\mathrm{A}\,\mathrm{N}\iint(-x^2 + y^2)\,dx\,.\,dy = \frac{4\,a^2b^2}{(a^2 + b^2)^2}\,\mathrm{N}\tau\mathrm{I}.$$

The shears are now easily calculated.

$$\mathrm{S} = \mathrm{N}\,(-\,2\,\mathrm{A}\,x + \tau x),\ \mathrm{T} = \mathrm{N}\,(-\,2\,\mathrm{A}\,y - \tau y),$$

$$\mathrm{S} = \mathrm{N}\,x\left(\tau - \tau\frac{a^2 - b^2}{a^2 + b^2}\right) = \mathrm{N}\,\tau\,x\left(\frac{2\,b^2}{a^2 + b^2}\right),$$

$$\mathrm{T} = \mathrm{N}\,y\left(-\,\tau - \tau\frac{a^2 - b^2}{a^2 + b^2}\right) = -\,\mathrm{N}\,\tau\,y\left(\frac{2\,a^2}{a^2 + b^2}\right),$$

S is greatest when $x = a$, then

$$\mathrm{S} = \mathrm{N}\,\tau\,\frac{2\,b^2a}{a^2 + b^2},\ \text{also}\ \mathrm{S}^2 + \mathrm{T}^2 = f^2 = \frac{4\,\mathrm{N}^2\tau^2}{(a^2 + b^2)^2}\left\{b^4x^2 + a^4y^2\right\}.$$

The maximum stress anywhere in the section is the value of T when $y = b$, viz. $\mathrm{N}\,\tau\,\frac{2\,a^2b}{a^2 + b^2}$. For f is a maximum when $b^4x^2 + a^4y^2$ is greatest. Now, whatever be the value of x for which the maximum occurs, for that value of x the maximum will be when y is greatest—namely, on the boundary. Put $x = a\cos.\theta$, $y = b\sin.\theta$, then $b^4x^2 + a^4y^2$ becomes $a^2b^2\,[b^2 + (a^2 - b^2)\sin.^2\theta]$, which is obviously greatest when $\theta = 90°$. Hence the stress is a maximum at the extremities of the minor axis.

Example.—**Torsion of a Shaft whose Section is an Equilateral Triangle.**—Let $3a$ be the vertical height of the triangle, then the equation of the boundary is $(x - a)\,(x - y\sqrt{3} + 2a)\,(x + y\sqrt{3} + 2a) = 0$, or $x^3 - 3xy^2 + 3a\,(x^2 + y^2) - 4a^3 = 0$. The function $\psi = \mathrm{A}\,(x^3 - 3\,xy^2)$ satisfies the differential equation, and $\mathrm{A}\,(x^3 - 3\,xy^2) - \frac{1}{2}\tau\,(x^2 + y^2)$ will be constant over the boundary if $\mathrm{A} = -\,\frac{\tau}{6\,a}$.

Hence $\psi = -\,\frac{\tau}{6a}\,(x^3 - 3\,xy^2)$, and $\phi = -\,\frac{\tau}{6a}\,(y^3 - 3x^2y)$ . . . . (1).

From this it is easy to find the strains and stresses. It will be found that the shear is 0 in the corners, and is a **maximum at the middle of the sides.**

313. Approximate Formulæ.—St. Venant worked out ϕ and the strains and stresses for a number of sections, and he found that if the section of a shaft is not too different in any two of its dimensions across the centre, the torsional rigidity (or twisting moment per unit angle of twist) is $\mathrm{A} = m\,\frac{s^4\,\mathrm{N}}{\mathrm{I}}$ where s is the area of the section and I its moment of inertia about its centre of gravity; N is the modulus of rigidity, and m is a number which does not greatly differ in different cases. $m = {\cdot}02533$ for an ellipse; and in the sections examined its lowest value was ·023, and its highest was ·026. Consequently, if we take $\mathrm{A} = \frac{1}{40}\,s^4\mathrm{N}/\mathrm{I}$ in all such cases, there is no great error.

314. Non-uniform Flexure.—In Art. 309 let all the constants be zero except β_1. The displacements are

$$u = -\tfrac{1}{2}\beta_1\left\{\sigma z(x^2 - y^2) + \tfrac{1}{3}z^3\right\},$$

$$v = -\sigma\beta_1 xyz,$$

$$w = \phi + \beta_1\left\{\tfrac{1}{2}z^2x - xy^2\right\}$$

where ϕ satisfies $\dfrac{d^2\phi}{dx^2} + \dfrac{d^2\phi}{dy^2} = 0$ at all points of a normal section; and at the boundary the condition

$$l\frac{d\phi}{dx} + m\frac{d\phi}{dy} = \beta_1\left[\tfrac{1}{2}l\sigma x^2 + (2+\sigma)(mxy + \tfrac{1}{2}ly^2) - l\sigma y^2\right].$$

The stresses at any point of a section z = constant, are

$$S = N\left(\frac{d\phi}{dy} - (2+\sigma)\beta_1 xy\right), \text{ parallel to the axis } y,$$

$$T = N\left(\frac{d\phi}{dx} - \tfrac{1}{2}\beta_1(\sigma x^2 + 2y^2)\right), \text{ parallel to the axis } x,$$

$$R = E\beta_1 zx, \text{ parallel to the axis } z.$$

The total force parallel to x (really the shearing force X at the section) turns out to be $EI\beta_1$, and the resultant total force parallel to y (call it a shearing force Y) is $E\beta_1\displaystyle\iint xy\,.\,dx\,.\,dy$, and this is 0 if the axes are principal axes; and we shall assume them to be so. The resultant stress parallel to z vanishes. The couple about the axis of y is $-z\beta_1 EI$. The couple about the axis of z may be written out. By a combination of this and previous solutions, we have the case of **any twisting and bending couples applied to a prism.** If we merely take the case of a prism being bent, being fixed at one end, and loaded at the other with a load W, we must make our twist 0; and hence we must use the solution in which α_1 was constant, together with this in which β_1 is constant, so that the two twists may just neutralise each other. We get this condition, we find, by taking α_1 and β_1, such that $\alpha_1 + \beta_1 l = 0$. In this case we have a total tangential force parallel to x of the amount $W = X = E\beta_1 I$, and a couple $E\beta_1 I\ (l - z)$, or $W\ (l - z)$, which we usually call a bending moment, due to a load W at the distance l from the fixed end of the beam. $\alpha_1 = -\beta_1 l = -Wl/EI$.

The equation to the centre line is $x = \dfrac{W}{EI}\left(\tfrac{1}{2}z^2 l - \tfrac{1}{6}z^3\right)$. We see, therefore, that **the ordinary theory is right** as to the shape of such a beam. The curvature α_1 is the bending moment at the fixed end divided by EI, the flexural rigidity. The displacements are, if there is no twist,

$$u = \frac{W}{2\,EI}\left[(l - z)\,\sigma\,(x^2 - y^2) + (l - \tfrac{1}{3}z)\,z^2\right],$$

$$v = \frac{W}{EI}\,\sigma\,(l - z)\,xy,$$

$$w = \frac{W}{2\,EI}\left[(z^2 - 2\,lz)\,x - 2\,xy^2\right] + \phi.$$

Notice from u and v that the change of shape of the section is as given in Art. 311. The strain $g = \frac{dw}{dz} = \frac{W}{EI}(z - l)\,x$ (that is, the tensile strain at any point of the section is proportional to x). $\frac{du}{dx} = e = \frac{W}{EI}\sigma\,(l - z)\,x$, $f = \frac{dv}{dy} = \frac{W}{EI}(l - z)\,x$. We may write out the shear strains a and b in terms of ϕ, etc. $c = 0$.

Now in cases that have been considered in which x and y are small compared with $l - z$, it is found that ϕ is of the third degree in x and y. So that a and b are small compared with e, f, and g. For the same reason, the term $2\,xy^2$ is not important in w above. And the **engineer's theory** based on the assumption that **a plane section remains plane,** may be taken as correct. When x and y are not small compared with $l - z$, we must find ϕ, and this is difficult. As w contains ϕ, although the tensile strain is proportional to x, the plane section does not remain plane if beams are short. St. Venant's solution assumes that W is distributed in a particular manner over the end section. But by the principle of equipollent loads the actual distribution of W is of very little consequence, except close to the end section itself, and hence is of no practical importance except in beams that are not long in comparison with the values of x and y in their sections.

315. Vertical loads are often applied to beams on their horizontal top surfaces. We know from the principle of equipollent loads that the actual distribution has little effect **except in the neighbourhood of the surfaces** to which the loads are applied. We can obtain a fairly clear notion of the effect by thinking of the load on a plane surface bounding an infinite elastic solid.

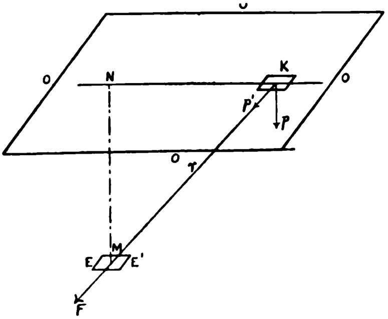

Fig. 205.

M. Boussinesq has given the solution.* The following brief memorandum may be useful, O O being the plane surface bounding the infinite isotropic solid. M is a point within, situated at a distance M N = z below the surface. K is any element of the surface, situated at the distance K M = r from the point M, and subject to a given exterior surface pressure Kp = p per unit area, having the component K$p^1 = p^1$ per unit area along K M. The pressure which a plane element K E^1 taken through M parallel to the surface O O will support per unit area in consequence of the pressure p will be found directed along K M produced, and is equal to M F = $\frac{3\,p^1 z}{2\,\pi r^3}$. If, as a particular case, the pressure K P = p be normal, then $p^1 = p$ cos. N M K $= p\frac{z}{r}$, and M F = $3pz^2/2\,\pi r^4$. If we want the vertical component of M F, we have $3pz^3/2\,\pi r^5$.

Generally.—Let w_1 be the normal force per unit area at any point on the plane bounding surface at the point $x = x^1$, $y = y^1$, $z = 0$, the axes of x and y being in the plane, and the axis of z being the normal to the plane drawn into the material.

$$\text{Let } \phi = \int\!\!\int w_1\,.\,r dx^1\,.\,dy^1, \text{ and } \mathrm{x} = \int\!\!\int w_1 \log.(z + r)\,dx^1\,.\,dy^1,$$

where r is the distance from x^1y^10 to xyz. Then u, v, and w being the displacements at x, y, z,

$$u = -\frac{1}{4\,\pi\,(\lambda + \mathrm{N})}\,\frac{d\,\mathrm{x}}{dx} - \frac{1}{4\,\pi\,\mathrm{N}}\,\frac{d^2\phi}{dz\,.\,dx},$$

$$v = -\frac{1}{4\,\pi\,(\lambda + \mathrm{N})}\,\frac{d\,\mathrm{x}}{dy} - \frac{1}{4\,\pi\,\mathrm{N}}\,\frac{d^2\phi}{dz\,.\,dy},$$

$$w = -\frac{1}{4\,\pi\,(\lambda + \mathrm{N})}\,\frac{d\,\mathrm{x}}{dz} - \frac{1}{4\,\pi\,\mathrm{N}}\,.\,\frac{d^2\phi}{dz^2} + \frac{\lambda + 2\,\mathrm{N}}{4\,\pi\,\mathrm{N}\,(\lambda + \mathrm{N})}\,\nabla^2\phi.$$

From u, v, and w all the strains and stresses may be calculated at any point.

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* The interested student may refer to a paper read by Prof. Carus Wilson before the Physical Society of London (June, 1891) on "The Influence of Surface Loading on the Flexure of Beams," and to another by Dr. Chree.

CHAPTER XVI.

BENDING.

316 In Fig. 206, C D is a **beam** carrying a weight W. We know that the beam transmits the weight to the walls, and that in doing so the beam is kept in a strained condition; we must consider what is the state of strain in the beam. To observe this it will be well to take a beam which is very visibly strained, a beam of indiarubber. A B is its appearance when lying on the table; draw upon it a number of parallel lines in chalk or pencil, *a b*, *c d*, *e f*, etc. Now if we support the beam at its two ends, and load it, we find that the lines *a b*, *c d*, etc., remain straight, but they are no longer parallel. We find the distance *a′ c′* to be less than *a c*, but *b′ d′* is greater than *b d*. In fact, *a′ c′* is **compressed**, *b′ d′* is **extended**. We find also that along the line E′ F′ there is **neither compression nor extension**. E′ F′ remains of its old length, although it is no longer straight. If we consider each cross section of such a beam we see that the upper part of it is in compression, the lower part of it is in extension, and there is a straight line in the middle where there is neither compression nor extension. This line is called the **neutral axis** of the cross section, and all these axes lie in a surface called the **neutral surface** of which E′ F′ is an edge view. Fig. 207 is a magnified drawing of the small portion of the beam between two cross sections. *c e f d* shows its original shape, *c′ e′ f′ d′* its shape when strained. Evidently there is more compression at *c′ e′* than at *l′ m′*.

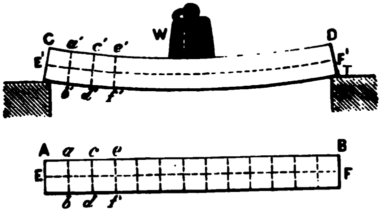

Fig. 206.

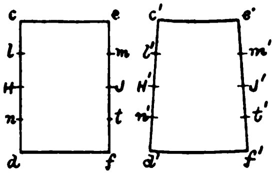

Fig. 207.

The compression becomes less and less as we come nearer H′ J′, then the extension begins, and becomes greater and greater as we get farther away from H′ J′ until we get to *d′ f′*, where it is greatest. If the material is likely to break in compression it will be most likely to break at *c′ e′*. If it is likely to break in tension it will be most likely to break at *d′ f′*.

317. If we know the compression or extension at any place, **we can calculate** what it is at any other place, for *the strain is evidently proportional to the distance from the middle.* Thus if at *e′* there is a compressive strain of ·002, that is, there is a compression of ·002 foot for every foot in length, then at *m′*, half-way between J′ and *e′*, there is only a strain of ·001. There is the same strain at *t′* the same distance below J′, but it is now an extension. The material resists being strained in this way, and the pushing and pulling forces which it exerts at the section *e′ f′*, Fig. 207, are just the forces required to balance all the other forces acting on the part *e′* D T *f′*.

As *e′* D T *f′* is a body kept at rest by forces, and which is no longer altering in shape, it is to be regarded as a **rigid body.*** Now what is the condition under which it is kept at rest?

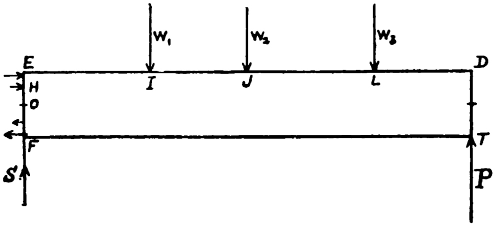

Fig. 208.

318. The beams used by us are almost never deformed so much as the beam shown in Fig. 206, and indeed our theories are only true on the assumption of exceedingly small changes of shape. Let, then, the part *e′* D T *f′* be drawn less deformed as E D T F in Fig. 208, and consider its equilibrium. We had better consider more weights than one, loading it.

The forces W_1, W_2, and W_3 represent loads, and P is the supporting force at the end T of the beam. In Art. 99 we saw that if all the **loads** on a structure are given, the **supporting forces** may be calculated.

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* In books on mechanics you may have read much about rigid bodies and the laws of their equilibrium, and you may have thought that such bodies had no existence; but you must remember that we can regard a quantity of water, or a piece of steel spring, or a rope, as a rigid body for the time being, *if it is being* acted on by forces, and is *no longer changing its shape.*

319. We are now considering only **horizontal beams** on which **the loads are only vertical** and the supporting forces vertical: Students had better work again here a few exercises, to find the supporting forces when the loads are given. Either neglect the weight of E D T F itself or imagine it represented by W_2. Suppose, then, P to be known.

The molecular forces acting on the surface E F (by the material to the left upon the material to the right of E F) **balance all the external forces** acting upon E D T F, namely, W_1, W_2, W_3, and P. Art. 98 gave the following three as the conditions of equilibrium.

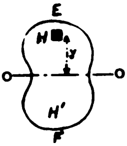


Fig. 209.

I. The upward tangential resultant of the molecular forces, which I shall call S, the **shearing force** at the section, is equal to $W_1 + W_2 + W_3 - P$. The student ought now to work a number of exercises. Given loads, find supporting forces; find S at many sections; show all answers in one diagram and call it the diagram of shearing force. Observe that we assert nothing as to how this shearing force is distributed over the section. We shall find in Art. 369 that it is most intensely distributed about the middle parts near the neutral axis O O.

II. As the loads are all vertical there is no **horizontal** component in the resultant of the molecular forces; in fact, all the pushing forces on E F balance all the pulling forces. Figs. 209, 210 show E F magnified, its actual shape and side elevation also. At H, a point in E F at the distance y from O O the neutral axis, the pushing force per square inch or the compressive stress being proportional to y, let us call it $c\ y$, when c is some constant; if a point is at H′ (Fig. 209), we shall call O H′ a negative value of y, so that a pulling force is regarded as the negative of a pushing force. Now at H let there be an exceedingly small portion of area a, the force on this is $c\ y\ a$, and we must have the sum of all such terms as $c\ y\ a$ for the whole area to be zero. This is only another way of saying that the pushing and pulling forces are equal.

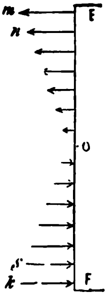


Fig. 210.

All the terms $c\,y\,a$ have the same multiplier c; hence what we state is that if every little portion of area a be multiplied by its y, the sum is zero. When this is so, Art. 109 tells us that O O, the **neutral axis, must pass through the centre of gravity of the area.** This is why we are always so anxious to find the centre of gravity of the section of a beam. The rules for finding the centre of gravity of the area are given in Art. 111. We are now about to find the value of c. Notice that if we know c we know the stress at any point in the section, and we particularly want to know c. O E or c. O F, the greatest stress.

III. The **moments** of all the molecular forces about any axis balance the moments of W_1, W_2, W_3 and P about the same axis. Now as these are all vertical forces, if we choose an axis at right angles to the plane of bending (the plane of the paper) in the plane E F, notice that about any such axis these moments will be the same; hence we speak of the moments of W_1, W_2, W_3 and P about any such axis as the **bending moment M about the section** E F. When it tends to make the beam convex upwards I call it positive. Hence $M = -P \cdot TF + W_1 \cdot IE + W_2 \cdot JE + W_3 \cdot LE$.

The student ought to practise the calculation of M for many sections of a beam and then show his answers in a diagram,

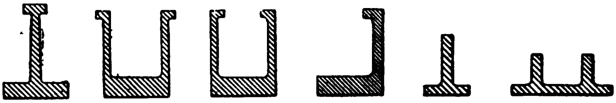


Fig. 211.

We have an easy graphical method of drawing such a diagram (*see* Art. 349). We call it a diagram of bending moment.

320. Very well, then, the sum of the moments of all such molecular forces as $c\,y\,a$ must be equal to M. Take the moments about O O, the neutral axis; the moment of $c\,y\,a$ is $c\,y\,a \times y$ or $c\,y^2\,a$. Now, when every little portion a of an area is multiplied by the square of its distance from a line and the sum taken, it is called the moment of Inertia I. of the whole area about the line, and it is easy to find I. for any area about the axis O O. Part of Chap. VII. is devoted to this subject of moments of inertia. We have, then, $c\,I = M$, or $c = M/I$; and hence the *compressive* stress p at points y inches from the neutral axis is

$p = M y/I$ . . . (1). As y is greatest at points like E or F, we have the greatest compressive and tensile stresses at these points. In fact, the greatest compressive stress in the section is at E, and its amount is O E. M/I . . . (2); the greatest tensile stress is at F, and its amount is O F. M/I . . . (3), and these two expressions give us the great laws of strength of beams. If we know the stresses which the material will stand, we know whether the section E F will withstand the bending moment M.

321. If the material is cast iron it is advisable to have O E = about 4½ times O F, because cast iron will stand about 4½ times as much compressive as tensile stress. Hence the usual economical cast-iron sections are as shown in Fig. 211, with centres of gravity near the bottom boundaries of the sections. Whereas in wrought iron and mild steel and other materials the resistance to compressive stress is much the same as the resistance to tensile stress, and consequently O E is made equal to O F, and the usual economical sections are as shown in Fig. 221.

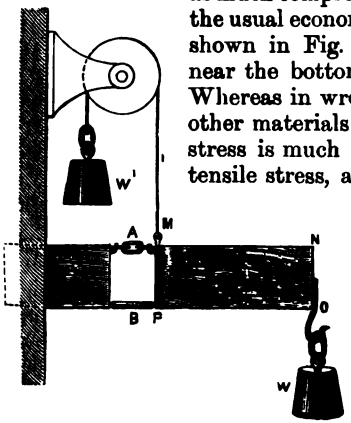


Fig. 212.

322. In the model, Fig. 212, which shows a beam fixed at one end and loaded at the other, part of the material has been removed, and instead of it we have inserted a chain or link A, which is only capable of exerting a pull, and a rod B, which is only capable of exerting a push. It is found that forces acting merely horizontally on M N O F are not sufficient to keep it at rest; we also need an upward force at M F, which is equal to the weight W, together with the weight of M N O F itself. We see then that at such a section as M F of a beam we need pulling and pushing forces, but also to satisfy the first condition given above we need the **shearing force** at M F. In fact, an upward force W′ must be exerted at M F equal to the weight W and also to the weight of M N O F. At M F the bending moment is W × O F, together with the weight of M N O F × the distance of its centre of gravity from M F. This is to be balanced by the pull in the chain A or the *push in the rod* B, for these are equal, multiplied by the distance between their lines of action. If a beam is long, the shearing force exerted by the material at a section of the beam is usually not so important to consider as the pushing and pulling forces, and in many cases it is neglected. When a beam is very short the shearing force becomes more important to consider.

323. We shall now take a case in which there is **only bending moment** to be balanced by the material at a section. Let A B (Fig. 213) be a strip of wood or metal originally straight, whose weight we shall neglect. Fix or solder to the ends stout pieces of metal, and by means of cords and weights, or in any other way, exert *couples* on these ends as shown. Consider now the equilibrium of any portion say, C D B (Fig. 214). At the section C D we know that pulling and pushing forces must be exerted by the material which exists at the left of C D on the material which exists at the right of C D, and the moments of these just balance the moment of the forces F and F, and this is evidently the same at any section of the strip. The bending moment at any section is then the moment of the couple or torque M acting at either end. Magnifying the section E F, as in Fig. 210, and representing the amounts of the pulling and pushing stresses by arrows, we see as before that as the sum of all the forces one way must be equal to the sum of all the forces acting the other way, and as the stress at each place is proportional to distance from O, the part where there is no stress or the neutral axis is as before, a line through O at right angles to the paper, and this must pass through the centre of gravity of the section. We see also that the stresses at all points of the section are given by (1), and if we particularly desire to know the greatest stresses they are given us by (2) and (3).

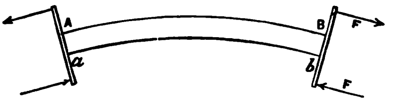


Fig. 213.

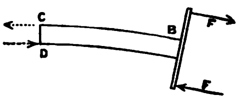


Fig. 214.

324. Unsymmetrical Bending.—If the bending moment M in a section is about an axis OX^1 through O the centre of gravity of the section, and if this is not a principal axis (see Art. 114), but makes the angle $X^1 O X = \alpha$ with a principal axis O X (about which the moment of inertia is I_1), the other being O Y (about which the moment of inertia is I_2), then M may be resolved into M cos. α about O X, and M sin. α about O Y. The stresses and strains due to these separately have now to be combined to obtain the actual stresses and strains due to M. Thus at a point whose co-ordinates referred to the principal axes are x and y, we have the total stress

$$f = \frac{M \cos. \alpha . y}{I_1} + \frac{M \sin. \alpha . x}{I_2}.$$

If f is put equal to 0, we find the position of the neutral line. It evidently is like O Q, and makes the angle $\beta = Q O X$ with O X, such that tan. $\beta = \frac{I_1}{I_2}$ tan. α. It is now easy to find the points which are at the greatest distance from O Q, and these are the points at which there is greatest stress.

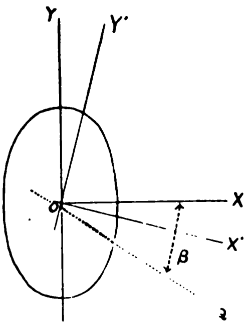


Fig. 215.

Exercise.—In a beam of rectangular section, if the axis about which the bending moment takes place is at right angles to a diagonal, show that the greatest stress is at corners, and is of the amount 6 M ÷ *bd* D if *b*, *d*, and D are the breadth, depth, and diagonal of the rectangle.

325. The line which passes through the centre of gravity of every cross section, being neither extended nor compressed, is of the original length of the strip. When the beam is bent as in the figure, A B becomes longer than this, and *a b* shorter, yet their ends are in the same planes A *a* and B *b*. Thus the strip may be considered as a bundle of fibres lying in arcs of circles which have the same centre and subtend the same angle at that centre. If we know their relative lengths we can tell where the centre of the circle is. Now we know the stress per square inch on a certain fibre, and we know its original length, hence we can calculate its present length (*see* Arts. 241 and 265), and its length is to the length of the neutral fibre as its radius is to that of the neutral fibre. In this way we find that *the radius of the neutral fibre is numerically equal to the modulus of elasticity of the material multiplied by the moment of inertia of the cross section, and divided by the bending moment at the section*, or **the curvature** $\frac{1}{r} = \frac{M}{E I} \ldots (4)$.

To put it in another way :—The curvature in Fig. 207 being angular change per unit length, is the angle between $c'\,d'$ and $e'\,f'$ if H′ J′ is unity. But this angle is, strain at $y \div y$, or $p \div \text{E}\,y$ . . . (5) and by (1) this becomes (4). The form (5) is sometimes useful in indicating the greatest curvature which may be given to a beam without hurt, by making p the greatest stress which the material will stand, y being the greatest distance of any point in the section from the neutral axis on the compression or tension side. In the same way it is easy to see that if a beam was already bent, having a curvature $\frac{1}{r_0}$, the **change of curvature** $\frac{1}{r} - \frac{1}{r_0} = \frac{\text{M}}{\text{E I}}$ . . . (6).

Example.—A straight strip of tempered steel, 0·7 inch broad, 0·1 inch thick (this represents the *depth* of a beam), is subjected to a bending moment of 100 pound inches : **find its radius of curvature.** Answer :—The moment of inertia of the section is 0·7 × ·1 × ·1 × ·1 ÷ 12, or ·0000583. The modulus of elasticity of steel is, say 36,000,000, and 36,000,000 × ·0000583 ÷ 100 gives 21 inches for the radius of curvature of the bent strip. The curvature is $\frac{1}{21}$. If the strip was already bent before the load was applied and had a radius of curvature 50 inches, then the change of curvature $\frac{1}{r} - \frac{1}{50} = \frac{1}{21}$; hence $r = 14{\cdot}8$, or if the new bending takes place in an opposite way to the old so that the new curvature is called negative for our purpose, $\frac{1}{50} - \frac{1}{r} = \frac{1}{21}$, so that $r = -\ 36{\cdot}2$ inches.

Exercise.—A beam is supported horizontally on two points, one under each end; C is a point of the beam one-fourth of its length from one of the points of support. Compare the curvature at C, supposing the beam to be uniformly loaded, with what it would be if the beam were without weight and the load concentrated at the middle point, the total load in both cases being the same. *Ans.*, As 3 to 4.

326. When a beam is loaded in any way, we know now how to find the bending moment at any place, and if we know the modulus of elasticity of the material, and the moment of inertia of the section, we can find the **curvature of the beam.** We *may* draw a bent beam, then, in the same way as we draw the *springs of Fig.* 216, but the beam is so little curved usually that we have difficulty in getting compasses long enough. In this case it is usual to diminish all the radii in some large proportion, remembering that the deflection of the beam as you draw it is increased in this proportion. For a beam fixed at one end and loaded at the other we get a curve just like the portion S T in Fig. 216 *c*, S being the fixed end and T the loaded end.

The following method is not perhaps so instructive for beginners, but it is the quicker and more accurate method. The

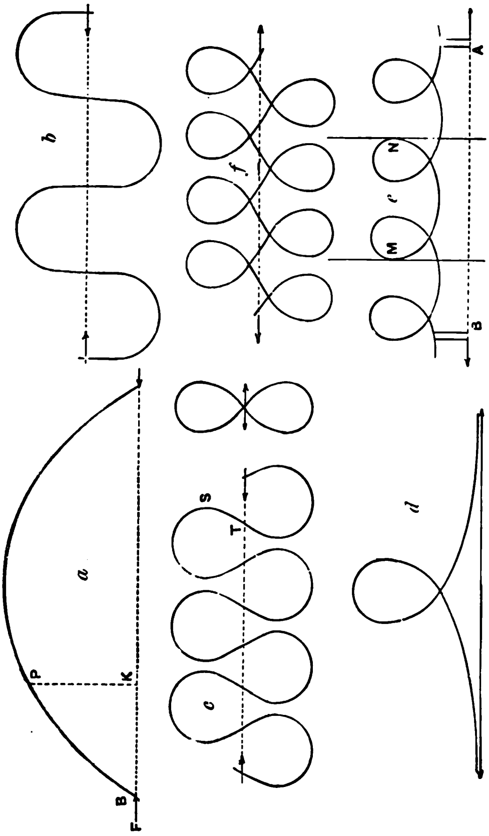


Fig. 216.

theory is given in Art. 358. Suppose A C D E B is a diagram showing the value of $M \div EI$ at every place, E being Young's modulus, and I the moment of inertia at each section. Treat it as if it were a diagram of loading, as described in Art. 350, and proceed as if you wanted to find M; in truth what you will find is a diagram which shows everywhere the *deflection* of the beam. That is, if you get *a b* of Fig. 349 horizontal, you will have found the *shape* of the beam turned upside down. The scale will be:—If the beam is drawn to a scale of 1 inch represented by *n* inches, and if $\frac{M}{EI}$ is drawn to a scale of unity in pounds and inches represented by 3 inches, and if an area of 1 square inch on the diagram of $\frac{M}{EI}$ in Fig. 237 is represented on the diagram of Fig. 236 by *v* inches, then a deflection of the beam of 1 inch is represented by a distance of $3n^2v/OH$ inches. O H is to be in inches.

327. Elastic Curve.—If we take a straight uniform strip of steel and subject it to two equal and opposite forces in the same straight line, the strip will assume one of the forms shown in Fig. 216, which all go under a common name—the *elastic curve.* Now consider the part P B, Fig. 216 *a*. Neglecting its own weight, it is acted on by a force F at B, and at P there must be an equal and opposite force to produce balance. There is a force at P tending to compress the steel, but what is of more importance is the fact that F produces a bending moment at P, and the amount of it is the force × the distance P K. Now our strip is everywhere of the same material and section, and the only thing that can alter is its curvature. This curvature at any place we know to be greater when the bending moment is greater, and less when the bending moment is less; in fact, the radius of curvature is inversely proportional to the bending moment, and this really comes to the fact that the radius of curvature at any place P is inversely proportional to the distance P K; or if the distance P K of any point from the line of force is called x, as $r = EI/M$ or $EI/F.x$, and as E I and F are constant, $r\,x$ is constant.

We can obtain the shapes shown in Fig. 216 in two ways: first, by taking a straight strip of steel and performing the operation; secondly, by drawing the curves in a series of arcs of circles. Suppose we have calculated, as in the above example, that the modulus of elasticity of the material multiplied by the moment of inertia of its cross-section is, say, 200 in inches and lbs., and suppose we know that the force acting at B is 10 lbs., [illegible] know that the radius of curvature at P is equal to 200 divided by the bending moment at P, which is 10 × P K. In fact, the radius of curvature at P is equal to 20 divided by P K or x. Choose now in Fig. 217 the point C as the middle point in the strip. Suppose C D or the value of x for C to be 4 inches, then the radius here is 5 inches. Take C O = 5 inches, and with O as centre describe a small arc, C E. Join E O and produce it. Now at E measure E F, the value of x for E, and suppose you find it 3·4 inches; divide 20 by 3·4, and we get 5·88 inches, and set this new radius off from E to O′. Take O′ as a new centre, and describe the short arc E G of any convenient small length, and in this way proceed until the curve is finished. This is not a very accurate method of drawing the curve unless the arcs are very short, and small errors are apt to have magnified evil effects, but I know of no better exercise to impress upon you the connection between radius of curvature of a strip and the bending moment which produces it. You are therefore supposed to have actually drawn one such curve at least before proceeding with your study of this subject. There is a way of diminishing errors, easy to discover for yourself if you are interested.

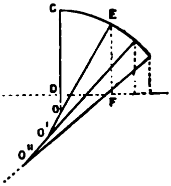


Fig. 217.

328. Parts of these curves happen to be the shape taken by liquids, because of their **capillary** action, between two parallel solid faces. They are also the shapes of the **arches** which are best fitted to withstand fluid pressure. Thus, for instance, in Fig. 218 the curve from M to N is of the shape of the curve Fig. 216 *e*, from M to N, the free water level being the line A B; and in Fig. 219 the middle line of the joints of the arch M to N is the same curve inverted. The water, whose pressure it resists, has as free water level the line A B in the relative position of the line of force to the elastic curve.

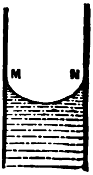


Fig. 218.

329. When a strip of elastic material is bent, it not only alters its shape in the well-known way, but it alters the form of its cross-section. On the convex side of the strip the breadth becomes concave, and on the concave side of the strip the breadth becomes convex. It is very easy to try this for yourself on a broad strip of steel or a bar of india-rubber. These **saddle shapes** of the surfaces are due to the fact that when each fibre is pulled it gets thinner as well as longer (*see* Art. 265), and when it is pushed it gets broader as well as shorter, and it is very curious that this action should not interfere perceptibly with the laws of bending as I have given them to you. In all probability it does so interfere, however, in the case of a very broad thin strip of material greatly curved.

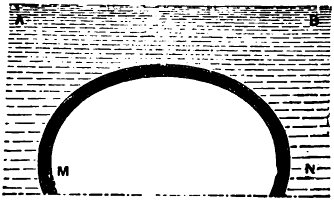


Fig. 219.

330. In any section of a beam, consider a rectangular portion of area of breadth b (parallel to the neutral line) and depth d. It evidently undergoes a change of shape of outline whose character does not depend upon what portion of the section it is in. But, for ease of calculation, let us take it to be symmetrical above and below the neutral line O O (Fig. 220). The curvature of the beam is $\text{M}/\text{E I}$. The compressive stress at a distance y from the neutral line is $f = \text{M}\,y/\text{I}$. This produces compressive strain, but it also produces a lateral thickening strain βf (*see* Art. 265). Hence the breadth b of the rectangle at any place y becomes broadened by the amount $b\,\text{M}\,y\beta/\text{I}$, and hence the straight sides A C and B D alter to the straight sides E H and G L. Also the dimensions parallel to A O and B O on the A B side of O O lengthen equally, and on the C D side they shorten equally. Consequently A B and C D become arcs of circles with the same centre. y is $\frac{1}{2}d$ at A B, and $-\frac{1}{2}d$ at C D. Consequently $\text{E G} = b\ (1 + \text{M}\,\frac{1}{2}d\beta/\text{I})$, $\text{H L} = b\ (1 - \text{M}\,\frac{1}{2}d\beta/\text{I})$. And if r is the radius of O′ N O′, $\frac{r + \frac{1}{2}d}{r - \frac{1}{2}d} = \frac{1 + \frac{1}{2}\text{M}\,d\beta/\text{I}}{1 - \frac{1}{2}\text{M}\,d\beta/\text{I}}$, and hence $\frac{r}{d} = \frac{\text{I}}{\text{M}\,d\beta}$, or $r = \text{I}/\text{M}\,\beta$. The student will recollect that we used a in Art. 265 to represent $\frac{1}{\text{E}}$; and β is the smaller number, such that β/a was called Poisson's ratio σ. We see now that the curvature of O′ N O′ is σ times the curvature of the neutral line of the beam. In the above case the curvature of either the top or bottom surface, or of the neutral surface, is anticlastic or saddle-shaped.

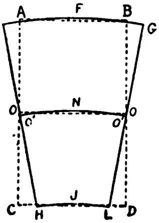


Fig. 220.

Exercise.—One side of a plate of metal is at θ_1°C and the other is at θ_2°C. The plate when cold is plane. What is now its curvature, and what is the greatest stress in the material if curvature be prevented? *Ans.*—$a^2 . (\theta_1 - \theta_2)^2/y^2$. (This measures the specific curvature of the surface, being the product of the two principal radii of curvature at any point *of the* surface.) Greatest stress $= \text{E}\,a\ (\theta_2 - \theta_2)$. (*See* Art. 5.)

CHAPTER XVII.

STRENGTH AT ANY SECTION OF A BEAM.

331. OBSERVE that in any section of a beam, the stress being greatest at places furthest away from O O (Fig. 220), these are places where the material is most useful in resisting bending. In some cases, as in railway girders, economy of material and its weight are so important that there is very little area of section except at E and F, where there are **two booms** or **flanges** each of area A: one where nearly all the compressive forces act (f A is the total compressive force if f is the compressive stress), and one where nearly all the tensile forces act (being equal to the same f A), and the sum of their moments about O (or any parallel line at right angles to the plane of bending), being f A d if d is the distance from E, the centre of one boom, to F, the centre of the other (and called the depth of the beam), is equal to the bending moment. In plate girders there is a thin web, or perhaps two (Fig. 221), holding the booms or flanges in their proper positions; in open-work girders there are diagonal braces to do this. The function of the **web** or **diagonal braces** is to resist the **shearing forces**, and we have good reason to know (*see* Art. 334) that the **booms** or **flanges** need be proportioned only to withstand the **bending moment.** In Fig. 100 we had a

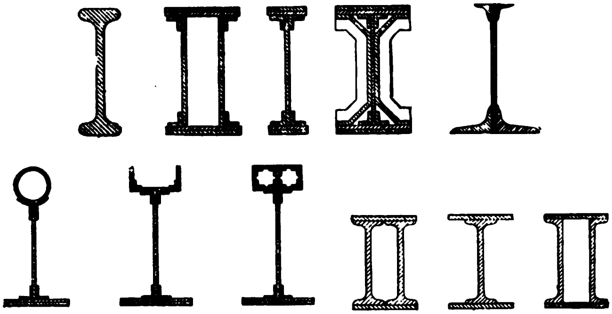


Fig. 221.

girder with few diagonal pieces. Now we already know that for absolute certainty in calculating the forces exerted by the pieces it was necessary to imagine pin joints not merely between the braces and the booms, but between one piece of boom and the next. Evidently this is nothing more than assuming that each piece can only exert a pulling or pushing force, and has no shear in a cross section. It will be found in Art. 369 that there is practically no shear in the cross section of the flanges of even a plate girder, and that the above easy practical rule ($f\text{A}d=\text{M}$) used so generally by engineers is quite legitimate.

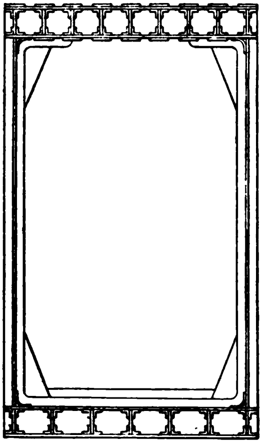


Fig. 222.

332. When the web is a plate it is hardly worth while to calculate whether it will resist the shearing force; we always find that the web which is thick enough (never less than $\frac{3}{8}$ inch) to let the girder be handled, and to let other girders be fastened to it and pieces to prevent buckling, is ever so much larger to resist shearing than is merely necessary for this purpose. Plate girders are used up to 75 and even 100-feet span for railway bridges. For great spans, as in the Menai tube of Fig. 222, this form is no longer thought to be economical.

In **open-work girders** it is necessary to make the calculation, and it is done in the following way. In built-up structures, if iron and timber are used, it is well to use timber for the struts (*see* Art. 372), because it is laterally large, and iron for the ties. In structures in which different materials are used, **expansion** is **different** in the different parts, and there ought to be no **redundant parts**. *Thus* if four pieces form a parallelogram and it has two *diagonal* pieces, of the six one is redundant. If all expand

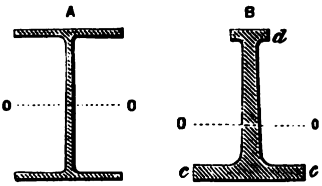


Fig. 223.

equally there is no great harm, but if one expands more than another great weakness may result.

Exercise.—A hinged square of four pieces of iron, each 6 feet long, has two diagonal pieces, one of brass and the other of iron. The iron bars are of equal cross section, and three times that of the brass one. There is no initial strain in them at 0° C. What are the forces in the pieces at 30° C.?

333. Redundant parts are often useful in stiffening a structure when the loads are apt to be very different at different times. When it is necessary to reject redundant parts we need no rules; common-sense will tell us, for example, that if the bars are long and slender we ought to reject the struts rather than the ties. When there are no redundant bars, the student follows the graphical method of Chap. VIII., or the corresponding analytical method which gives the same answer. When there are redundant bars he must make **certain assumptions**. Thus, if a table with fairly uniformly distributed load upon it has 20 legs, we usually assume that each leg has a twentieth of the load if they are all equal in length and the floor equally yielding everywhere. Probably this is wrong, and if we had sufficient information we might see reasons from theory or common-sense to suspect more load on some of the legs than on others. But we can do no better. If in Fig. 224 B C and D E are the booms in compression and tension, if B D is h inches, the vertical distance between the centres of area, and if the sectional area of each is A, then, as we have seen, f A h = the bending moment at the section B D. We are supposed to know S, the total shearing force at the section. This is the force with which the material to the left of B D acts upwards on the material to the right of B D. If the push or pull in a diagonal piece is P, the upward component of this is P sin. θ if θ is its inclination to the horizon, and the total shearing force is resisted by the vertical components in the diagonal pieces. Thus if all the diagonal bars are of iron, and if they are equally inclined

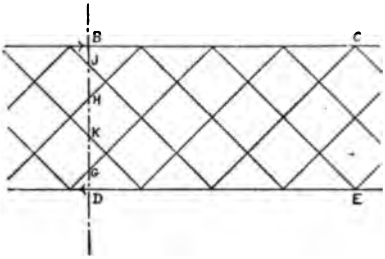


Fig. 224.

at 45° to the vertical, if the force in each of them is P, the upward component of each is P cos. 45° or $P/\sqrt{2}$; if there are n of them crossing the vertical plane B D then $n\ P/\sqrt{2}$ is the whole shearing force S. If S is known, P may be calculated. When a figure is drawn it is easy to see which piece exerts a push and which exerts a pull.

Exercise.—In Fig. 224 there are four diagonal pieces crossing the section at 45°. If the bending moment at B D is 3×10^8 pound-inches and B D is 5 feet, find the area of the boom at B or at D. If the shearing force at B D is 4×10^5 pounds find the probable push or pull in each of the four diagonal pieces and distinguish struts from ties. *Ans.*, $5 \times 10^6 \div f$; $7{\cdot}07 \times 10^4$.

If instead of Fig. 224 we have a section of a Warren girder with only one diagonal piece crossing the section at an angle of 60° with the horizontal, what is the push or pull in it? *Ans.*, $4{\cdot}62 \times 10^5$.

Exercise.—If only one diagonal piece at 45° meets a boom and a vertical piece, show that the force in the vertical piece is equal to the total shearing force in the girder section at the place.

As the student must by this time have drawn diagrams of shearing forces, he knows that when the load of a beam is uniformly distributed over it the shearing force is greatest at the left end, diminishes to zero at the middle, and increases positively to the right-hand end of the beam. The diagonal pieces are therefore large at the ends of a beam and small in the middle, and especially in large girders, for much of the load of any large girder must be distributed uniformly.

334. We see that when, as in **large girders**, we think greatly of economy and we know our loads and that they are vertical, we have flanges or booms whose sizes may be calculated with a fair amount of correctness. Even in smaller beams to carry vertical loads, it is convenient to look at what occurs at a section from this point of view; that **the flanges resist the bending moment and the web the shearing stress.** Taking, in Fig. 223 for example, h the vertical distance in inches from the middle of the top flange to the middle of the bottom; A_c the area of the top flange, A_t the area of the bottom one, in wrought iron we make $A_c = A_t$, because f_c is much the same as f_t, but in cast iron f_c is $4\frac{1}{2}$ times f_t, and hence we make $4\frac{1}{2}\ A_c = A_t$; or the *bottom* flange, which is in tension, $4\frac{1}{2}$ times the area of the top one. The bending moment is f_c A_c or f_t A_t multiplied by h.

Exercise.—The top and bottom flanges of a rolled section of wrought iron are $8'' \times \frac{5}{8}''$. The web is of same thickness. The height over all is $12''$. What is the bending moment when the greatest tensile stress is 10,000 lbs. per square inch?

Work this in two ways. I. The tensile or compressive force in each flange is $8 \times \frac{5}{8} \times 10{,}000 = 50{,}000$ lbs. The value of h is $12'' - \frac{5}{8}''$ or $11\frac{3}{8}$. Hence the bending moment is $11\frac{3}{8} \times 50{,}000$, or 569,000 pound-inches. II. The moment of inertia of the section about its neutral axis is $\frac{8 \times 12^3}{12} - \frac{7\frac{3}{8} \times (10\frac{3}{4})^3}{12} = 388.$ This divided by 6 is z, the strength modulus, and 10,000 z, or 647,000, is the bending moment.

335. The above example gives a fairly good idea of the error in adapting **the usual practical rule** for a railway girder where there is almost no web to a rolled girder where there is a web. The web is here distinctly of importance in resisting bending moment. Some engineers, instead of taking h from centre to centre of flange, take it the height over all; but even if we take $h = 12''$ in the above we still have an answer which is about 7 per cent. too small. The error is on the safe side.

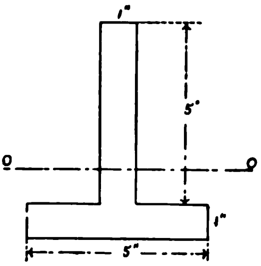
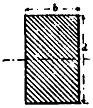
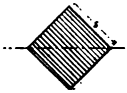
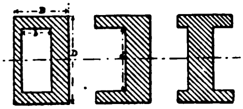
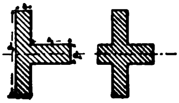


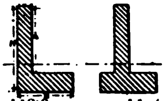
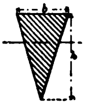
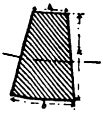
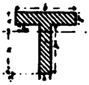
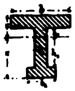
Fig. 225.

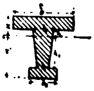
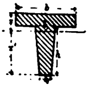
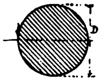
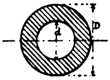
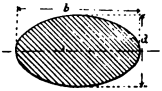
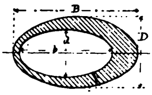
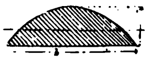
Nevertheless, it is always better to keep to **the correct rule** of Art. 320 in girders which have an important web, and in all mechanical engineering calculations we keep to the correct rule. I therefore give here a list of the values of I and of Z for various forms of section.

Exercise.—Show that the centre of gravity O of the area in Fig. 225 is 2 inches above the bottom. Take it as x inches. The middle of the bottom rectangle is $x - \frac{1}{2}$ from O and the middle of the top one is $3\frac{1}{2} - x$ from O. Hence $(x - \frac{1}{2})\ 5 = (3\frac{1}{2} - x)\ 5$ or $x = 2$ inches. The moment of inertia of the top rectangle about axis O O is $\frac{5 \times 1^3}{12} + 5\,(1\frac{1}{2})^2 = 21\frac{2}{3}$. The moment of inertia of the bottom rectangle about axis O O is $(1\frac{1}{2})^2 = 11\frac{2}{3}$; and the sum is $I = 33\frac{1}{3}$. Z_1 for the top is Z_2 for the bottom is $I \div 2 = 16\frac{2}{3}$.

TABLE VI.

I Moment of Inertia of Section.	$Z = I/y$ Strength Modulus o
 $\frac{bd^3}{12}$	$\frac{1}{6}bd^2$
 $\frac{s^4}{12}$	$\frac{\sqrt{2}\,s^3}{12}$
 $\frac{BD^3 - bd^3}{12}$	$\frac{BD^3 - bd^3}{6D}$
 $\frac{b_1d_1^3 + b_2d_2^3}{12}$	$\frac{b_1d_1^3 + b_2d_2^3}{6d_1}$

I **Moment of Inertia of Section.**	$z = I/y$. **Strength Modulus of Section.**
 $\frac{(BH^2 - bh^2)^2 - 4BHbh(H-h)^2}{12(BH - bh)}$	$\frac{(BH^2 - bh^2)^2 - 4BHbh(H-h)^2}{6(BH^2 + bh^2 - 2bHh)}$
 $\frac{bh^3}{36}$	$z' = \frac{bh^2}{24}$ $z'' = \frac{bh^2}{12}$
 $\frac{b^2 + 4bb_1 + b_1^2}{36(b + b_1)}h^3$	$z' = \frac{b_1 + 4bb_1 + b_1^2}{12(2b + b_1)}h$ $z'' = \frac{b + 4bb_1 + b_1^2}{12(b + 2b_1)}h$
 $\left[\frac{1}{3}b(x^3 - h_3^3) + b_1\left(h_3^3 + x^{13}\right)\right]$	$z' = \frac{I}{x^1}$ $z'' = \frac{I}{x}$
 $\frac{1}{3}\left[\frac{b_1 - b_2}{4(h_1 + x^1)}\left(x^{1^4} - h_1^4\right) + b(x^3 - h_1^3) + h_2\left(h_1^3 + x^{1^3}\right)\right]$	$z' = \frac{I}{x^1}$ $z'' = \frac{I}{x}$

I Moment of Inertia of Section.	$z = I/y$ Strength Modulus of Sec
$\frac{1}{3}\left[b\,(x^3 - h_1^3) + b_1\,(h_1^3 + h_2^3) + b_2\left(x^{1^3} - h_2^{\ 3}\right)\right]$ 	$z' = \frac{I}{x^1}$ $z'' = \frac{I}{x}$
$\frac{1}{3}\left[\frac{b_1 - b_2}{4\,(h_1 + h_2)}\,(h_2^4 - h_1^4) + b\,(x^3 - h_1^3) + b_2\,(h_1^3 + h_2^3) + b_3\left(x^{1^3} - h_2^{\ 3}\right)\right]$ 	$z' = \frac{I}{x^1}$ $z'' = \frac{I}{x}$
 $\frac{\pi}{64}\,D_4$	$\frac{\pi}{32}\,D^3$
 $\frac{\pi}{64}\,(D^4 - d^4)$	$\frac{\pi}{32} \cdot \frac{D^4 - d^4}{D}$
 $\frac{\pi}{64}\,bd^3$	$\frac{\pi}{32}\,bd^2$
 $\frac{\pi}{64}\,(B\,D^3 - bd^3)$	$\frac{\pi}{32}\left(\frac{B\,D^3 - bd^3}{D}\right)$
(Parabolic segment)  $\frac{8}{175}\,bh^3$	$z' = \frac{4}{35}\,bh^2$ $z'' = \frac{8}{105}\,bh^2$

Exercise.—Find the greatest load that may be uniformly distributed on a cast-iron girder having top and bottom flanges united by a web, of the following dimensions:—Width of upper flange, 3 inches; of lower flange, 9 inches; total depth, 12 inches; thickness of each flange and of the web being 1 inch; distance between the points of supports, 10 feet; when the greatest admissible stress in the compression flange is 6 tons per square inch, and that in the tension flange $1\frac{1}{2}$ tons per square inch.

Ans., 9·8 tons.

336. Proportion of Depth to Length in Railway Girders.—It is usually assumed that maximum economy of weight in booms and diagonal pieces leads to a most economical ratio of depth d to length l, but we must confess that we feel dissatisfied with the easy mathematical statements sometimes deduced on this subject from incomplete data. Take it that in girders of the same style the diagonal pieces make some known angle with the horizontal. Let us take 45°, for example. Then each bar of length $d\sqrt{2}$ and cross-section a withstands a shearing force $S = af \times \sqrt{2}$, where f is the shear stress, and has a weight $d\sqrt{2}\,.\,a\,(\cdot 28)$, or the weight is $d\sqrt{2}\,(\cdot 28)\,\frac{S}{f\sqrt{2}}$. The weight of the corresponding pieces of boom is $2dA\,(\cdot 28)$, where $Afd = M$, where f is also the tensile stress. Total weight of a bay is therefore

$$2d\,(\cdot 28)\,\frac{M}{fd} + \left(\frac{\cdot 28}{f}\right)2dS \;.\;.\;.\;.\; (1),$$

or the weight in pounds per inch run of the girder is

$$\frac{\cdot 56}{f}\left(\frac{M}{d} + S\right) \;.\;.\;.\;.\; (2).$$

We see, therefore, that if the inclination of the diagonal pieces is fixed, the greater d is the better; and there is no most economical depth of a girder derivable—at all events, from these simple considerations. It is true, however, that making the depth great, if the inclination is constant, means that we are increasing the length of the unsupported part of the compression boom and struts, and they may need more lateral bracing; or, as it is better to put it, the cost of the compression members per pound will increase. Again, the weight of the platform will also increase. These are, however, questions of a different kind, difficult to settle by elementary mathematical equations when systems are so different. Writing down such general expressions as we may, there is evidence that there is greater economy in weight, whatever there may or may not be in cost, in letting the depth get less where the bending moment is less, instead of keeping it constant.

The most important matter, how the natural period of vibration of a bridge ought to come in, seems never to be brought forward in these calculations. Professor Milne finds that the horizontal transverse deflection is the most serious motion of a railway bridge. It begins when the train is perhaps 200 feet or more away; it becomes accentuated with every passing carriage, and when the whole train has passed there is the natural vibration of the bridge, which continues for some time. These vibrations are due to lurching of carriages and impact of wheel-flanges against the rails. Sometimes a light waggon seems to produce much larger effects than the heavy locomotive, and there is some speed of train with which the vibration is much more serious than with quicker or slower speeds. Bridges have not yet been studied from this point of view, and engineers must for the present rely upon their large factors of safety. We are sure that more attention ought to be paid to these lateral vibrations, which seem to be greatly accentuated when gusts of wind are acting laterally.

The Board of Trade rule is 5 tons per square inch on wrought iron and $6\frac{1}{2}$ on steel. This is not sufficiently safe. It is well to say 5 in tension on iron, 4 in compression, and steel, say, $\frac{1}{3}$ greater. In some large bridges it is estimated that stresses due to wind are greater than those due to rolling loads.

The force acting on the rails in the direction of their length is sometimes as much as $\frac{1}{7}$th of the weight of a train if the train is quickly stopped.

As for the vertical motion, its consideration will probably lead to some rule connecting maximum deflection y_1 under static load, and length of girder l. There is a vague sort of understanding that y_1 shall be something between $\frac{1}{800}$th and $\frac{1}{1200}$th of l. If beams are of uniform strength and depth, d, the curvature is constant, being $2f/\mathrm{E}d$ (*see* Art. 362), f being the greatest stress in every section; and hence in girders supported at the ends the deflection y_1 is equal to $l^2f/4d\mathrm{E}$. If, now, we take y_1 to be $l/1{,}200$, $\frac{l}{1{,}200} = \frac{l^2f}{4d\mathrm{E}}$ or $\frac{l}{d} = \frac{\mathrm{E}}{300f}$. Taking E/f as 3,000, $\frac{l}{d} = 10$. . . . . (3).

It is not quite fair to say that the calculation of the probable loading of railway bridges is more scientific in America than in Britain. British engineers differ much more in their assumed loads than American, but in neither case can it be said that there is a scientific basis for the rules in use. It is very important that the student should know this, because there is sometimes a pretence of accuracy of treatment in bridge calculations which is quite misleading. The following rules are more common than others, and may well be used in academic problems.

In ordinary railway bridges the greatest possible **rolling load** may be taken as if it were a static load of w_1 tons per foot-run for a double line, where $w_1 = 3\frac{1}{2} + 176/l$, if l is the span in feet. This is really on some such assumption as that a rolling load must be multiplied by $1\frac{3}{4}$ to convert it into the equivalent static load. The diminution with length is due to the fact that the engine weight is more intense per foot than other parts of the train. The **weight of the platform** may be taken as $w_2 = 0{\cdot}7 + {\cdot}0072\,l$ tons per foot of the span for a double line. w_2 increases with the span because of greater wind-bracing and the greater width of larger spans necessary for lateral stability.

Half of every term may be taken for a single line, and used even as low as for 15-feet spans. A railway girder is usually built with so much negative deflection or "camber," as it is called, that it will become just about level when loaded.

From (2) of Art. 336, taking M as proportional to wl^2, if w is the total load per foot run, and S as proportional to wl, the whole weight, w_3, of the girders per foot run is equal to $\left(a\frac{wl}{d} + bw\right)l$, where a and b are two constants. If we take it, as is usual, that l/d is nearly constant (say 10, as above mentioned), this becomes $w_3 = lw/c$, where c is some constant. Now,

$$w = w_1 + w_2 + w_3 = \frac{cw_3}{l}.$$

$$\text{Hence } w_3 = \frac{w_1 + w_2}{\frac{c}{l} - 1} \text{ or } \frac{l}{c - l}(w_1 + w_2) \ldots . (4),$$

$$\text{and } w = (w_1 + w_2)/\left(1 - \frac{l}{c}\right).$$

Whatever may be the worth of this reasoning, it gives us a rule which is taken to hold in all railway girders. The value of c is about 1,000 feet for plate web girders, 1,200 for lattice girders of ordinary construction, and 1,400 feet for bow-string and well-designed lattice girders.

337. Small pistons for cylinders up to 20 inches diameter are packed so as to be steam-tight in the following way. The method is quite wrong. Suppose the cylinder is to be 15 inches diameter; there are two or more grooves turned in the piston block, about ½ inch or more broad and ½ inch deep, to receive cast-iron rings. Many such rings may be made at the same time. A hollow cylinder of cast iron is turned up about 15½ inches outside and 14½ inches inside, and many rings are cut from it. Each ring has a piece cut out, so that when it is sprung into place in the piston groove it may be jammed into the cylinder, its ends coming now close together. The cylinder keeps it smaller than its unstrained size, and the pressure all round is assumed to be uniform.

Exercise for Students.—Prove that the pressure is not uniform all round.*

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* To show that the pressure cannot be uniform all round in the usual method of manufacture. The ring unloaded is of a circular shape; loaded, it is circular and of smaller radius. The bending moment must therefore be constant. Let us try what distribution of pressure p will produce a constant bending moment; that is, using the above symbols, if M_o is the bending moment at the end where $\theta = 0$, $M_o + r^2 \int_o^\theta p \sin. \phi \,.\, d\phi = M$, a constant. Differentiating with regard to θ, we find $p \sin. \theta = 0$; and as this is true for all values of θ, we must have $p = 0$ everywhere. That is, the shape can only be maintained by couples applied at the two ends. It is quite a usual thing to see workmen beating such a ring out of shape near the joint, after it has been manufactured, because it is known that, so far from pressing uniformly all round, it does not even fit the cylinder, but concentrates its pressure at certain points.

The following method of making a piston ring produces uniform pressure all round, even if the section of the ring varies very greatly. A ring is cast which is larger than what is required; a piece is cut from it, and the two free ends are brought together by a clamp. This clamp pulls on both ends, and the more nearly the resultant pull is tangential to the mean cylindric surface at the end the more nearly perfect will the ring be. The ring is now turned up to the size of the cylinder and finished. It is unclamped, and may be sprung into place.

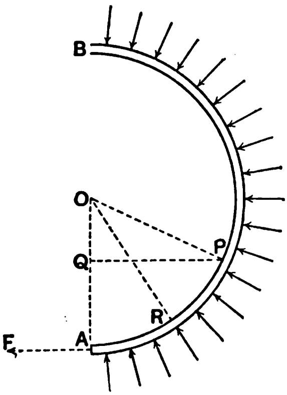


Fig. 226.

To prove that the pressure must be uniform all round. In Fig. 226 let B P A be part of a piston ring constrained to be of the size of the cylinder, of radius r. It may either be kept in its present shape by the equal and opposite forces F at its ends, or by pressures p lbs. per inch of its length. Let us suppose at first that p may not be the same all round, being some function of the angle $AOP = \theta$. Draw PQ perpendicular to OA. F without p produced bending moments at all the sections of the ring; the pressures p without F must produce the same bending moments. Let $AOP = \theta$. Let $AOR = \phi$, where P and R are any two places on the ring, and A is one end. Bending moment at P due to F is $M = F \cdot AQ$, or

$$M = Fr\,(1 - \cos.\theta) \; \ldots \; (1).$$

The pressure at R on the element $r \,.\, \delta\phi$ is $pr\delta\phi$, and the bending moment due to this at P is $pr^2 \,.\, \delta\phi \,.\, \sin\,(\theta - \phi)$, and we require

$$\int_0^\theta pr^2 \sin.(\theta - \phi) \,.\, d\phi = Fr\,(1 - \cos.\theta) \; \ldots \; (2).$$

In the integral p is a function of ϕ. Now we know that in genera

$$\int_0^a f(x) \,.\, dx = \int_0^a f(a - x) \,.\, dx,$$

and hence equation (2) is

$$r^2 \int_0^\theta p \sin.\phi \,.\, d\phi = Fr\,(1 - \cos.\theta) \; \ldots \; (3).$$

Differentiating with regard to θ, we have $r^2 p \sin.\theta = Fr \sin.\theta$, or

$$p = F/r, \text{ a constant} \; \ldots \; (4).$$

It will be observed that the pressure p is uniform all the way round, even if the ring varies greatly in section. It is quite true that the proof assumes the thickness to be everywhere inconsiderable. If, however, we assume a uniform thickness, it is easy to see that if the resultant force F exercised by the clamp acts on each end exactly at the mean radius, there is absolutely a constant pressure per inch all the way round. In Japan, twenty years ago, Professor R. H. Smith told me that the above proposition was true. I worked out the proof very easily. I do not think that it has been published before.

The usual method of making the rings is very much more mischievous than it may appear to be on a hurried examination. It seems to me that if the correct method is adopted, care being taken as to the proper method of applying the clamp, piston rings may be made in this way for the very largest cylinders, and it is evident that there must be a very great reduction in the cost of large pistons in consequence.

338. **Curvature.**—The curvature of a circle is the reciprocal of its radius; and of any curve, it is the curvature of the circle which best agrees with the curve. The curvature of a curve is better given as "the angular change (in radians) of the direction of the curve per unit length." Now draw a very flat curve, with very little slope i. Observe that the change in i or $\frac{dy}{dx}$ in going from a point P to a point Q is almost exactly a change of angle [change in $\frac{dy}{dx}$ is really a change in i, the tangent of an angle; but when an angle is very small, the angle, its sine and its tangent, are all equal]. Hence, the increase in $\frac{dy}{dx}$ from P to Q, divided by the length of the curve PQ, is the average curvature from P to Q; and as PQ is less and less, we get more and more nearly the curvature at P. But the curve being very flat, the length of the arc PQ is really δx, and the change in $\frac{dy}{dx}$ divided by δx, as δx gets less and less, is the rate of change of $\frac{dy}{dx}$ with regard to x, and the symbol for this is $\frac{d^2y}{dx^2}$. Hence we may take $\frac{d^2y}{dx^2}$ as the curvature of a curve at any place, when it is everywhere nearly horizontal.

If the beam was not straight originally, and if y' was its small deflection from straightness at any point, then $\frac{d^2y'}{dx^2}$ was its original curvature. We may generalise the following work for beams not straight to begin with by using $\frac{d^2}{dx^2}(y-y')$ instead of $\frac{d^2y}{dx^2}$ everywhere.

It is easy to show that a beam of uniform strength—that is, a beam in which the maximum stress f (if compressive, positive ; if tensile, negative) in every section is the same—has the same curvature everywhere, if its depth is constant.

If d is the depth, the condition for constant strength is that $\frac{M}{I} \cdot \frac{1}{2}d = \pm f$, a constant. But $\frac{M}{I} = E \times$ curvature; hence curvature $= \frac{2f}{E \cdot d}$.

Example.—In a beam of constant strength, if $d = \frac{1}{a + bx}$, then $\frac{d^2y}{dx^2} = \frac{2f}{E}(a + bx)$. Integrating, we find $\frac{E}{2f} \cdot \frac{dy}{dx} = c + ax + \frac{1}{2}bx^2$, and, again, $\frac{E}{2f} \cdot y = e + cx + \frac{1}{2}ax^2 + \frac{1}{6} \cdot bx^3$, where e and c must be determined by some given conditions. Thus, if the beam is fixed at the end, where $x = 0$, and $\frac{dy}{dx} = 0$ there, and also $y = 0$ there, then $c = 0$ and $e = 0$.

339. In a beam originally straight, we know now that if x is distance measured from any place along the beam to a section, and if y is the deflection of the beam at the section and I is the moment of inertia of the section, then

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \dots (1),$$

where M is the bending moment at the section, and E is Young's modulus for the material.

We give to $\frac{d^2y}{dx^2}$ the sign which will make it positive if M is positive. If M would make a beam convex upwards and y is measured downwards, then (1) is correct. Again, (1) would be right if M would make a beam concave upwards and y is measured upwards.

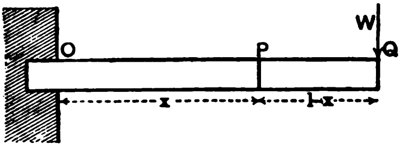


Fig. 227.

Example 1.—**Uniform beam of length l fixed at one end** loaded with weight W at the other. Let x be the distance of a section from the fixed end of the beam. Then $M = W(l-x)$; so that (1) becomes

$$\frac{EI}{W}\frac{d^2y}{dx^2} = l - x \dots (2).$$

Integrating, we have, as E and I are constants,

$$\frac{\text{EI}}{\text{W}}\frac{dy}{dx} = lx - \tfrac{1}{2}x^2 + c.$$

From this we can calculate the slope everywhere.

To find c we must know the slope at some one place. Now, we know that there is no slope at the fixed end, and hence $\frac{dy}{dx} = 0$, where $x = 0$; hence $c = 0$. Integrating again,

$$\frac{\text{EI}}{\text{W}} y = \tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3 + \text{c}.$$

To find c, we know that $y = 0$ when $x = 0$, and hence $\text{c} = 0$; so that we have for the shape of the beam—that is, the equation giving us y, the deflection for any point of the beam—

$$y = \frac{\text{W}}{\text{EI}}\left(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3\right) \ldots . (3).$$

We usually want to know y when $x = l$, and this value of y is called D, the maximum deflection of the beam; so that

$$\text{D} = \frac{\text{W}l^3}{3\,\text{EI}} \ldots . (4).$$

Example 2.—**A beam of length l loaded with W at the middle and supported at the ends.** Observe that if half of this beam in its loaded condition has a casting of cement made round it so that it is rigidly held, the other half is simply a beam of length $\frac{1}{2}l$, fixed at one end and loaded at the other with $\frac{1}{2}$ W, and, according to the last example, its maximum deflection is

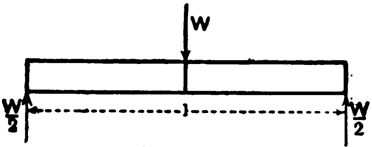


Fig. 228.

$$\text{D} = \frac{\frac{1}{2}\text{W}\left(\frac{1}{2}l\right)^3}{3\,\text{EI}} \text{ or } \frac{\text{W}l^3}{48\,.\,\text{EI}} \ldots . (5).$$

The student ought to make a sketch to illustrate this method of solving the problem.

Example 3.—**Beam fixed at one end** with load w per unit length spread over it. The load on the part PQ is $w \times$ PQ or $w\,(l - x)$. The resultant of the load acts at midway between P and Q, so, multiplying by $\frac{1}{2}(l - x)$, we find M at P, or

$$\text{M} = \tfrac{1}{2}w\,(l - x)^2 \ldots . (6).$$

Using this in (1), we have $\frac{2\,\text{EI}}{w}\frac{d^2y}{dx^2} = l^2 - 2lx + x^2$. Integrating, we have $\frac{2\,\text{EI}}{w}\frac{dy}{dx} = l^2x - lx^2 + \frac{1}{3}x^3 + c$. This gives us the slope everywhere. Now $\frac{dy}{dx} = 0$ where $x = 0$, because the beam i there. Hence $c = 0$. Again integrating,

$$\frac{2\text{EI}}{w} y = \tfrac{1}{2} l^2 x^2 - \tfrac{1}{3} l x^3 + \tfrac{1}{12} x^4 + \text{C};$$

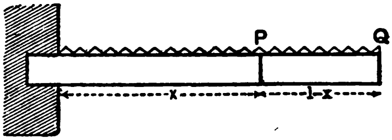


Fig. 229.

and as $y = 0$ where $x = 0$, $\text{C} = 0$, and hence the shape beam is

$$y = \frac{w}{24\,\text{EI}} (6 l^2 x^2 - 4 l x^3 + x^4) \;\ldots\ldots\; (7).$$

y is greatest at the end where $x = l$, so that the ma deflection is

$$\text{D} = \frac{w}{24\,\text{EI}} 3 l^4, \text{ or } \text{D} = \frac{1}{8} \frac{\text{W} l^3}{\text{EI}} \;\ldots\ldots\; (8)$$

if $\text{W} = wl$, the whole load on the beam.

Example 4.—Beam of length l **loaded uniformly** with unit length, **supported at the ends.** Each of the sup forces is half the total load. The moment about P of $\frac{1}{2} wl$, distance PQ, is against the hands of a watch, and I ca direction negative: the moment of the load $w\,(\frac{1}{2} l - x)$ average distance $\frac{1}{2}$ PQ is therefore positive, and hence the b moment at P is

$$- \left\{ \tfrac{1}{2} wl\,(\tfrac{1}{2} l - x) - \tfrac{1}{2} w\,(\tfrac{1}{2} l - x)^2 \right\}, \text{ or } - \left\{ \tfrac{1}{8} wl^2 - \tfrac{1}{2} wx^2 \right\} \ldots$$

so that, from (1), $\text{EI} \frac{d^2 y}{dx^2} = - \left\{ \frac{1}{8} wl^2 - \frac{1}{2} wx^2 \right\}$. Integrati

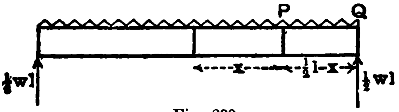


Fig. 230.

have $\text{EI} \frac{dy}{dx} = - \frac{1}{8} wl^2 x + \frac{1}{6} wx^3 + c$, a formula which enables find the slope everywhere c is determined by our kno that $\frac{dy}{dx} = 0$ where $x = 0$, and hence $c = 0$. Integrating

$\mathrm{EI}y = \frac{1}{16}wl^2x^2 + \frac{1}{24}wx^4 + \mathrm{c}$, and $\mathrm{c} = 0$, because $y = 0$ where $x = 0$. Hence the shape of the beam is

$$y = \frac{w}{48\,\mathrm{EI}}(3l^2x^2 - 2x^4) \;\ldots\ldots\; (10).$$

y is greatest where $x = \frac{1}{2}l$, and is what is usually called the maximum deflection D of the beam, or $\mathrm{D} = \frac{5\,\mathrm{W}\,l^3}{384\,\mathrm{EI}}$ if $\mathrm{W} = lw$, the total load.

Example 5.—In any beam, whether supported at the ends or not, if w is constant, integrating (4) of Art. 357, we find

$$\frac{d\mathrm{M}}{dx} = b + wx, \text{ and } \mathrm{M} = a + bx + \tfrac{1}{2}wx^2 \;\ldots\ldots\; (5).$$

In any problem we have data to determine a and b. Take the case of a uniform beam uniformly loaded, and merely supported at the ends. Measure y upwards from the middle, and x from the middle. Then $\mathrm{M} = 0$ where $x = \frac{1}{2}l$ and $-\frac{1}{2}l$, $0 = a + \frac{1}{2}bl + \frac{1}{8}wl^2$, and $0 = a - \frac{1}{2}bl + \frac{1}{8}wl^2$. Hence $b = 0$, $a = -\frac{1}{8}wl^2$, and (5) becomes

$$\mathrm{M} = -\tfrac{1}{8}wl^2 + \tfrac{1}{2}wx^2 \;\ldots\ldots\; (6),$$

which is exactly what we used in Example 4, where we afterwards divided M by EI, and integrated twice to find y.

With regard to the following important practical problem :—When the sizes of an angle iron are given to find the least radius of gyration of its section about a line through its centre of gravity, I give students a number of angle irons every year; for each of them they find the least radius of gyration. From the tabulation of these results I hope to get an empirical formula. I had hoped to be able to publish this now, but I have not yet obtained a sufficient number of results.

CHAPTER XVIII.

SOME WELL-KNOWN RULES ABOUT BEAMS.

340. WHEN beams have the same section everywhere we look for the place where the bending moment is greatest, as that is the place where fracture tends most to take place, and we find the cross section to withstand this greatest bending moment. We shall now consider such a **uniform beam loaded and supported in various ways.**

Thus, the load may be hung from one end of the beam, the other end being rigidly fixed, say by being built into a wall. When we say that the end of a beam is fixed, we mean that it is rigidly held in position, whereas when we say that a beam is supported at its ends, we mean that it is merely held up there. In Table VIII. six ways are shown in which the same length of beam is supposed to be loaded. The total load is supposed to be the same in every case, and the length from A to B is supposed to be the same. Then, we see that when the beam is fixed at both ends, and the load spread over it, it is twelve times as strong as when one end is fixed, and the whole load hung from the other end. This means that if, with the beam fixed at one end, a load of one ton, hung at the other end, breaks the beam, then, when fixed at both ends, and the load spread uniformly over it, the same sized beam will carry 12 tons. Hence, if experiments are made on the strength of the beam when loaded in any of these ways we know what its strength ought to be when loaded in any of the other ways. Now a great many experiments have been made upon beams of rectangular section supported at both ends and loaded in the middle, the third case given in the Table; and from these experiments we know how to find the load which such a beam will carry. Having found this, we know that when loaded and supported in a different way the beam will carry more or less according to the numbers the column headed "Strength."

341. If M is the bending moment and Z the strength modulus of the section, and f the stress which the material will stand, $M = Zf$ . . . . (1).

Let us take as an example beams of rectangular *section,* breadth b, depth d; the strength modulus is $b\,d^2/6$, so that $M = f\,b\,d^2/6$ .... (2).

Now our theory is based on the idea of perfect elasticity; we cannot, therefore, assume that M is the bending moment which will break a beam if f is the ultimate tensile or compressive stress, because our theory cannot hold beyond the elastic limit, but we find by experiment that the breaking, bending moment is proportional to $b\,d^2$. Thus if rectangular beams of the same material, but of different lengths (l feet), breadths (b inches), and depth (d inches) are supported at the ends and loaded in the middle, we find that the breaking load W lb. follows with fair accuracy the rule—$W = c\,b\,d^2/l$ (3) where c is given in the following table and stands for 1/18 of our old f. It is well to try to remember that doubling the breadth of a beam doubles its strength, but doubling its depth gives four times the strength.

TABLE VII.

BEAMS SUPPORTED AT THE ENDS AND LOADED IN THE MIDDLE.

Nature of Material.	c to Calculate Strength.	e to Calculate Deflection.
Teak	820	·00018
Oak	450 to 600	·00044 to ·00020
English Oak . . .	557	·0003
Ash.	675	·00026
Beech	518	·00031
Pitch Pine . . .	544	·00035
Red Pine . . .	450	·00023
Fir.	370	·0005 to ·0002
Larch	284	·00041
Deal	600	·00023
Elm	337	·00061
Cast Iron . . .	2,540	·000024
Wrought Iron . .	3,470	·000016
Hammered Steel . .	6,400	·000013
Marble	150	...
Good Sandstone . .	50 to 80	...

The numbers given in this table are merely the average values found by various experimenters. You may wish, however, to find for yourself whether they are correct or not. You are designing a beam of pitch-pine, say; then take a rod of pitch-pine, 1 foot long, 1 inch broad, 1 inch deep; support it at the ends, and load in the middle till it breaks: the Table says that the load will be 544 lbs., but you may find it to be more or less than this. Remember also that it is near the middle that your beam is likely to break; this, then, ought to be the soundest and most evenly grained part of the timber if possible and the specimens which you try ought to be as nearly as possible the same kind of timber.

If your beam is loaded or supported in any of the other five ways described in Table VIII., you will multiply the breaking load which you have found by the number called strength in Table VIII. The reason is obvious.

342. In certain standard cases we like to state algebraically the amount of bending moment at any section of a beam. We shall do this in the six standard cases, so well known to all carpenters, shown in pages 413 to 415.

I. Beam of length l inches fixed at one end A, loaded only with W at the other end B. At a section x inches from B the bending moment is evidently $M = W x$. The shearing force is $S = - W$. The diagram, Case I., shows M. Notice that M is greatest at A, and is then $W l$.

II. Beam fixed at A (Case II.,), load w lb. per inch of its length or total load W spread uniformly. Note that the load on P B, if P is a section x inches from B, would be $w x$, and the resultant of this acts at the distance $\frac{1}{2} x$ from the section, so that its moment is $\frac{1}{2} x \times w x$ or $\frac{1}{2} w x^2$. It is greatest at A, being there $\frac{1}{2} w l^2$. S the shearing force at P is $- w x$. Note that $w l = W$, so that the bending moment anywhere is $\frac{1}{2} \frac{W}{l} x^2$ and i $\frac{1}{2} W l$ at the end A. The shearing force at the end A i numerically greater than anywhere else, being $- W$.

III. Beam of length l, load W in middle supported at th ends. The supporting forces are each $\frac{1}{2}$ W. At a section x inche from either end th bending moment i $- \frac{1}{2} W x$, being − $W l$ at the middle and the shearing force S is $- \frac{1}{2}$ W from A to the middle, when it suddenly changes and becomes $+ \frac{1}{2}$ W from the middle to B.

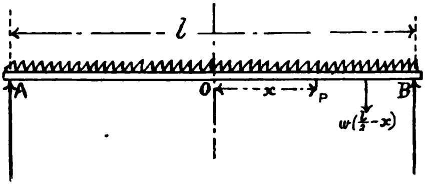


Fig. 231.

IV. Beam of length l supported at the ends, load w lb. per *inch of* its length or total load W spread uniformly (se

Fig. 231). The supporting forces at A and B are each $\frac{1}{2}$ W or $\frac{1}{2} w\, l$. At P if O P $= x$, the bending moment is the moment of the supporting force about P minus the moment of the load on P B, or, since O B $= \frac{1}{2} l$ and P B $= \frac{1}{2} l - x$,

$$M = - \left\{\tfrac{1}{2} w\, l \left(\tfrac{1}{2} l - x\right) - w \left(\tfrac{1}{2} l - x\right) \tfrac{1}{2} \left(\tfrac{1}{2} l - x\right)\right\},$$

and this simplifies to $M = - \frac{1}{8} w \left(l^2 - 4 x^2\right)$. It is numerically greatest at the middle where x is o, being there $- \frac{1}{8} w\, l^2$ or $-\frac{1}{4}$ W l. The shearing force at P is $\frac{1}{2} w\, l - w \left(\frac{1}{2} l - x\right)$, or $w\, x$.

The student ought himself to draw all the diagrams of M (bending movement) and S (shearing force).

The diagrams of M for cases V. and VI. are shown in the figures. (Case VI. will be worked out in Art. 360). They are simply the diagrams of III. and IV. with the average value of M subtracted from every ordinate—that is, the whole diagrams are lowered by this amount. The diagrams of S are exactly the same whether a beam is merely supported at the ends or is fixed, if the loading is symmetrical (*see* Art. 362)—that is, the fixing does not alter the actual supporting forces at the ends.

We have, in fact, the following rule for finding the bending moment diagram for a uniform beam symmetrically loaded, fixed at the ends. Find the diagram of bending moment as if the beam were merely supported at the ends: raise it by a distance equal to its average height. We now have the diagram of the bending moment when the ends of the beam are fixed.

The shearing force diagram is not altered by fixing the ends.

If for any **two kinds of loading** we have the diagrams of M and S, then for the two kinds of loading applied at the same time we simply add algebraically the ordinates of the separate diagrams.

TABLE VIII.

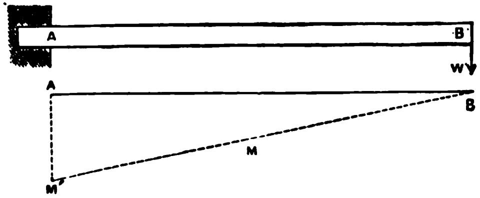
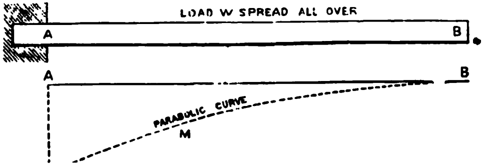
Nature of Support and Loading.	Sketches of Beams and Diagrams of Bending Moment.	Strength.	Deflection.
Beam fixed at one end. Length A B = l. Load W at the free end. Greatest bending moment is at A, and is W multiplied by length of beam A B. Shearing force is – W everywhere.	CASE I. 	$\frac{1}{4}$	16
Beam fixed at one end. Length A B = l. Load W spread all over. Greatest bending moment is at A, and is W multiplied by half length of beam. Shearing force is – W	CASE II.  LOAD W SPREAD ALL OVER A B A B PARABOLIC CURVE M	$\frac{1}{2}$	6

TABLE VIII. *(continued.)*

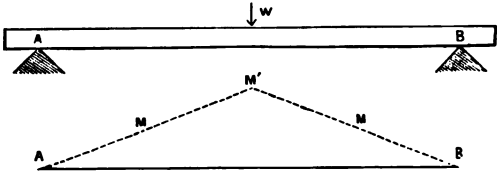
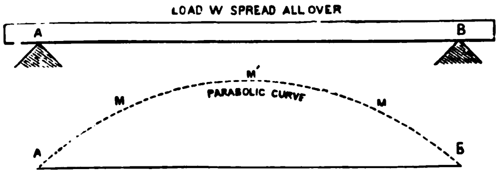
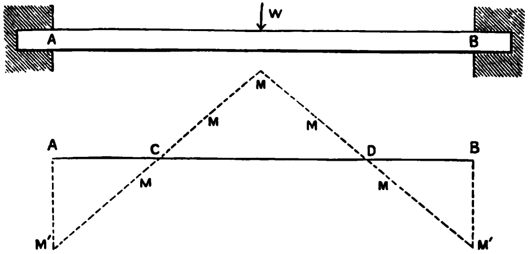
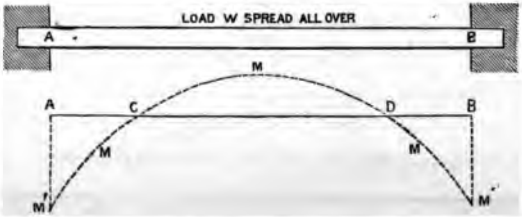
Nature of Support and Loading.	Sketches of Beams and Diagrams of Bending Moment.	Strength.	Deflection.
Beam supported at both ends. Length l = A B. Load W in middle. Greatest bending moment occurs in the middle, and is $-\frac{1}{4}Wl$. Shearing force is $\frac{1}{2}$ W from B to middle, $-\frac{1}{2}$ W from A to middle.	CASE III. 	1	1
Beam supported at both ends. Length l = A B. Load W spread all over. Greatest bending moment occurs in the middle, and is $-\frac{1}{8}Wl$. Shearing force is $\frac{1}{2}$ W at B, $-\frac{1}{2}$ W at A, 0 at middle.	CASE IV.  LOAD W SPREAD ALL OVER A B M′ PARABOLIC CURVE M M A B	·2	·625

TABLE VIII. (*continued.*

Beam fixed at both ends. Uniform length l A B. Load W at middle. Greatest bending moment occurs both at middle and at the ends, and is $-\frac{1}{8}$W at middle, and $+\frac{1}{8}$W l at the ends. There is no bending moment at the points C and D, and these are called points of inflection. Shearing force is $\frac{1}{2}$W from B to middle, $-\frac{1}{2}$W from A to middle.	CASE V.  W A B A C D B M M M M M M′ M′	2	·25
Beam *fixed* at both ends. Uniform length l = A B. Load W spread all over. Greatest bending moment occurs at the *ends*, and is $+\frac{1}{12}$W l. There is no bending moment at the points C and D, and these are called points of inflection. Shearing force is $\frac{1}{2}$W at B, $-\frac{1}{2}$W at A, O at middle.	CASE VI.  LOAD W SPREAD ALL OVER A B M A C D B M M M′ M′	3	·125

343. At the Imperial College of Engineering, in Japan, we had a testing-machine with which I made a great many experiments with my students. It increased the load on a beam at a uniform rate, and registered the load and deflection of the beam at every instant—that is, it drew a curve, each point of which showed the deflection and the load which produced it. Mr. George Cawley, instructor in mechanical engineering at the college, lithographed a number of these curves, taken by himself; and although the experiments were made on Japanese wood, so that the actual amounts of load and deflection are not of general interest, yet the shapes of the curves are so interesting as to be worthy of publication. With only one exception, two beams were broken and two curves taken for each kind of wood. The mean of these two curves has been given in Fig. 232 —that is, a curve lying between the two. The specimens were all free from knots. They were all 28 inches long and $1\frac{3}{4}$ inch square. The distance O W represents one ton, and the distance O D represents a deflection of 2 inches, so that the scale of the diagram is known. The load was in each case added to at a uniform rate, beginning with O, and the rate at which it increased was one ton in two minutes, and we see from the figure that practically only in three cases did the breaking of the beam take more than two minutes. The end of each curve shows where the specimen broke; it is easy to see where the curve ceases to be a straight line—that is, where the law, "Deflection is proportional to load," ceases to be true; and this point is therefore the *elastic limit*. In some cases the load corresponding to the elastic limit is less than half the breaking load, and in some cases greater than this, but usually it may be seen that it is about one-half.

344. What about **beams that are not rectangular** in section? Suppose we have a beam of the same section everywhere, whose strength and stiffness we know, and suppose we want to know the strength and stiffness of another beam which has the same form of section—that is, suppose the new section is such that all the old *lateral* dimensions are increased in a certain ratio—then the strength and stiffness increase in this ratio; if all the old *vertical* dimensions are increased in a certain ratio, then the strength increases as the square of this ratio, and the stiffness increases as the cube of this ratio. The effect of change of length is just the same as it was with rectangular beams, and

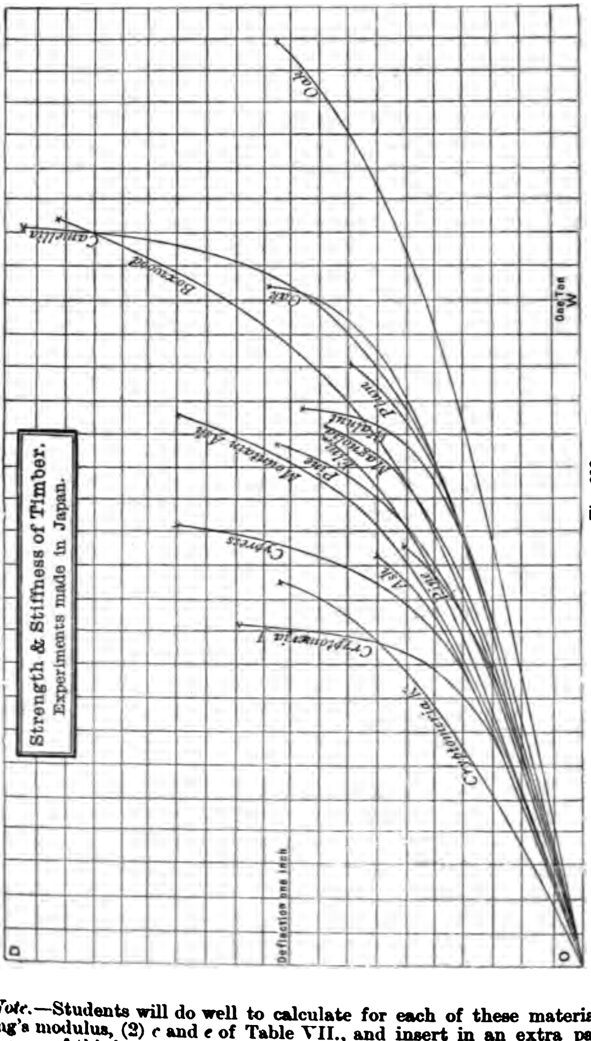


Fig. 232.

Note.—Students will do well to calculate for each of these materials Young's modulus, (2) *c* and *e* of Table VII., and insert in an extra page their copy of this book.

we know the effect produced by different methods of supporting and loading the beam from Table VIII.

From Arts. 341 and 342 it is evident that the load which a beam will carry without breaking is proportional to the *strength modulus* of its section divided by the length of the beam. The deflection of the beam is proportional to the load multiplied by the cube of the length, divided by the moment of inertia of the cross section.

345. **Beams of uniform strength** are those in which the nature of the loading is exactly known, and every section is made just of such a shape and size as to be equally ready to break with all the other sections. There is no difficulty in making Z, the strength modulus, exactly proportional to M. Thus taking up the four well-known cases already described; let us design beams of rectangular section everywhere of breadth b or depth d, or of circular section of diameter D, which shall be of uniform strength. Note that for a **rectangular section** $Z \propto bd^2$, and for a **circular section** $Z \propto D^3$. In the case of the circular section, the plan and elevation of the beam are of the same shape.

Case 1.—$M = Wx$, so that $bd^2 \propto x$. Keep b constant, the elevation showing d is a parabola. Keep d constant, the plan showing b is a triangle. $D^3 \propto x$, so that plan and elevation are what is sometimes called the cubic parabola; anyhow, it is easy to draw.

Case 2.—$M \propto x^2$, so that $bd^2 \propto x^2$. Keep b constant, the elevation showing d is a triangle. Keep d constant, the plan showing b is a parabola. $D^3 \propto x^2$; the plan and elevation are easily drawn.

Case 3.—From the middle to each end, this beam is the same as the beam of Case 1.

Case 4.—$M \propto (l^2 - 4x^2)$, so that $bd^2 \propto (l^2 - 4x^2)$. Keep b constant, the elevation showing d is an ellipse. Keep d constant, the plan showing b is two parabolas. $D^3 \propto (l^2 - 4x^2)$, so that the plan and elevation are easily drawn.

We cannot treat Cases 5 and 6 in the same way, because we only know M in these cases on the assumption that the beams are of the same section everywhere (*see* Art. 362).

EXERCISES.

Rolled girder section, or two equal flanges and web, like A, Fig. 223. In Table VI. we see that the moment of inertia is

$$I = \left\{ bd^3 - (b - b_1)\, d_1^3 \right\} \div 12,$$

and the strength modulus Z is I divided by $\frac{1}{2}d$, if d is the depth over all, d_1 the depth *between* the flanges, b breadth of either flange, b_1 the thickness of web.

1. Let the student calculate I and Z in all the cases of the following Table, page 421.

2. Taking $f = 20$ tons per square inch for iron, and 30 tons for steel, find fZ in each case. This is the greatest bending moment in inch-tons which each section will stand.

3. Show that a beam l feet long, supported at the ends and loaded in the middle, will break when the load in tons is $f\text{Z} \div 3\,l$. Calculate this for $l = 10$ feet in each case of the table.

4. Beams 10 feet long. For one ton at the middle the deflection in inches is $\text{D} = \frac{1}{48} \cdot \frac{2{,}240 \times 120^3}{30 \times 10^6 \times \text{I}}$, or $2{\cdot}688 \div \text{I}$. Find this in each case. I take E the same for iron and steel—namely, 30×10^6 lbs. per square inch.

5. An iron beam of the rolled girder section of the table, 18 inches deep, 25 feet long, *fixed at the ends*. What is the breaking load if spread uniformly?

Ans., The load for a 10-foot beam in the table is 89 tons; for a 25-foot beam, supported at the ends, and loaded in the middle, it is $\frac{89}{25} \times 10$, or 35·6 tons. Table VIII. shows that fixing at the ends and loading all over allows us three times as much breaking load, or 106·8 tons.

6. What is the mid-deflection of the beam of Exercise 5, fixed at the ends, when the load spread all over is 20 tons?

Ans., For a 10-foot beam it would be $\cdot 00223 \times \cdot 125$, according to the "deflection" column of Table VIII.; and for a 25-foot beam we multiply by $25^3 \div 10^3$, and the answer is ·0044 inch.

7. Compare the strength to resist bending of a wrought-iron I section when it is placed like this: I, and like this: ⊢⊣. The flanges of the beam are each 6 inches wide and 1 inch thick, and the web is $\frac{3}{4}$ inch thick, and measures 8 inches between the flanges. *Ans.*, 4·57; 1.

8. What is the greatest stress in a bar which is subject to a bending moment of 4,000 inch-pounds (1) if the section is a circle of $\frac{3}{4}$ inch radius; (2) if of I form, 2 inches deep and 1 inch wide, the web and flanges each being $\frac{3}{8}$ inch thick. *Ans.*, 5·4 tons; 3·1 tons.

9. The dimensions of the section of a cast-iron girder are the following: top flange, 4 by $1\frac{1}{2}$ inches; bottom flange, 12 by $1\frac{3}{4}$ inches; web, 16 by $1\frac{1}{2}$ inches. Determine the position of the neutral axis, and calculate the moment of inertia of the section. Find, also, the moment of resistance, the greatest permissible tensile and compressive stresses being $2\frac{1}{2}$ and $7\frac{1}{2}$ tons per square inch respectively. If the girder be 20 feet long, and is supported at its two ends, find also the greatest safe load which it will carry when uniformly distributed along its length.

Ans., $7\frac{1}{8}$ inch from top; 2,280·5 inch-units; 800 ton-inches; $26\frac{2}{3}$ tons.

10. Find the moment of resistance to bending of a beam of wrought iron which has a section like that of the third figure in Table VI., taking $b = 4$ inches; depth, 5 inches; the thickness of metal everywhere, $\frac{1}{2}$ inch; and $f = 20$ tons per square inch. What is the greatest load, placed at the centre, which a 10-foot beam of this section will stand when supported at both ends? *Ans.*, 60·45 inch-tons; 2 tons.

11. In a wrought-iron girder, supported at both ends, the section is that shown in the third figure of Table VI. $\text{B} = 8$ inches, $b = 7$ inches, $d = 10$ inches, $\text{D} = 12$ inches. The length is 24 feet. Find the greatest

ROLLED GIRDER SECTION.

Depth *d*.	Breadth of Flange *b*.	Thickness of Web b_1.	Thickness of Flange.	Area in Square Inches.	Weight per Foot in lbs.		Moment of Inertia I.	Strength Modulus Z.	Breaking Load w Tons at Middle of 10-Foot Beam.		Bending Moment of Section in Inch-Tons.		Deflection in Inches of 10-Foot Beam for 1 Ton in Middle.
					Iron.	Steel.			Iron.	Steel.	Iron.	Steel.	
20	8·26	·76	·97	29·67	97·2	100	1,825	182·5	122	183	3,650	5,475	·0015
18	7·10	·71	·94	24·67	81·6	84	1,208	134·2	89	134	2,673	4,010	·0022
16	6·06	·64	·82	19·15	62·7	64·5	731	91·4	61	91·4	1,828	2,742	·0037
14	5·87	·50	·81	15·76	51·5	53	494	70·6	47	70·6	1,412	2,118	·0044
12	6·23	·73	·87	18·45	60·2	62	404	67·3	44·9	67·3	1,346	2,019	·0067
10	6·16	·66	·70	14·27	46·7	48	221·5	44·3	29·5	44·3	886	1,329	·0121
8	4·02	·42	·56	7·4	24·3	25	74·2	18·5	12·3	18·5	370	555	·036
7	3·7	·32	·46	5·36	17·5	18	42·4	12·1	8·1	12·1	242	363	·063
6	3·09	·39	·50	5·06	16·5	17	27·5	9·2	6·1	9·2	184	276	·098
5	4·35	·35	·58	6·39	20·9	21·5	26·4	10·6	7·1	10·6	212	318	·102
4	3·23	·48	·41	4·16	13·6	14	9·82	4·91	3·3	4·91	98·2	147·3	·274
3													

uniformly distributed load which the girder will safely bear, taking $f = 9{,}000$ lbs. per square inch. Calculate the deflection at the middle of the beam. *Ans.*, 12·29 tons; 0·447 inches.

12. A steel plate girder, 70 feet span, 7 feet deep, has a uniform load of 1 ton per foot, a detached load of 10 tons at the middle, and loads of 5 tons each at 15 feet on each side of the centre. Find the diagram of bending moment, and state its amount at the detached loads. Calculate the best areas of cross-section of the booms at the middle. Take 6 tons per square inch in compression and 7 tons in tension as safe stresses. *Ans.*, 21·13 square inches; 18·1 square inches.

13. A hollow tube of wrought iron 20 feet long, 3 inches outside and $2\frac{1}{2}$ inches inside diameter. Find its weight. What is its deflection with its own weight? What further weight on its middle will it carry safely if $f = 4\frac{1}{2}$ tons per square inch? *Ans.*, 145·2 lbs.; ·44 inch; 158 lbs.

14. The supporting forces of a 35-foot beam are 20 and 15 tons. There is a load uniformly spread of 10 tons, and one detached load. Find the detached load and its position, and find the bending moment at the detached load, and also at the middle.

Ans., 25 tons at 21 feet from one end; 252 foot-tons; 218·7 foot-tons.

15. A beam 70 feet long, with weights of 5, 2, 4, 6, and 5 tons at distances 10, 20, 30, 35, and 60 feet from one end. Find the bending moment at each of these places, and find the supporting forces.

Ans., 117·1, 184·3, 231·4, 234, and 102·9 foot-tons; 11·7 tons; 10·3 tons.

16. If we take it that f really means some kind of ultimate stress which ought to be considered in the formula for a rectangular section:

$$\text{Greatest bending moment} = f \,.\, \frac{bd^2}{6},$$

we know that in a beam of length l inches, the breaking load at the middle being W, the beam being supported at the ends, the breaking bending moment must be $Wl/4$, and hence we have $Wl/4 = f\frac{bd^2}{6}$, or $W = \frac{2}{3}f \,.\, \frac{bd^2}{l}$. In the beams of Table VII., $b = 1$ inch, $d = 1$ inch, $l = 12$ inches. Calculate f for each of the materials mentioned in Table VII.

17. A beam 12 inches long, 1 inch broad, 1 inch deep, has a deflection D for a load of 1 lb. The values of D are given in Table VII. for many materials. They are there called e. Calculate the Young's modulus in each case. Here $D = \frac{wl^3}{48\,EI}$ becomes

$$E = \frac{wl^3}{48\,I e} = \frac{1 \times 12^3 \times 12}{48 \times 1 \times e} = \frac{432}{e}.$$

Thus, for English oak $e = \cdot 0003$, or 3×10^{-4}, and hence $E = 1{,}440{,}000$. Divide, therefore, 432 by every number in the "deflection" column of Table VII. to find Young's modulus.

18. A bar of wrought iron 3 inches broad and $1\frac{1}{4}$ inch thick is supported in a horizontal position at two points $2\frac{1}{2}$ feet apart. What deflection at the middle will be caused by placing there a load of 15 cwts.? *Ans.*, 0·064 inch.

19. A straight bar of wrought iron 1 inch × 1 inch in section is

loaded as a tie bar with 5 tons. It is found that the portion between two points on it, 4 feet apart, elongates ·019 inch. What is the value of E? If the bar be subject to a bending moment of 1,800 inch-pounds, what would be the radius of curvature? Find also the greatest stress and deflection if the bar be supported at points 4 feet apart and loaded with 120 lbs. *Ans.*, $28{\cdot}3 \times 10^6$; 109 ft.; 8,040 lbs.; 0·0123 inch.

20. A square bar ¾ inch × ¾ inch is subjected to a bending moment of 350 inch-pounds. What is the greatest stress in the bar, and the radius of the circle into which it is bent, E being taken at 2,000,000 lbs. per square inch? *Ans.*, 4,978 lbs.; 12·5 ft.

21. A bar of deal 6 feet long, 2 inches broad, and 3 inches deep, supported at the ends, is broken by a weight of 1,200 lbs. suspended at the centre. What uniformly distributed load would a beam of the same length and material bear if the depth were 4 inches and breadth 3 inches? *Ans.*, 6,400 lbs.

22. Sketch the diagrams of shearing force and bending moment for a uniform beam, supported at both ends, 30 feet long, weighing 10 lbs. per foot run, and carrying a load of 1 ton at a point 12 feet from one end. Give the numerical values of the shearing force and bending moment for the middle transverse section of the beam. *Ans.*, 896 lbs.; 174,780 inch lbs.

23. A beam 42 feet span supports five wheels of a locomotive. The fore wheel is 1 foot from the left end, and the distances between the wheels, in order, are 5, 8, 10, and 7 feet, and the loads transmitted to the beam, in order, are 5, 5, 11, 12, and 9 tons. Find the maximum bending moment. *Ans.*, 3,612 inch tons.

346. In designing railway girders the theory of bending is found to lead to useful rules; but in machine design some judgment is necessary in applying rules, and want of judgment is conspicuous sometimes in the writers of books on this subject. What we would urge upon students is the necessity for a great respect being shown to what are sometimes called "rule-of-thumb-proportions": great respect, tempered by criticism. When all the people of an old trade have made a pedestal or hanger of much the same proportions, we must remember that these proportions have been reached at considerable cost in trials and failures.

347. The Teeth of Wheels.—When toothed wheels drive each other, their teeth tend to break like little beams fixed at one end. It is usual in considering their strength to regard the pressure between two teeth as acting at a corner, because this may accidentally occur, and it is the most trying condition. There are usually two pairs of teeth in contact at once, so we consider that only half the total horse-power has ever to be transmitted by one pair of teeth. This transmitted horse-power, multiplied by 33,000, divided by the circumferential velocity of the wheel in feet per minute, is, of course, the force in pounds which each tooth has to withstand (*see* Art. 41). Imagine the tooth to tend to break at a section a L c d b, Fig. 233, making 45° with the depth a H, just as we know it would break if the corner were struck smartly with a hammer. This consideration leads to the rule that the square of the pitch is proportional to the force, divided by the greatest safe stress per square inch to which the material may be subjected. If teeth are so carefully trimmed up that we imagine the load always to be distributed over the whole breadth, it is easy to see that the pitch ought to be proportional to the force per inch breadth of wheel. Taking the first case, it is easy to see that the practical rule becomes

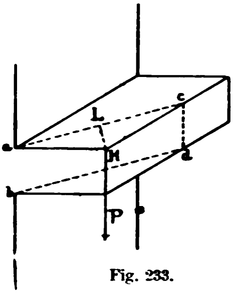


Fig. 233.

$$p = c\sqrt{\text{H}/v}$$

where H is the horse-power transmitted at a velocity of v feet per minute. If p is the pitch in inches, c may be taken as 7 for well-made cast-iron wheels working without shock; c is 9 for ordinary mill wheels; c is 11 in wheels subjected to shocks; c is also usually taken as 11 in mortise wheels. At high speeds c is usually taken greater as shocks are probably greater.

Exercise.—A spur wheel making 100 revolutions per minute has 60 cogs of 3 inch pitch (the circumference is therefore 180 inches, or 15 feet, so that $v = 1{,}500$ feet per minute); what power will it transmit safely? Here vp^2/c^2 is $1{,}500 \times 9 \div c^2$, so that the horse-power is—

275 in the most favourable case,
167 in mills,
112 when there is much shock.

When teeth are well trimmed we may take it that the above numbers for c ought to be diminished by 25 per cent. if the breadth of the wheel is four times the pitch. In very broad bevel wheels the above velocity is to be taken as the average velocity. In ordinary or narrow bevel wheels it is the velocity of the inner *parts* of the teeth.

348. Similar Structures Similarly Loaded.—If a girder is loaded mainly by its own weight, then any other girder made to the same drawings but on a different scale would be a similar structure similarly loaded; and this is the name given to all structures made from the same drawings but to different scales, if their loads are in the same proportions to the weights of the structures themselves. It will be found that in all such cases the stress at similar places is proportional to the size of the structure —that is, the weakness of the structure is in direct proportion to its size.

This is easily seen if we imagine the structure to be such a simple one as a rod, A, Fig. 234, carrying a weighty ball, W. If there is another such arrangement, of twice the size in every direction, the area of cross section of the rod would be four times as great; but the load to be carried would be eight times as great, and therefore the stress per square inch at a section would be twice as great—that is, the larger rod and ball would be twice as weak. As the stress would be twice as great and the length of the rod twice as great, the extension would be four times as great. The extension of the rod per foot in length would only be twice as great. In the same way a beam of cast iron, 1 inch square and 1 foot long, is 1,700 times too light to break with its own weight, whereas a beam of cast iron whose length, breadth, and depth are in the same proportion, if 1,700 feet long and 1,700 inches square in section, would break with its own weight. The deflection of similar beams similarly loaded is proportional to the square of their dimensions; but the deflection per foot of length is only proportional to their dimensions.

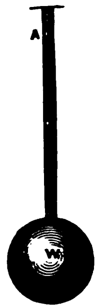


Fig. 234.

Imagine similar beams of the same material whose similar dimensions are as 1 to s and the loads are as 1 to s^3; evidently bending moments will be as 1 to s^4; moments of inertia of cross sections will be as 1 to s^4; curvatures at similar places will be equal; deflections will be as 1 to s^2; stresses will be as 1 to s.

If we dare take the loading of similar ships as 1 to s^n where n is something between 2 and 3, bending moments will be as 1 to s^{n+1}; curvatures will be as 1 to s^{n-3}; deflections will be as 1 to s^{n-1}; stresses will be as 1 to s^{n-2}.

In machines, accelerating forces are proportional to $S \times$ masses $\times N^2$ if N is the number of revolutions per second or per minute. Hence in similar machines the forces are proportional to $S^4 N^2$, bending moments are proportional to $S^5 N^2$, curvatures to $S N^2$, deflections to $N^2 S^3$, stresses to $S^2 N^2$. If, therefore, the stresses due to mere accelerations in machines are to be the same, $S N$ must remain constant—that is, if the size of a machine is doubled its speed must be halved.

348a. Strength of Ships.—Let AC represent the length of a ship. Imagine the ship divided by sections at equidistant points in AC. Let the ordinates of the curve W represent the weights of these portions, and the ordinates of the curve B represent the buoyancy or weights of water displaced by them. The two areas must be equal, and their centre of gravity lie on the same ordinate. The ship is "water-borne" at D and E. We take the vertical distances between ABD^1 and AWD^1, and between CBE^1 and CWE^1, as representing the downward load per foot or per inch on a beam AD, and on CE and the vertical distances between D^1BE^1 and D^1WE^1 as upward load per foot or per inch on the part of the beam DE. If we set off these distances, therefore, as ordinates from a line AC (Fig. 234A), we have the curve of positive and negative loading of a beam. We can use them to obtain the shearing force and bending moment at every section of the beam or ship.

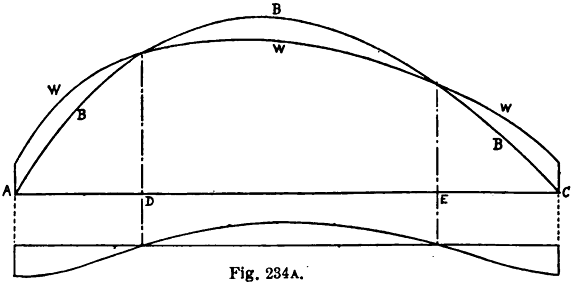


Fig. 234A.

Even when lying level, it is evident that the loads on a ship, regarded as a beam, are in other directions besides the vertical. Shipbuilders must also take account of the loading due to the inertia of the parts of the ship in her various kinds of motion. When these loading forces are known it is not difficult to calculate the strength. Besides the strength as a whole, it is important to see how each part communicates the load on it to the general system. The complete problem is therefore a complicated one, which, however, can all be worked out according to the principles given in this book. The general student is supposed to know the above simple method of obtaining loading and, therefore, bending moment and shearing force at each section of the ship as if it were a beam.

CHAPTER XIX.

DIAGRAMS OF BENDING MOMENT AND SHEARING FORCE.

349. WHEN **a beam is loaded in any way** whatever, it is easy to obtain by a graphical method the **diagram of bending moment**; in fact, in finding the supporting forces one finds the diagram of bending moment. Let A B (Fig. 235) represent the length of a beam which has three vertical loads—1, 2, 3. To find the vertical supporting forces at A and B, draw the unclosed force polygon, K L (Fig. 236)—before the student arrives at this part of the book he will probably have drawn other force polygons where all the sides were really in the same straight line—1, 2, and 3 (Fig. 236), representing in direction and magnitude the three loads of Fig. 235. Choose any point O. Join O K, O 1 2, O 2, 3, and O L (O 2, 3, means the line joining O with the point where the sides 2 and 3 meet). Now draw the link polygon (Fig. 235), beginning at any point *a*, in the vertical from A, and ending in the point *b*. Now *a b* is the side closing the link polygon. Draw O N (Fig. 236) parallel to *a b* (Fig. 235). Then L N is the amount of the supporting force at B, and N K is the amount of the supporting force at A. Also draw any vertical line, S T (Fig. 235). Then *the length* S T, *intercepted by the sides of the link polygon, represents the bending moment of the beam at any point* P on some scale which it is easy to find.

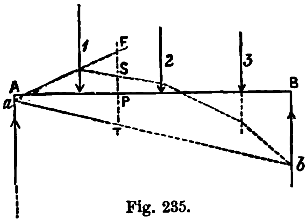


Fig. 235.

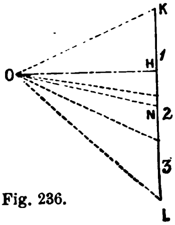


Fig. 236.

To prove this: Draw O H horizontally. The moment at any point P due to the supporting force, N K at A, is N K × A P; and this is equal to O H × F T; for, by similar triangles,

a T : T F : : O N : N K,

and therefore A P : T F : : O H : N K.

This second proportion gives NK . AP equal to OH . TF. In the same way the moment at P due to the force 1 is OH . FS; and hence the true moment at P, being the difference of these, is OH . ST. Let the student prove as an exercise that if the beam is drawn to a scale of 1 inch represented by x inches, and if the loads are drawn to a scale of 1 pound represented by y inches, then ST is the bending moment at P, on a scale such that 1 pound-inch is represented by $\frac{x \cdot y}{\text{OH}}$ inches, OH being measured in inches.

If the load is not concentrated at a number of points, it is usual to imagine it divided into a number of loads, each of which acts at one point. The diagram of bending moment is drawn in the way which I have just described, and then for the polygon with its straight sides we substitute a curve which touches all the sides of the polygon.

After you have found a diagram of bending moment, if you wish to see the effect of additional loads, draw a diagram for these loads as if they acted alone, but take care that the horizontal distance OH is the same as before. Add together the ordinates of your two diagrams to get your new diagram of bending moment for all the loads.

350. It is very unfortunate that this subject is usually taken up from an academic standpoint. Students discuss much theory sufficiently well for examination purposes, but they do not understand the most elementary things. They seldom state on a drawing what is the actual bending moment in pound or ton-inches at any section. Nor, indeed, do they ever seem to use these drawings for practical purposes. Instead of one diagram covering a large sheet of paper, we have the usual diagram looking like a mere book illustration in size.

A student will find that unless he works a number of exercises graphically he cannot comprehend this subject. Let him take such an exercise as this. A beam AB is loaded with 5 tons at C, 7 tons at D, 2 tons at E where AC is 8 feet, AD is 13 feet, AE is 18 feet, AB is 24 feet. It is also loaded with 1 ton per foot from A to C, 0·5 tons per foot from C to D, 0·7 tons per foot from D to E, and 0·8 tons per foot from E to B. It is worth while making a diagram for the distributed load, its ordinate showing the amount of load per foot or per inch.

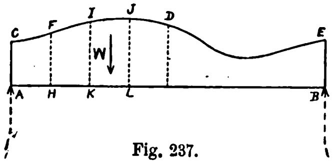


Fig. 237.

Take also a case with no detached loads but only a diagram showing w lb. per inch as shown in Fig. 237. We divide the area up into a convenient number of parts, CFHA, FIKH, &c. Find the centre of gravity of IJLK and let the arrow there represent the area IJLK—that is, the load on the portion KL. The student ought to state to what scale his area represents load. Thus we represent the whole by a number of detached loads; we draw the diagram of bending moment which is a polygon, but it is to be noticed that lines FH, IK, JL, &c., produced, meet the polygon at points which are on the true bending moment diagram for the distributed load, and so the curve is easily drawn touching the polygon at these points.

351. It is proved in Art. 339 that if D is the greatest distance moved by any point in a beam, and this is called the beam's deflection, and if the cross-section is the same everywhere, W the load, L the length, I the moment of inertia of the section, and E the modulus of elasticity,

$D = \frac{W L^3}{3 E I}$ for a beam fixed at one end and loaded at the other.

$D = \frac{3}{8} \frac{W L^3}{3 E I}$ for a beam fixed at one end and loaded uniformly.

$D = \frac{1}{16} \frac{W L^3}{3 E I}$ for a beam supported at the ends and loaded in the middle.

$D = \frac{5}{8} \frac{1}{16} \frac{W L^3}{3 E I}$ for a beam supported at the ends and loaded uniformly.

The third of these formulæ is the one most needed. It is by means of this formula that the modulus of elasticity is generally determined. Thus in careful experiments with an iron beam 1 inch broad, 1 inch deep, carried on supports 24 inches asunder, suppose we find that a load of 2,000 lbs. produces a deflection of one-quarter of an inch. Now, I for the beam is $\frac{1 \times 1 \times 1 \times 1}{12}$ or $\frac{1}{12}$. The third formula given above becomes

$$\cdot 25 = \frac{1}{16} \frac{2{,}000 \times 24 \times 24 \times 24}{3 E \times \frac{1}{12}},$$

and from this we find that E is 27,648,000 lbs. per square inch.

Again, taking the third of the cases shown, I find that 560 lbs. produced a deflection of 0·22 inch in a beam of wood 24 inches long, 1¾ inch square, supported at the ends. Here

$$I = 1{\cdot}75 \times 1{\cdot}75 \times 1{\cdot}75 \times 1{\cdot}75 \div 12, \text{ or } \cdot 781,$$

$$\text{and} \quad \cdot 22 = \frac{1}{16} \frac{560 \times 24 \times 24 \times 24}{3 E \times \cdot 781},$$

from which we find that E is 938,656 lbs. per square inch.

Again, from Table VII., we see that a beam of teak 12 inches long, 1 inch broad, 1 inch deep, gets a deflection of ·00018 inch for a load of 1 lb. Here the moment of inertia of the cross-section is $\frac{1}{12}$ and $\cdot 00018 = \frac{1}{16} \frac{1 \times 12 \times 12 \times 12}{3 E \times \frac{1}{12}}$, from which we find that E for teak is 2,400,000 lbs per square inch.

352. Take a small beam, A B, Fig. 238, supported at the ends, and load it in the middle. Measure carefully the **deflection** or lowering of the middle point. This is called the deflection of such a beam.. Now this distance will usually be small, and so we had better magnify it by letting the string C W pass over the little axle E, which carries a long pointer. This pointer will show on the scale P K a magnification of the deflection. We shall find that the more load we place at C the greater is the deflection; and in fact the deflection is proportional to the load, until our loads become great enough to produce permanent set, when (Art. 244) the deflections increase more rapidly than the load. If now we use a beam of the same material but of double the breadth, then for the same load we shall get one-half the old deflection. If we use a beam of double the depth, then for the same load we shall get only one-eighth of the old deflection. Also, if we double the length of our beam, using the same load, we shall get eight times the old deflection. A very instructive series of experiments may be made very easily in this subject, and we shall not thoroughly understand the matter unless we *make* a few such experiments. It is found that a beam of pitch

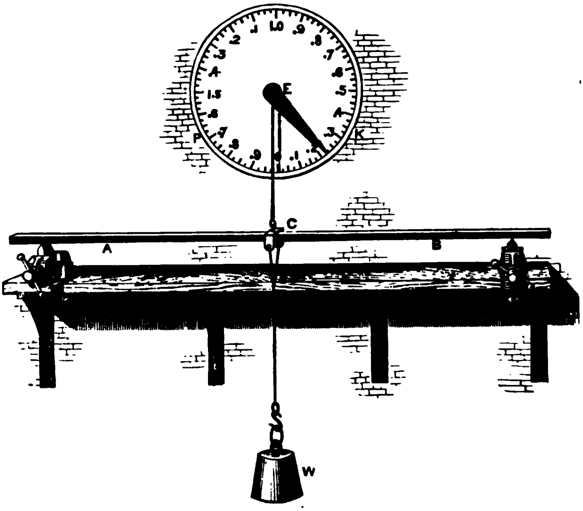


Fig. 238.

pine, 1 foot long, 1 inch broad, and 1 inch deep, supported at its two ends and loaded in the middle, is deflected ·00035 inch by a load of 1 lb. This explains the numbers given in Table VII. It is found that if the same beam is fixed at one end and loaded at the other (first case of Table VIII.), the deflection is 16 times as great, whereas if the beam is fixed at both ends and the load is spread uniformly (last case of Table VIII.), the deflection is only ·125, or one-eighth as great. This explains the "deflection" column of Table VIII.

The rule, then, to find the deflection in inches of any beam loaded in any of the ways shown in Table VIII. is this:—*Multiply together the cube of the length in feet, the total load in pounds, the number called deflection in Table VIII., and the number called deflection in Table VII., and divide the product by the breadth of the beam in inches, and by the cube of the depth in inches.*

Example.—A beam 20 feet long, 10 inches broad, 15 inches deep, of pitch pine, fixed at one end and having spread all over it a total load of 4,000 lbs.—what is its deflection? Here the number in Table VIII. is 6, and in Table VII. it is ·00035; hence we have 20 × 20 × 20 × 4,000 × 6 × ·00035 divided by 10, and again divided by 15 times 15 times 15, which gives as answer 1·99 inch. The end of the beam would be deflected this distance.

A beam is said to be stiff if its deflection is small, and we say that the **stiffness** of a beam supported and loaded in the various ways shown in Table VIII. is for the various cases $\frac{1}{16}, \frac{1}{6}$, 1, 1·6, 4, 8. In fact, a beam of a certain length carrying a certain load is 128 times stiffer when it is fixed at the ends and loaded uniformly than when it is fixed at one end and loaded at the other end.

It is well to remember that when we double the breadth of a beam we double its strength and also its stiffness, but if we double its depth we get four times the strength and eight times the stiffness. Beams required to be very stiff ought to be very deep. Care must be taken, however, that they are laterally supported, else they will buckle. If you double the length of a beam you get half the strength, but you only get one-eighth of the stiffness.

353. The student must see that the heights of the points on the force polygon; for example, the heights of the points between 1 and 2, 2 and 3, &c. (Fig. 236), above N, give him the ordinates of his shearing force diagram so that he can obtain this diagram by horizontal projection. When the loads are distributed, at points like H, K and L of Fig. 237, the ordinates are the same as if the loads were detached, and they may be joined by a curve.

We shall now consider a number of cases where arithmetic and algebra help out our graphical methods of working.

354. If the loads on a structure are W_1, W_2, etc., the stresses and strains everywhere are the sum of those that would be produced if each load acted alone. We find this true in all cases that we try, unless, indeed, in certain cases where instability is produced. It is usually supposed to be a mere statement of the mathematical law of superposition of small effects. We know, at all events, that the bending moment at any part of a beam due to loads W_1, W_2, etc., acting together, is the sum of the bending moments due to each acting singly.

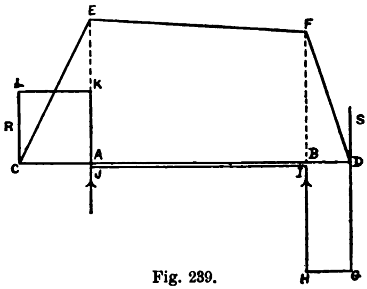


Fig. 239.

Exercise.—In Fig. 239 loads R = 10 tons and S = 15 tons act at the ends C and D; supporting forces P and Q act at A and B. CA = 5 feet, AB = 15 feet, BD = 3 feet. Draw the diagrams of bending moment and shearing force.

Answer.—CEFDC is the diagram of bending moment, where AE represents 50 foot-tons and BF represents 45 foot-tons. The supporting forces, which are easily found graphically and analytically, are P = 10·33 tons, Q = 14·67 tons. Hence the diagram of shearing force is DGHIJKLCD, where DG = − 15, BI = − 0·33, AK = + 10, all in tons.

Exercise.—In Fig. 240 a weightless beam rests on supports

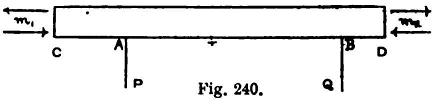


Fig. 240.

at A and B; torques m_1 and m_2 are applied at C and D. Find the bending moment and shearing force everywhere.

Evidently P and Q are equal and opposite; $P \times AB = m_1 - m_2$, or $P = \frac{m_1 - m_2}{AB}$, $Q = -\frac{m_1 - m_2}{AB}$. If P is an upward force, Q must be a force holding the beam down at B. Thus, let $m_1 = 30$ ton-feet, $m_2 = 17$ ton-feet; let AB be 15 feet. Then

P = 13/15 ton, Q = − 13/15 ton. S is zero from D to B, − 13/15 ton from B to A; and again zero from A to C. The bending moment from D to B is constant and equal to m_2; from B to A it increases to m_1, and remains constant from A to C.

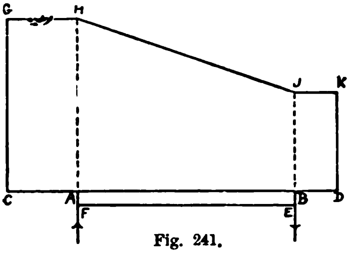


Fig. 241.

C G H J K D C is the diagram of bending moment, where C G = A H represents m_1, and J B = K D = m_2. D B E F A C D is the diagram of shearing force where B E = A F = − 13/15.

Exercise.—Let the beam of Fig. 240 have moments m_1 and m_2 applied at its ends, as shown in Fig. 241; also let it have a load of W_1 lbs. (say 5 tons) at T (say that A T is 6 feet). This of itself would produce bending moment, as shown at A L B of Fig. 242, where T L is − 18 ton-feet, and would produce supporting forces P = 3 tons, Q = 2 tons. The shearing force diagram is B X V T U N A B, where B X = T V = 2 tons, and A N = T U = − 3 tons.

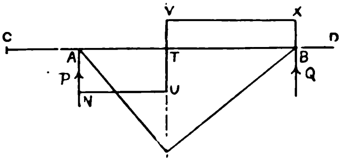


Fig. 242.

Now imagine the beam loaded with 1 ton per foot on C A and 0·5 tons per foot on B D. If C D′ = x, the bending moment at D′ is $1 \times x \times \frac{1}{2} x$ or $\frac{1}{2} x^2$, and at A it is $\frac{1}{2}$ (C A)^2 or 12·5 ton-feet. The bending moment curve C E is parabolic. Similarly the bending moment curve D F is parabolic, the moment being $\frac{1}{2} \times 0{\cdot}5 \times 3^2$ or 2·25 ton-feet at B, shown at B F. The bending moment due to the loads on the ends would be C E F D over the whole beam, and the supporting forces would be P = 5·6833, Q = 0·8167. The shearing force diagram is D I J K A E C D (Fig. 245), where B I = − $1\frac{1}{2}$ tons, B J = A K = − 0·6833 ton, A E = 5 tons.

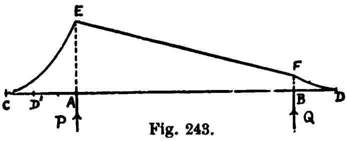


Fig. 243.

Now imagine the beam to have a uniformly distributed load of $\frac{1}{3}$ ton per inch between A and B. The supporting forces required by this are P = 2·5 and Q = 2·5 tons. The bending moment diagram is a parabolic curve A E B, where

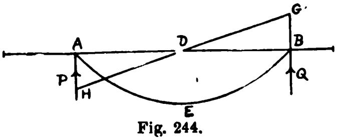


Fig. 244.

D B = − 9·375 ton-feet. The shearing force diagram is B G D H A B (Fig. 244), where B G = 2·5 tons, A H = − 2·5 tons.

Let the student work out all these diagrams. and add all the ordinates together.

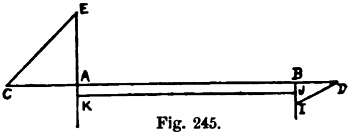


Fig. 245.

Now let him obtain the same result by one graphical construction. The continuous loading he will break up into a number of detached loads, and the following example will show him how to work the problem. Let the beam K L N P be loaded and supported as shown in Fig. 246. Use Bow's lettering.

355. In Fig. 246a the loads are given, half-barbs being used for the arrow-heads; the missing corner of the force polygon A B C D E F G H I is I. Choose a pole 0 and join with A B C, etc. We cannot yet draw 0 I. Now start at K, or any point in the line of load A B (Fig. 246), and draw a line through the space A parallel to 0 A (Fig. 246a) to meet (produced) the force A I in Q. Also draw K X through the space B parallel to 0 B to meet the load-line B C in X. Draw X W through C parallel to 0 C, and so on to R. Now join R and Q. This is parallel to the missing line 0 I (Fig. 246a), which may now be drawn and I found. H I and I A (Fig. 246a) are the amounts of the supporting forces. The diagram Q A X W V U T S R Q is the diagram of bending moment,

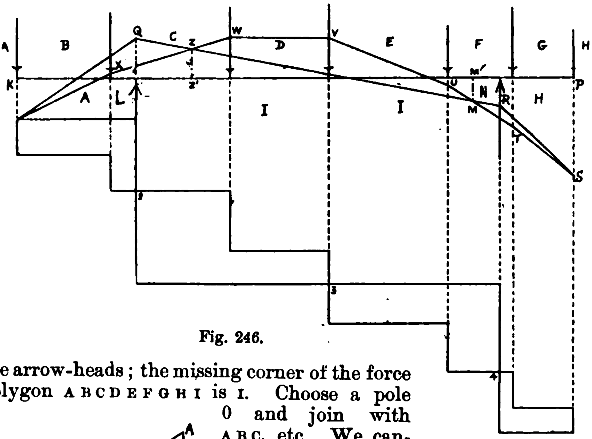


Fig. 246.

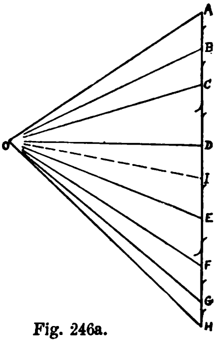


Fig. 246a.

vertical distances such as V Y representing the bending moment at each corresponding section. The scale of measurement may be computed by Art. 349. We have positive bending moment

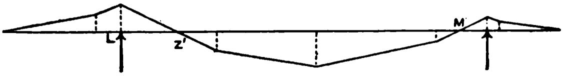


Fig. 247.

(tending to make the beam convex upwards) between K and Z' and between M' and P, and we have negative bending moment between Z' and M'. Z' and M' are places of no bending moment, where the beam has no curvature; they are called points of inflexion. The shear force diagram 1, 2, 3, 4, 5 presents no difficulty.

It is usual, when we finish the work, to draw the diagram of bending moment as in Fig. 247, the ordinates being measured, not from a broken line as in Fig. 246, but from a horizontal line.

Travelling Loads.

356. I. Suppose the load W (Fig. 248), to travel over the beam from A to B. When W is at any point C, between A and D, the

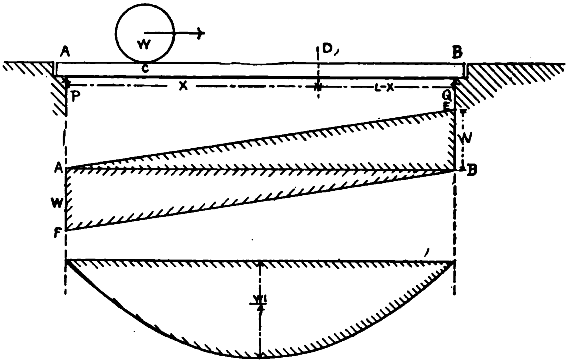


Fig. 248.

shearing force at D (say S_D) is positive, and equal to the reaction Q. This positive shear at D increases with Q as the load approaches D, and in the limit, when W is very near to D, it has the value $\frac{W}{L}x$.

At the instant the load passes D, the shearing force at D diminishesb y the amount W, and becomes $\frac{W}{L}x - W$, or $-\frac{W}{L}(L - x)$, thus becoming negative, and equal to the reaction at P; and as the load moves on towards B, the negative shear at D diminishes numerically. We thus see that the greatest positive shearing force at any place occurs when the load is just to the left of the place, and the greatest negative shear when the load is just to the right.

Expressed algebraically:—Max. $+ \text{s}_\text{D} = \frac{\text{W}}{\text{L}} \text{x}$; max. $- \text{s}_\text{D} = - \frac{\text{W}}{\text{L}}(\text{L} - \text{x})$. These are the equations to straight lines, and the corresponding diagrams are set out in Fig. 248: that for maximum positive shear is the triangle A B E, and that for maximum negative shear the triangle B A F.

Next consider the maximum bending moment at D (say max. M_D).

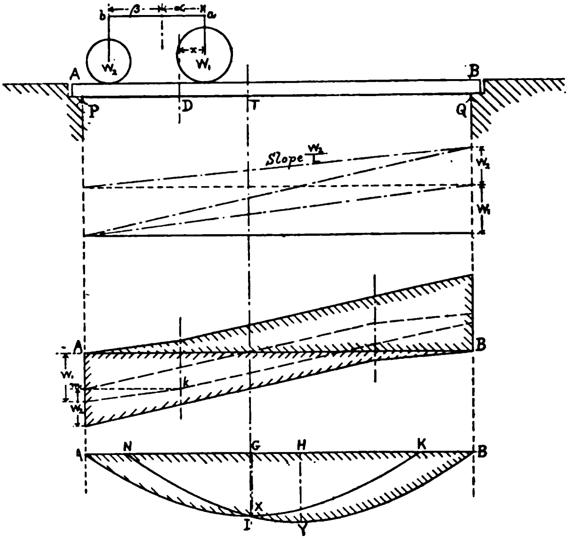


Fig. 249.

As the load advances from A towards D, M_D increases, since its value is Q (L − x), and Q increases. When the load passes D, M_D diminishes, since its value is now P x, and P diminishes. Therefore the maximum value of M_D occurs when the load is at D, and its value is given by the equation

$$\text{max. } \text{M}_\text{D} = \text{P x or Q} (\text{L} - \text{x}) = \text{W} \frac{\text{L} - \text{x}}{\text{L}} \text{x}.$$

This is the equation to a parabola passing through A and B, axis vertical, the ordinate of the vertex (*see* Fig. 248) being W L/4.

II. Two travelling loads W_1 and W_2 at a fixed distance *l* apart; draw the diagrams of maximum positive and negative shearing force and maximum bending moment.

As the front of the first load W_1 approaches D, $+ \text{S}_\text{D}$ increases, since its value $= \text{Q}$; when W_1 passes D, S_D undergoes a sudden diminution by the amount W_1; and as W_1 moves from D towards B the value of $+\ \text{S}_\text{D}$ increases, on account both of the advance of W_1 towards B and the approach towards D of the other load W_2. This increase goes on until W_2 crosses D, when there is a second sudden drop in the value of $+\ \text{S}_\text{D}$, followed by a gradual increase until W_2 arrives at B. We see that the maximum positive or negative S_D occurs when the front or back respectively of W_1 or W_2 is at D.

To draw the diagram: Draw the diagrams of S for the front and back of each of the loads W_1 and W_2 in its passage across the beam; the boundary line is the diagram required. The diagram for front of W_1 will be a straight line through A, the equation of which is $+\ \text{S} = \text{Q} = \text{W}_1 \dfrac{\text{X}}{\text{L}}$, until W_2 comes on, when the slope suddenly alters, the S at front of W_1 then being

$$+\ \text{S} = \text{Q} = \text{W}_1 \frac{\text{X}}{\text{L}} + \text{W}_2 \frac{\text{X} - l}{\text{L}} = (\text{W}_1 + \text{W}_2) \frac{\text{X}}{\text{L}} - \text{W}_2 \frac{l}{\text{L}}.$$

The slope now is $\dfrac{\text{W}_1 + \text{W}_2}{\text{L}}$, and the diagram of S in front of W_1 continues a straight line in this direction. Next, for S at back of W_1. This is equal to S at front $-\ \text{W}_1$ for all positions, and the diagram is a broken line parallel to the one just drawn at a distance below it, equal to W_1. Next, for S at front of W_2. The S between W_1 and W_2 is the same at all points; that is, when W_2 is just coming on, the S in front of it is the same as the S at back of W_1. Through K draw the horizontal line *k m*, then *m* is the starting-point for the S diagram in front of W_2. The initial slope is $\dfrac{\text{W}_1 + \text{W}_2}{\text{L}}$: this extends to a point distant horizontally l from B, when W_1 passes off the beam, and the slope diminishes to $\dfrac{\text{W}_2}{\text{L}}$. The diagram for the rear of W_2 is parallel to the line just drawn and at a distance below it, equal to W_2. As a test of accuracy, it is to be noted that this broken line should end at B.

Consider now the maximum bending moment. Let $\text{W}_1 + \text{W}_2$ at t be the resultant of W_1 at a and W_2 at b. Let $ta = \alpha$ and $tb = \beta$. Then $\dfrac{\alpha}{\beta} = \dfrac{\text{W}_2}{\text{W}_1}$, that position being shown in Fig. 249, in which the loads W_1 and W_2 are one on each side of D. Let the distance of line of load W_1 from D be x. The maximum M_D evidently occurs either when W_1 or W_2 is at D, or for some intermediate position. Consider an intermediate position as shown. $\text{M}_\text{D} = \text{Q}(\text{L} - \text{X}) - \text{W}_1 x = (\text{W}_1 + \text{W}_2) \dfrac{\text{X} + x - \alpha}{\text{L}} (\text{L} - \text{X}) - \text{W}_1 x = (\text{W}_1 + \text{W}_2) \dfrac{\text{X} - \alpha}{\text{L}} (\text{L} - \text{X}) + x \left\{ (\text{W}_1 + \text{W}_2) \dfrac{\text{L} - \text{X}}{\text{L}} - \text{W}_1 \right\}$. It will be seen that so long as the expression in the last bracket is positive, M_D is a maximum when x is greatest—that is, when $x = l$, or, that is, when load W_2 is at D. And $\left\{ (W_1 + W_2) \frac{L - X}{L} - W_1 \right\}$ is positive so long as $\frac{L}{L - X} < \frac{W_1 + W_2}{W_1}$. When $\frac{L}{L - X}$ is $> \frac{W_1 + W_2}{W_1}$, the expression in the last bracket is negative, and the value of M_D is therefore a maximum when x is zero—that is, when load W_1 is at D.

To draw the diagram, take point T in A B such that $\frac{AT}{BT} = \frac{W_2}{W_1}$. Let A T be called the field of W_2, and B T the field of W_1; then M_D is a maximum for any point D in the field A T when the load W_2, which governs that field, is directly over point D, and for any point of the field B T when W_1 is at the point, provided in each case the load can completely traverse its field without the other load leaving the beam, which condition requires l to be not greater than the smaller of the two values $\frac{W_1}{W_1 + W_2} L$, or $\frac{W_2}{W_1 + W_2} L$.

Curve for field B T. Putting $x = o$ in the expression for M_D, we have maximum $M_D = \frac{W_1 + W_2}{L} (X - a)(L - X)$, which is the equation to a parabola passing through B and a point N distant a from A; and by symmetry the curve for field A T is a parabola passing through A and a point K distant β from B; the two parabolas will be found to intersect at I directly under T. The distance X of the mid point of B N from A is $a + \frac{L - a}{2} = \frac{L + a}{2}$. Putting this value in the equation to the curve, we have the ordinate $HY = \frac{W_1 + W_2}{L} \left[\frac{L + a}{2} - a\right]\left[L - \frac{L + a}{2}\right] = \frac{W_1 + W_2}{4L}(L - a)^2$; the corresponding value for G X, or depth of other parabola A K, is $GX = \frac{W_1 + W_2}{4L} (L - \beta)^2$. When the distance l is greater than the shorter of the

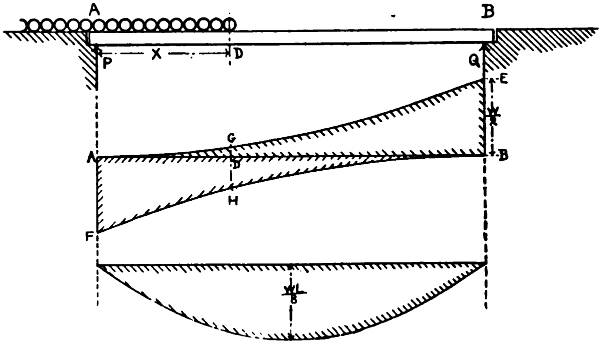


Fig. 250.

two fields, there is then a third parabola through A and B corresponding to the greater of the two loads taken alone by method shown in Case I.

III. A load as of a travelling train, w lb. per unit length, comes upon a girder A B from the left, covering it from end to end, and then leaving it. Show that the greatest positive shearing force at a section D occurs when the front of the train reaches D, and that the greatest negative shearing force occurs when the rear of the train leaves D. Also find the maximum bending moment at D.

We have seen that any load produces positive S if it is to the left of D, and negative S if it is to the right of D. Hence, to produce the greatest positive S at D, there ought to be no load to the right of D, and for greatest negative S at D there ought to be no load to the left of D, so the proposition is proved. When the train covers A D, S at D = Q, or $\frac{w\,x^2}{2\,L}$ . . . . (1), being shown by D G in Fig. 250. When the train covers D B, S at D is $-$ P, or $-\frac{w}{2\,L}(L - x)^2$ . . . . (2), being shown at D H in Fig. 250. B E and F A are numerically equal to half the load when it covers the span. The curves A E and B F are parabolic, the equations of which are given in (1) and (2) respectively. The student ought now to add to the ordinates of Fig. 250. the shearing force due to a uniformly distributed constant load, and write out in words actually what S is as a train rolls on, covers, and rolls off the bridge.

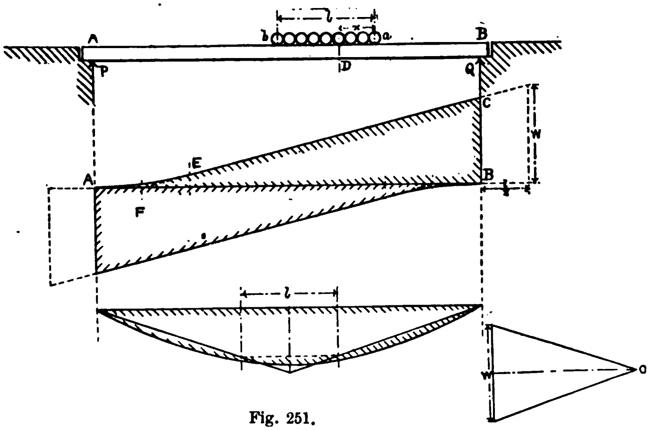


Fig. 251.

Next consider the maximum M at D. Any load anywhere on a girder increases the bending moment anywhere. Hence where load due to a travelling train comes upon a girder, covers the girder, and leaves it, the bending moment at any place is never greater than what it is when the whole girder is covered. Therefore, the maximum bending moment will occur when the girder is fully loaded. A curve to this is known to be a parabola through A and B, the depth of vertex being $\frac{\mathrm{W\,L}}{8}$.

IV. Travelling load of uniform intensity w lb. per unit length, of length l less than the span L. The maximum positive S occurs when the front of the load is at D, and maximum negative when the rear of the load is at D; so that while the load is only partially on the beam the diagram of maximum + S will be a parabola similar to that of Case III., the equation being for a distance l from A maximum + S = $\frac{w\,\mathrm{X}^2}{2\,\mathrm{L}}$. When the load comes wholly on the beam, as in Fig. 251, the diagram alters; maximum + S = Q = $\frac{wl}{\mathrm{L}}\left(\mathrm{X}-\frac{l}{2}\right)$ or $\frac{\mathrm{W}}{\mathrm{L}}\left(\mathrm{X}-\frac{l}{2}\right)$. The equation to a straight line slope $\frac{\mathrm{W}}{\mathrm{L}}$, intersecting A B at a distance $\frac{l}{2}$ from A, and therefore tangential to the parabola. The diagram of positive S is most readily set out by first drawing the line E C, as shown in figure, then drawing the parabola to touch it at E, and the line A B at A. A similar curve set out downwards from B will be the diagram of maximum negative S.

Maximum bending moment at D. It is easily seen that the maximum $\mathrm{M_D}$ occurs either when the front a or the rear b of the load is at D, or for some intermediate position, as in Fig. 251. To find value of $\mathrm{M_D}$ for an intermediate position let the condition of the co-ordinates be as in Fig. 251.

$$\mathrm{M_D} = \mathrm{Q}\,(\mathrm{L}-\mathrm{X}) - \frac{wx^2}{2} = \frac{wl}{\mathrm{L}}\left[\left(\mathrm{X}+x-\frac{l}{2}\right)(\mathrm{L}-\mathrm{X})\right] - \frac{wx^2}{2}$$

For a maximum, $\frac{d\,\mathrm{M_D}}{dx} = 0$ (X being constant), ∴ maximum $\mathrm{M_D}$ occurs when $\frac{wl}{\mathrm{L}}(\mathrm{L}-\mathrm{X}) - wx = 0$, or $\frac{\mathrm{L}-\mathrm{X}}{\mathrm{X}} = \frac{x}{l-x}$; *i.e.* when $\frac{\mathrm{B\,D}}{\mathrm{A\,D}} = \frac{a\,\mathrm{D}}{b\,\mathrm{D}}$, or the maximum $\mathrm{M_D}$ occurs for such a position of the load that D divides it and the beam into segments, the ratio of which are respectively equal.

Putting this value of x, namely $x = l.\frac{l-\mathrm{X}}{\mathrm{L}}$, into the expression for $\mathrm{M_D}$, we have maximum $\mathrm{M_D}$

$$= \frac{wl}{\mathrm{L}}\left[\mathrm{X} + \frac{l(\mathrm{L}-\mathrm{X})}{\mathrm{L}} - \frac{l}{2}\right](\mathrm{L}-\mathrm{X}) - \frac{wl^2(\mathrm{L}-\mathrm{X})^2}{2\,\mathrm{L}^2}$$

$$= \frac{\mathrm{W}}{\mathrm{L}}\left(1-\frac{l}{2\,\mathrm{L}}\right)\mathrm{X}\,(\mathrm{L}-\mathrm{X})$$

equation to a parabola passing through A and B, the depth of the vertex being

$$\therefore\ \mathrm{M_C} = \frac{\mathrm{W\,L}}{4}\left(1-\frac{l}{2\,\mathrm{L}}\right) = \frac{\mathrm{W\,L}}{4} - \frac{\mathrm{W}\,l}{8}\ (\textit{see}\ \text{Fig. 251}).$$

CHAPTER XX.

MORE DIFFICULT CASES OF BENDING OF BEAMS

357. We do not usually trouble ourselves as to whether we call the bending moment which makes a beam convex upwards positive or negative, representing it by upward-drawn or by downward-drawn ordinates. There is the same sort of choice in regard to shearing force. But for the sake of having plus signs in the following expressions (1) and (2), we had better adhere to the following definitions. A section which is being sheared is supposed to be at the positive distance x to the right of the zero point. Loads are positive forces; supporting forces are negative. Positive shearing force S means that the material to the left of a section acts with downward force on the material to the right of the section. Positive bending moment M causes a beam to be convex upwards; positive y at a place is a downward displacement. When this is the case we know that

$$\frac{dy}{dx} = i \dots (1), \qquad \frac{di}{dx} = \frac{d^2y}{dx^2} = \frac{\text{M}}{\text{EI}} \dots (2).$$

In the same way we can show that

$$\frac{d\,\text{M}}{dx} = \text{S} \dots (3), \qquad \frac{d\,\text{S}}{dx} = w \dots (4).$$

To prove (3) and (4), consider the equilibrium of the portion of beam between the sections A B and C D. At A B there is the bending moment M and shearing force S, and at C D there are M + δM and S + δS, and O O′ is δx. The forces acting on this portion of beam are shown in Fig. 252. The load being w per unit length, the resultant load here is $w \,.\, \delta x$. Hence, considering the vertical forces, $\delta \text{S} = w \,.\, \delta x$ or $d\,\text{S}/dx = w \dots (4)$. Taking moments about O′, $\text{M} + \text{S} \,.\, \delta x + \frac{1}{2} w (\delta x)^2 = \text{M} + \delta \text{M}$ or $\delta \text{M}/\delta x = \text{S} + \frac{1}{2} w \,.\, \delta x$; and in the limit, as δx gets smaller and smaller, $\frac{d\,\text{M}}{dx} = \text{S} \dots (3)$.

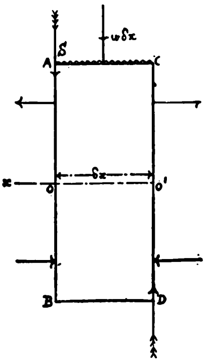


Fig. 252.

358. We saw (Art. 349) that if we have a diagram of w we can find easily, graphically, the diagram of M. We now see that if the value of M at every section be divided by the value of E I there, and if we treat this diagram, showing M/E I everywhere, exactly as we treated the w diagram, we obtain y.

This graphical method of working is quick and accurate. If we only possessed an accurate mechanical integrator, such that when given a curve showing x and v, v being a function of x, we could at once draw another curve whose ordinate for a particular value of x_1 represented the area of the v curve up to that place from some datum value of x, we could easily solve more difficult problems. I have often done this by counting squares on squared paper; also I have worked to obtain a number of points in the new curve by using a **planimeter** a number of times. We see now that if we know w, the integral of w shows S, the integral of S shows M, the integral of $\frac{\mathrm{M}}{\mathrm{EI}}$ shows i, and the integral of i shows y, the shape of the beam. When we integrate, however, we must settle the starting value of the new ordinate, and this is what usually gives trouble. Thus the starting value of S (when we integrate w) is not zero, but depends upon the supporting forces.

The student must see very clearly that the change in i, in going from one value of x to another, is equal to the area of the M/EI curve between those places.

359. I have found the arithmetical method of Art. 214 very satisfactory. Its accuracy depends, of course, on the number of ordinates taken. A student ought to test for himself the accuracy of the method on some such exercise as the following, of which he knows the answer.

A beam of rectangular section 1½ inch broad, 2 inches deep, and of length 15 inches, is fixed at one end and is loaded uniformly with 10 lb. per inch. Its E is 25×10^6, find M everywhere and y. Here $\mathrm{I} = 2{\cdot}25 \times 4^3 \div 12$, or $\mathrm{I} = 12$. x is distance in inches measured from the free end. Imagine the free end to be on our left and the rest of the beam on our right. The table gives the whole work, and needs almost no explanation.

The student who thinks will have no difficulty in working out a problem of this kind. To find such a number as M for $x = 9$, for example, we add the average value of 130 and 140 to the previous M, and so get 855.

We know that i is 0 when $x = 15$, and so we subtract $3{,}752 \times 10^{-8}$ from all the found values of $i + c$. In the same way to get the last column we add 39,801 to every number of the previous column.

Let us compare our results with the true answers, which the student can easily work out as in Art. 339: $\mathrm{S} = 50 + 10\,x$, $\mathrm{M} = 50\,x + 5\,x^2$,

$$\frac{dy}{dx} = \frac{1}{3 \times 10^8}\left\{-11{,}250 + 25\,x^2 + \frac{5}{3}\,x^3\right\}$$

$$y = \frac{1}{3 \times 10^8}\left\{119{,}531 - 11{,}250\,x + \frac{25}{3}\,x^3 + \frac{5}{3}\,x^4\right\}$$

Thus when $x = 15$, $\mathrm{M} = 1{,}875$, as in the table. When $x = 0$, i or $dy/dx = -3{,}750 \times 10^{-8}$, whereas the table gives $-3{,}752 \times 10^{-8}$.

Again, when $x = 0$, $y = 39{,}844 \times 10^{-8}$, whereas the table gives $39{,}801 \times 10^{8}$, which is the same for all practical purposes. After having worked this example a student must feel confidence in using this method of integration which gives us answers so readily.

x	w	S	M	$\frac{M}{EI}$ $\times 10^8$	$i + c$ $\times 10^8$	i $\times 10^8$	$y + c'$ $\times 10^8$	y $\times 10^8$
0	10	50	0	0		-3,752	0	39,801
1	10	60	55	18	9	-3,743	-3,747	36,054
2	10	70	120	40	38	-3,714	-7,476	32,325
3	10	80	195	65	91	-3,661	-11,163	28,638
4	10	90	280	93	170	-3,582	-14,785	25,016
5	10	100	375	125	279	-3,473	-18,312	21,489
6	10	110	480	160	421	-3,331	-21,714	18,087
7	10	120	595	198	600	-3,152	-24,956	14,845
8	10	130	720	240	819	-2,933	-27,998	11,803
9	10	140	855	285	1,082	-2,670	-30,800	9,001
10	10	150	1,000	333	1,391	-2,361	-33,315	6,486
11	10	160	1,155	385	1,750	-2,002	-35,497	3,304
12	10	170	1,320	440	2,162	-1,590	-37,293	2,508
13	10	180	1,495	498	2,631	-1,121	-38,648	1,153
14	10	190	1,680	560	3,160	-592	-39,505	296
15		200	1,875	625	3,753	0	-39,801	0

360. Beams Fixed at the Ends.—If the loading on a symmetrical beam is symmetrical so that we find, graphically or any other way, the diagram of bending moment *m* as if it were supported at the ends, we know that equal and opposite torques are needed to fix the ends. Thus, if A B is, say, a beam of uniform section and has a loading of any kind whatsoever which will produce (the beam being only supported at the ends) a diagram of bending moment *m* such as is shown in A C D E B (Fig. 253), and if equal and opposite torques are applied to fix the ends such as alone would produce the diagram of bending moment shown to scale in B G F A, then the algebraic sum of the two (for the A C D E B is negative and A B G F A positive) is shown in A F C′ D′ E′ G B A, being positive from A to C′ and from E′ to B, so that in these parts the beam is convex upwards; and being negative from C′ to E′, where the beam is concave upwards. We want, then, to know exactly how much G B or A F must be to fix the ends of the beam. Now, the difference of slope of beam between A and B is nothing if the ends are both fixed, and therefore the total area of the bending moment diagram must be zero (since E I is constant, we say M instead of M/E I). Hence, the area of the portion C′ D′ E′ being negative, must be numerically equal to the sum of the areas A C′ F + B E′ G. In fact, the average ordinate of the M curve must be zero, and therefore we raise the m curve A C D E B A by its *average* height to get the M curve.

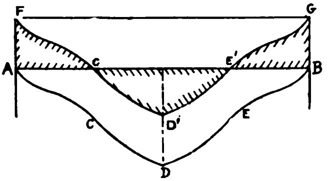


Fig. 253.

Example 1.—Beam of length l, with load w in the middle, fixed at the ends. The diagram of m is A D B A (Fig. 254), where D D″ represents $\frac{1}{4}$ wl; raise it therefore by the amount $\frac{1}{8}$ wl, and we find A F C′ D′ E′ G B A. Evidently A F = − D″ D′ = G B = $\frac{1}{8}$ wl. The beam is convex at the ends, concave in the middle, equally ready to break at ends and middle, and the points of inflexion are half-way between ends and middle (*see* Art. 342)

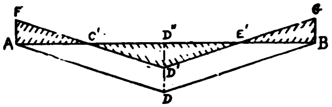


Fig. 254.

Example 2.—Beam A B of length l, with total load w spread uniformly. A D B shows the m curve, a parabola, the diagram of bending moment if the beam were merely supported at the ends, D D″ being $\frac{1}{8}$ wl. The average ordinate of A D B A is $\frac{2}{3}$ D D″. Raising the diagram by this amount, we find the true diagram A F C′ D′ E′ G B A of M for the beam fixed at the ends. There is $\frac{1}{12}$ wl at each end and $-\frac{1}{24}$ wl at the middle (*see* Art. 342). The points of inflexion are at C′ and E′, no longer exactly half-way between ends and middle.

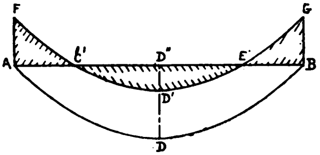


Fig. 255.

361. Students will do well to work at least one complicated example of a uniform beam fixed at the ends with symmetrical loading. If the beam varies in section, but is symmetrical—that is, if at two points equally distant from the two ends the sections are the same, and if the loading is symmetrical, first obtain the *m* curve graphically and measure *m* at a number of equi-distant points. Thus, for some beam of 20 feet long, let us suppose that the values of *m* as measured are given in the following table. We need not use E, as we will suppose it to be constant.

Distance from end in feet.	$-m$ in ton-feet.	Values of I in inches to the 4th power.	$\frac{m}{I}$	$\frac{1}{I}$	M.
1	15	500	·03	·002	17·85
3	25	300	·0833	·00333	7·85
5	35	250	·140	·004	− 2·15
7	40	320	·125	·003125	− 7·15
9	44	360	·122	·002778	−11·15
11	44	360	·122	·002778	−11·15
13	40	320	·125	·003125	− 7·15
15	35	250	·140	·004	− 2·15
17	25	300	·0833	·00333	+ 7·85
19	15	500	·03	·002	+17·85
		Total... ...	1·0006	·03046	
		Average ...	0·10006	·003046	

We see that the area of the $\frac{M}{I}$ curve is to be zero, and if m_1 is the unknown bending moment applied at each end to fix it, $M = m + m_1$; so that the average value of $\frac{m + m_1}{I}$ must be zero, or the average value of $-\frac{m}{I}$ must be equal to m_1 multiplied by the average value of $\frac{1}{I}$. Hence m_1 is equal to the average value of $\frac{m}{I}$ divided by the average value of $\frac{1}{I}$.

In the above case this is 0·10006 ÷ 0·003046, so that m_1 = 32·85 ton-feet; and M = 32·85 + m (algebraical sum) is the true bending moment everywhere.

362. The shearing force diagram in the symmetrical cases is the same whether a beam is fixed or only supported at the ends; and as $\frac{dM}{dx}$ is not altered by the fixing, the shearing force is everywhere the same in the beam, and the deflection due to shear is the same.

The solution just given is applicable to a beam of which the I of every cross-section is settled beforehand in any arbitrary manner, so long as I and the loading are symmetrical on the two sides of the middle. Let us give to I such a value that the beam shall be of **uniform strength** everywhere; that is, that $\frac{M}{I} z = f \ldots (2)$, wher z is the greatest distance of any point in the section from the neutral axis on the compression or tension side, and f is the constant maximum stress in compression or tension to which the material is subjected in every section. Taking $z = \frac{1}{2} d$, where d is the depth of the beam, (2) becomes $\frac{M}{I} d = \pm 2f \ldots (3)$, the + sign being taken over parts of the beam where M is positive, the − sign when M is negative. As the area of the M/I curve from end to end of the beam is to be zero, and $M/I = 2f/d$, we see that the area of a curve showing everywhere the value of $1/d$ ought to be zero. $\int \pm \frac{2f}{d} dx = 0 \ldots (4)$, the positive sign being taken from the ends of the beam to the points of inflexion, and the negative sign being taken between the two points of inflexion. We see, then, that to satisfy (4) we have only to solve the following problem. In the figure EATUCGE is a diagram whose ordinates represent the values of $\frac{1}{d}$ or the reciprocal of the depth of the beam which may be arbitrarily fixed, care being taken, however, that d is the same at points which are at the same distance from the centre. EFGE is a diagram of the values of what the bending moment m would be if the beam were merely supported at its ends. We are required to find a point P such that the area of EPTA = area of POO′T, where O is in the middle of the beam. When found, this point P is a point of inflexion, and PR is what we have called m_1. That is, m − PR is the real negative bending moment M at every place, or the diagram EFG must be raised vertically till R is at P to obtain the diagram of M. Knowing M and d, it is easy to find I through (3).

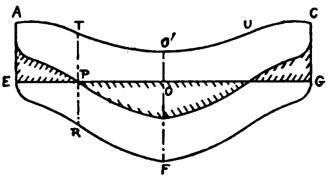


Fig. 256.

It is evident that if such a beam of **uniform strength** is also of **uniform depth,** the points of inflexion are half-way between the middle and the fixed **ends.**

363. In the most general way of loading, the bending moments required at the ends to fix them are different from one another.

Thus in Fig. 257 let AFCGBA be what the bending moment m would be if the beam were merely supported at its ends: let fixing moments m_1 = AH and m_2 = EB be applied at the ends, producing of themselves (*see* Art. 354) a bending moment diagram shown by AHEBA, or, if AD is x, then RD, the bending moment produced by the end couples is $m_1 + \frac{m_2 - m_1}{l}x$, if l is the length AB. Let us call this $m_1 + bx$. Hence M or RD − DS is

$$M = m_1 + bx - m \dots . (1),$$

or $$E\frac{d^2y}{dx^2} = \frac{m_1}{I} + \frac{b}{I}x - \frac{m}{I} \dots . (2).$$

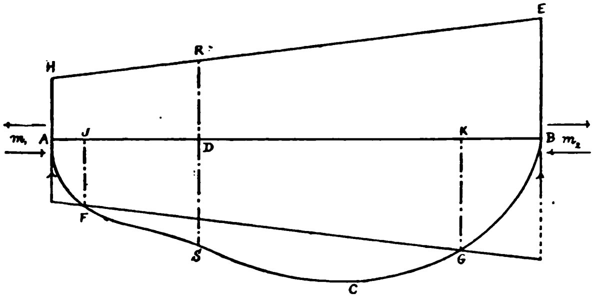


Fig. 257.

Now, as I is known, let A J_1 J_2 J_3 J_4 B A be drawn to represent the value of $\frac{1}{I}$ everywhere. Let it be integrated; that is, let such an ordinate as D K^1 (call it Y) represent the area of A J_1 J_2 D, and so obtain A K_1 K_2 K_3 B A, and let Y_1 be the value of this when x = A B. Also integrate $\frac{x}{I}$, and say that the ordinate of the resulting curve

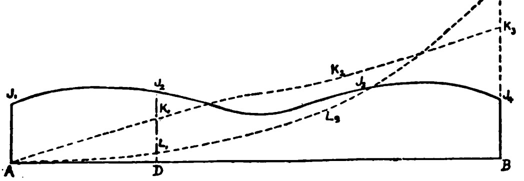


Fig. 258.

A L_1 L_2 L_3 B A is X, and let X_1 be the value of this when x = A B. Integrate also the curve whose ordinate is $\frac{m}{I}$ everywhere, calling the answer μ, and μ_1 its value for $x = \text{AB}$; then (2) becomes, since $\frac{dy}{dx}$ is the same at both ends,

$$0 = m_1 \text{Y}_1 + b\,\text{X}_1 - \mu_1 \ldots . (3).$$

Again integrating Y, X, and μ (but it is only necessary to get the areas over the whole length at once), calling the answers $\mathbf{Y}_1$, $\mathbf{X}_1$, and M_1, we see that, since at the two ends y is 0,

$$0 = m_1 \mathbf{Y}_1 + b\,\mathbf{X}_1 - \text{M}_1 \ldots . (4).$$

The two unknowns, m_1 and b, can now be found from (3) and (4).*

We give in Art. 365 an example completely worked out. The column headed m represents the bending moment in ton-feet due to a given set of loads, if the beam were merely supported at its ends. These values may be found, of course, by the graphical method of Art. 349. The values of I are supposed to be given us, and they are in inches to the fourth power.

364. We shall now consider a beam fixed at one end, B, and merely supported at the other, A, which is on the same level as B. If, as before, m is the bending moment at any place, D, which would exist if the beam were supported at each end, and if m_2 is the fixing couple, the true bending moment is

$$\text{EI}\frac{d^2y}{dx^2} = \frac{m_2}{l}x + m \ldots . (1).$$

Take, first, a simple case, a uniform beam uniformly loaded with w lb. per inch. It is easy to prove, as in Art. 339, that $m = -\frac{1}{2}wlx + \frac{1}{2}wx^2$, and

$$\text{EI}\frac{d^2y}{dx^2} = \frac{m_2}{l}x - \tfrac{1}{2}wlx + \tfrac{1}{2}wx^2 \ldots . (2).$$

Hence

$$\text{EI}\frac{dy}{dx} = \frac{m_2}{2\,l}x^2 - \tfrac{1}{4}wlx^2 + \tfrac{1}{6}wx^3 + c \ldots . (3),$$

---

* We have used the symbols μ, X, Y, μ_1, X_1, Y_1, M, $\mathbf{X}$, $\mathbf{Y}$, M_1, $\mathbf{X}_1$, $\mathbf{Y}_1$, fearing that students are still a little unfamiliar with the symbols of the calculus. Perhaps it would have been better to put the investigation in its proper form, and asked the student to make himself familiar with the usual symbol, instead of dragging in fresh symbols.

After (3) above, write as follows :—

$$\text{E}\left[\frac{dy}{dx}\right]_0^l = m_1\int_0^l \frac{dx}{\text{I}} + b\int_0^l \frac{x\,.\,dx}{\text{I}} - \int_0^l \frac{m}{\text{I}}dx = 0 \ldots . (4).$$

Again integrating between limits,

$$\text{E}\Big[\,y\,\Big]_0^l = m_1\int_0^l\int_0^l \frac{dx}{\text{I}} + b\int_0^l\int_0^l \frac{x\,.\,dx}{\text{I}} - \int_0^l\int_0^l \frac{m\,.\,dx}{\text{I}} = 0 \ldots . (5).$$

The integrations indicated in (4) and (5) being performed, the unknowns m_1 and b can be calculated and used in (1). The student must settle for himself which is the better course to take—to use the formidable-looking but really easily understood symbols of this note, or to introduce the letters whose *meaning* one is always forgetting.

and

$$\text{EI}\,y = \frac{m_2}{6\,l}x^3 - \tfrac{1}{12}\,wlx^3 + \tfrac{1}{24}\,wx^4 + cx \;.\;.\;.\;.\; (4).$$

We need not add a constant here, because y is 0 when x is 0. In (3) and (4) we insert the conditions that

$$\left.\begin{aligned} \frac{dy}{dx} &= o \text{ when } x = l \\ y &= o \text{ when } x = l \end{aligned}\right\} \quad (5),$$

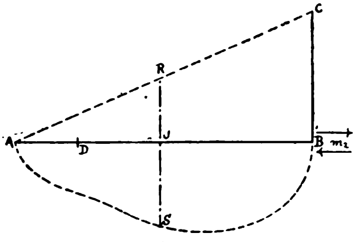


Fig. 259.

and we find

$$0 = \frac{m_2 l}{2} - \tfrac{1}{12}\,wl^3 + c \;.\;.\;.\;.\; (6),$$

$$0 = \frac{m_2 l}{6} - \tfrac{1}{24}\,wl^3 + c \;.\;.\;.\;.\; (7).$$

From these we find that $c = \frac{1}{48}\,wl^3$, $m_2 = \frac{1}{8}\,wl^2$, so that the true bending moment everywhere is given by (2), and the shape of the beam by (4). The slope of the beam at the supported place is $wl^3/48\,\text{EI}$.

If the loading is of any kind whatsoever, and if the section varies in any way, it is necessary to be able to integrate a function when given as the ordinate of a curve. We have, as before, in (1

$$\text{E}\frac{d^2y}{dx^2} = \frac{m_2}{l} \cdot \frac{x}{\text{I}} + \frac{m}{\text{I}} \;.\;.\;.\;.\; (8).$$

Let the integral of $\frac{x}{\text{I}}$ be called x, and the integral of $\frac{m}{\text{I}}$ be called μ, and let curves be drawn showing the values of x and μ. Let their values when $x = l$ be x_1 and μ_1. Then

$$\text{E}\frac{dy}{dx} = \frac{m_2}{l}\text{x} + \mu + c \;.\;.\;.\;.\; (9).$$

We can now integrate (9) again, and obtain $\text{E}y$; but if we do not actually wish to find the shape of the beam, we need only use the two ideas—first, that $y = 0$ when $x = 0$, and this shows that no new constant needs to be added; secondly, that $y = 0$ when $x = l$; and hence, if the areas of the whole x and μ curves from $x = 0$ to $x = l$ are $\mathbf{X}_1$ and M_1, we have

$$0 = \frac{m_2}{l}\mathbf{X}_1 + \text{M}_1 + cl \;.\;.\;.\;.\; (10).$$

Also, using in (9) the statement that $\frac{dy}{dx}$ is zero when $x = l$, we have

$$0 = \frac{m_2}{l} \mathrm{X}_1 + \mu_1 + c \; \ldots . (11).$$

From (10) and (11) we can find c and $\frac{m_2}{l}$. Using the value of m_2 so found, we find the bending moment everywhere given in (1).*

If in the above exercise we imagine the supported end at A to be raised a distance a above its original position, or that B has settled downwards by this amount, find (supposing the beam, fixed at one end, B, to be unloaded) what upward force at A (call it P) will cause it to rise through the distance a. We have only to assume that there is an additional supporting force of this amount, and that the bending moment due to it acts as well as the other—in fact, instead of (1), we have as the bending moment

$$\mathrm{E\,I}\frac{d^2y}{dx^2} = \frac{m_2}{l} x + m - \mathrm{P} x.$$

365. It is very important that the student should work out carefully such an exercise as the following. A beam is given with loads, and we know m and I at every place. The integrations to be performed are much the same whether it is a case of a beam fixed at the ends or fixed at one end and merely supported at the other, and therefore we give both. The two results are stated immediately after the table.

---

* Without using the letters μ, X, μ_1, X_1, etc., the above investigation is

$$\mathrm{E}\frac{dy}{dx} = \frac{m_2}{l}\int \frac{x}{\mathrm{I}} dx + \int \frac{m}{\mathrm{I}} dx + c \; \ldots . (9).$$

Integrating (9) between the limits 0 and l, and recollecting that y is zero at both limits,

$$0 = \mathrm{E}\Big[y \Big]_0^l = \frac{m_2}{l}\int_0^l \int_0 \frac{x}{\mathrm{I}} dx + \int_0^l \int_0 \frac{m}{\mathrm{I}} dx + cl \; \ldots . (10).$$

Also using in (9) the statement that $\frac{dy}{dx}$ is zero when $x = l$, we have

$$0 = \mathrm{E}\left[\frac{dy}{dx}\right]^l = \frac{m_2}{l}\int^l \frac{x}{\mathrm{I}} dx + \int^l \frac{m}{\mathrm{I}} dx + c \; \ldots . (11).$$

The integrations in (10) and (11) being performed, the unknowns $\frac{m_2}{l}$ and c can be calculated. The true bending moment everywhere is what we started with, $m + \frac{m_2}{l} x$.

x	m	I	$\frac{1}{\text{I}}$	$\text{Y} = \int \frac{1}{\text{I}} dx$	$\frac{x}{\text{I}}$	$\text{X} = \int \frac{x}{\text{I}} dx$	$\frac{m}{\text{I}}$	$\mu = \int \frac{m}{\text{I}} dx$	M Beam Fixed at Ends.	M Beam Fixed at one End, Supported at the Other.
	−	+	+	+	+	+	−	−		
				·0						
·5	8	500	·00200		·00100		·01600		+ 22·30	− 7·13
				·00200		·00100		·01600		
1·5	15	450	·00222		·00333		·03330		+ 14·73	− 12·38
				·00422		·00433		·04930		
2·5	21	400	·00250		·00625		·05250		+ 8·16	− 16·64
				·00672		·01058		·10180		
3·5	25	350	·00286		·01001		·07150		+ 3·59	− 18·90
				·00958		·02059		·1733		
4·5	28	320	·00313		·01404		·08740		+ ·03	− 20·15
				·01271		·03467		·2607		
5·5	32	300	·00333		·01826		·1066		− 4·54	− 22·41
				·01604		·05289		·3673		
6·5	34	300	·00333		·02159		·1132		− 7·11	− 22·66
				·01937		·07448		·4805		
7·5	36	320	·00313		·02340		·1123		− 9·67	− 22·92
				·02250		·09788		·5928		
8·5	37	350	·00286		·02431		·1058		− 11·24	− 22·18
				·02536		·1222		·6986		
9·5	35	400	·00250		·02375		·0875		− 9·81	− 18·43
				·02786		·1459		·7861		
10·5	32	400	·00250		·02625		·0800		− 7·38	− 13·69
				·03036		·1722		·8661		
11·5	31	400	·00250		·02875		·0775		− 6·95	− 10·94
				·03286		·2009		·9436		
12·5	30	380	·00263		·03288		·0789		− 6·51	− 8·20
				·03549		·2338		1·0225		
13·5	28	360	·00278		·03753		·0774		− 5·08	− 6·46
				·03827		·2713		1·0999		
14·5	26	330	·00303		·04393		·0788		− 3·65	− ·71
				·04130		·3153		1·1787		
15·5	24	300	·00333		·05161		·0799		− 2·22	+ 3·03
				·04463		·3669		1·2586		
16·5	18	330	·00303		·05000		·0545		+ 3·22	+ 10·78
				·04766		·4169		1·3131		
17·5	12	380	·00263		·04602		·0316		+ 8·65	+ 18·52
				·05029		·4629		1·3447		
18·5	5	400	·00250		·04625		·0125		+ 15·08	+ 27·26
				·05279		·5092		1·3572		
19·5	4	500	·00200		·03900		·0080		+ 15·52	+ 30.01
				·05479		·5482		1·3652		
			$\text{Y}_1 =$	·5748	$\text{X}_1 =$	4·062	$\text{M}_1 =$	15·276		

Beam Fixed at the Ends.—$\mu_1 + m_1 \mathrm{Y}_1 + \mathrm{P}\,\mathrm{X}_1 = 0$, where m the bending moment where $x = 0$, m_2 where $x = l$, $\mathrm{P} = \frac{m_2 -}{l}$ or $-1\cdot365 + \cdot0548\, m_1 + \cdot548\,\mathrm{P} = 0$; also $-\mathrm{M}_1 + m_1 \mathbf{y}_1 + \mathrm{P}\,\mathbf{x}_1 =$ or $-15\cdot276 + \cdot5748\, m_1 + 4\cdot062\,\mathrm{P} = 0$. Hence, $m_1 = 30\cdot$ $\mathrm{P} = -\cdot5673$, $m_2 = 19\cdot23$. Adding $\mathrm{P}\,x$ to m, we find the t bending moment M given in the tenth column of the table. It w be found that if every value of M be divided by the correspondi value of I, the algebraic sum for the whole beam is $-\cdot0$ instead of 0—a very close approximation to accuracy. T student may easily proceed to find the shape of the beam.

Beam Fixed at One End.—Where $x = l$ and the bendi moment is m_2, supported merely at the other, where $x = 0$,

$$-\mu_1 + \mathrm{P}\,\mathrm{X}_1 + c = 0,$$

where c is $\mathrm{E}\frac{dy}{dx}$ at $x = 0$, or $-1\cdot365 + \cdot5482\,\mathrm{P} + c = 0$;

$$-\mathrm{M}_1 + \mathrm{P}\,.\,\mathbf{x}_1 + cl = 0,$$

$$\text{or} \quad -15\cdot276 + 4\cdot062\,\mathrm{P} + 20\,c = 0.$$

Hence $\mathrm{P} = \frac{m_2}{l} = 1\cdot744$, or $m_2 = 34\cdot88$, and $c = \cdot4091$. Add $\mathrm{P}\,x$ to m everywhere, we find the true bending moment M give the eleventh column of the table. The student may easily proc to find the shape of the beam.

366. As we have proved (Art. 357), since

$$\frac{dy}{dx} = i, \quad \frac{di}{dx} = \frac{\mathrm{M}}{\mathrm{E\,I}}, \quad \frac{d\mathrm{M}}{dx} = \mathrm{S}, \quad \frac{d\mathrm{S}}{dx} = w,$$

we have a succession of curves which may be obtained fr knowing the shape of the beam y by differentiation, or which m be obtained from knowing w, the loading of the beam, by integr tion. Knowing w, there is an easy graphical rule for finding knowing $\frac{\mathrm{M}}{\mathrm{E\,I}}$, we have the same graphical rule for finding Some rules that are obviously true in the w to M construction a need no mathematical proof may at once be used without math matical proof, in applying the analogous rule from $\frac{\mathrm{M}}{\mathrm{E\,I}}$ to y. Th the area of the $\frac{\mathrm{M}}{\mathrm{E\,I}}$ curve between the ordinates x_1 and x_2 is t increase of i from x_1 to x_2, and tangents to the curve showing t shape of the beam at x_1 and x_2 meet at a point which is vertical in a line with the centre of gravity of the portion of area the curve in question. The whole area of the $\frac{\mathrm{M}}{\mathrm{E\,I}}$ curve in span H J is equal to the increase in $\frac{dy}{dx}$ from one end of the span the other, and the tangents to the beam at its ends H J meet in point P, which is in the same vertical as the centre of gravity of the whole $\frac{M}{EI}$ curve. These two rules may be taken as the starting-point for a treatment of the most difficult problems in beams by graphical methods.

If the vertical from this centre of gravity is at the horizontal distance HG from H and GJ from J, then P is higher than H by the amount HG × i_H, the symbol i_H being used to mean the slope at H; J is higher than P by the amount GJ × i at J. Hence J is higher than H by the amount $HG \cdot i_H + GJ \cdot i_J$, a relation which may be useful when conditions as to the relative heights of the supports are given, as in continuous beam problems.

367. Theorem of Three Moments.—For some time railway engineers, instead of using separate girders for the spans of a bridge, fastened together contiguous ends to prevent their tilting up, and so made use of what are called **continuous girders.** It is easy to show that if we can be absolutely certain of the positions of the points of support, continuous girders are much cheaper than separate girders. Unfortunately a comparatively small settlement of one of the supports alters completely the condition of things. In many other parts of applied mechanics we have the same difficulty in deciding between cheapness with some uncertainty and a greater expense with certainty. Thus there is much greater uncertainty as to the nature of the forces acting at riveted joints than at hinged joints, and therefore a structure with hinged joints is preferred to the other, although, if we could be absolutely certain of our conditions, an equally strong riveted structure might be made which would be much cheaper.

Students interested in the theory of continuous girders will do well to read a paper published in the "Proceedings of the Royal Society," cxcix., 1879, where they will find a graphical method of solving the most general problems. We shall take here, as a good example of the use of the calculus, a uniform girder resting on supports at the same level, with a uniform load distribution on each span. Let ABC be the centre line of two spans, the girder originally straight, supported at A, B, and C. The distance from A to B is l_1 and from B to C is l_2, and there are any kinds of loading in the two spans. Let A, B, and C be the bending moments at A, B, and C respectively, counted positive if the beam is convex upwards.

At the section at P at the distance x from A let m be what the bending moment would have been if the girder on each span were quite separate from the rest. We have already seen that by introducing couples m_2 and m_1 at A and B (tending to make the beam convex upwards at A and B) we made the bending moment at P really become what is given in Art. 363. Our $m_2 = A$, $m_1 = B$, and hence the bending moment at P is

$$m + A + x\frac{B - A}{l_1} = EI\frac{d^2y}{dx^2} \dots (1),$$

where m would be the bending moment if the beam were merel supported at the ends; and the supporting force at A is lessened b the amount

$$\frac{\mathrm{A} - \mathrm{B}}{l_1} \;.\;.\;.\;.\; (2).$$

Assume EI constant and integrate with regard to x, and we hav

$$\int m \,.\, dx + \mathrm{A}x + \tfrac{1}{2}x^2\frac{\mathrm{B} - \mathrm{A}}{l_1} + c_1 = \mathrm{EI} \,.\, \frac{dy}{dx} \;.\;.\;.\;.\; (3).$$

Using the sign $\int\int m \,.\, dx \,.\, dx$ to mean the integration of th curve representing $\int m \,.\, dx$, we have

$$\int\int m \,.\, dx \,.\, dx + \tfrac{1}{2}\mathrm{A}x^2 + \tfrac{1}{6}x^3\frac{\mathrm{B} - \mathrm{A}}{l_1} + c_1 x + e = \mathrm{EI} \,.\, y \;.\;.\;.\;.\; (4$$

As y is 0 when $x = 0$ and it is evident that $\int\int m \,.\, dx \,.\, dx =$ when $x = 0$, e is 0. Again, $y = 0$ when $x = l_1$. Using t symbol μ_1 to indicate the sum $\int\int m \,.\, dx \,.\, dx$ over the whole spa

$$\mu_1 + \tfrac{1}{2}\mathrm{A}\,l_1^2 + \tfrac{1}{6}\,l_1^2\,(\mathrm{B} - \mathrm{A}) + c_1 l_1 = 0 \;.\;.\;.\;.\; (5).$$

From (3) let us calculate the value of $\mathrm{EI}\,\frac{dy}{dx}$ at the point B, a let us use the letter a_1 to mean the area of the m curve over t span, or $\int_0^{l_1} m \,.\, dx$; so that $\mathrm{EI}\,\frac{dy}{dx}$ at B is

$$a_1 + \mathrm{A}\,l_1 + \tfrac{1}{2}\,l_1\,(\mathrm{B} - \mathrm{A}) + c_1 \;.\;.\;.\;.\; (6).$$

But at any point Q of the second span, if we had let $\mathrm{BQ} = x$, v should have had the same equations as (1), (3), and (4), using t letters B for A and C for B and the constant c_2. Hence, makin this change in (3), and finding $\mathrm{EI}\,\frac{dy}{dx}$ at the point B where $x =$ we have (6) equal to c_2 or

$$c_2 - c_1 = a_1 + \mathrm{A}\,l_1 + \tfrac{1}{2}\,l_1\,(\mathrm{B} - \mathrm{A}) \;.\;.\;.\;.\; (7),$$

and instead of (5) we have

$$\mu_2 + \tfrac{1}{2}\,\mathrm{B}\,l_2^2 + \tfrac{1}{6}\,l_2^2\,(\mathrm{C} - \mathrm{B}) + c_2 l_2 = 0 \;.\;.\;.\;.\; (8).$$

Subtracting (5) from (8), after dividing by l_1 and l_2, we have

$$c_2 - c_1 = \frac{\mu_1}{l_1} - \frac{\mu_2}{l_2} + \tfrac{1}{2}\mathrm{A}\,l_1 - \tfrac{1}{2}\mathrm{B}\,l_2 + \tfrac{1}{6}\,l_1\,(\mathrm{B} - \mathrm{A}) - \tfrac{1}{6}\,l_2\,(\mathrm{C} - \mathrm{B}) \;.\;.\;.\;.\; (9).$$

The equality of (7) and (9) is

$$\mathrm{A}\,l_1 + 2\,\mathrm{B}\,(l_1 + l_2) + \mathrm{C}\,l_2 = 6\left(\frac{\mu_1}{l_1} - a_1 - \frac{\mu_2}{l_2}\right) \ldots . (10),$$

an equation connecting A, B, and C, the bending moments at three consecutive supports. If we have any number of supports, and at the end ones we have the bending moments 0 because the girder is merely supported there, or if we have two conditions given which will enable us to find them in case the girder is fixed or partly fixed, note that by writing down (10) for every three consecutive supports we have a sufficient number of equations to determine all the bending moments at the supports.

Example.—Let the loads be w_1 and w_2 per unit length over two consecutive spans of lengths l_1 and l_2. Then

$$m = -\tfrac{1}{2}\,wlx + \tfrac{1}{2}\,wx^2, \quad \int m\,.\,dx = -\tfrac{1}{4}\,wlx^2 + \tfrac{1}{6}\,wx^3,$$

and $a_1 = -\dfrac{w_1}{12}\,l_1^3$, $\displaystyle\iint m\,.\,dx\,.\,dx = -\tfrac{1}{12}\,wlx^3 + \tfrac{1}{24}\,wx^4,$

and $\mu_1 = -\tfrac{1}{24}\,w_1 l_1^4, \quad \mu_2 = -\tfrac{1}{24}\,w_2 l_2^4.$

Hence $\dfrac{\mu_2}{l_2} + a_1 - \dfrac{\mu_1}{l_1}$ becomes $-\tfrac{1}{24}\,w_2 l_2^3 - \dfrac{w_1}{12}\,l_1^3 + \tfrac{1}{24}\,w_1 l_1^3,$

or $-\tfrac{1}{24}\,(w_2 l_2^3 + w_1 l_1^3),$

and hence the theorem becomes in this case

$$\mathrm{A}\,l_1 + 2\,\mathrm{B}\,(l_1 + l_2) + \mathrm{C}\,l_2 = \tfrac{1}{4}\,(w_2 l_2^3 + w_1 l_1^3) \ldots . (10).$$

If the spans are similar and similarly loaded, then

$$\mathrm{A} + 4\,\mathrm{B} + \mathrm{C} = \tfrac{1}{2}\,wl^2 \ldots . (11).$$

Case 1.—A uniform and uniformly loaded beam rests on three equidistant supports. Here $\mathrm{A} = \mathrm{C} = 0$ and $\mathrm{B} = +\tfrac{1}{8}wl^2$. $m = -\tfrac{1}{2}w\,(lx - x^2)$, and hence the bending moment at a point P distant x from A is $-\tfrac{1}{2}w\,(lx - x^2) + 0 - \dfrac{x}{l}\,\tfrac{1}{8}\,wl^2$. The supporting force at A is lessened from what it would be if the part of the beam A B were distinct by the amount shown in (2) $\dfrac{\mathrm{A} - \mathrm{B}}{l}$ or $\tfrac{1}{8}\,wl$. It would have been $\tfrac{1}{2}\,wl$, so now it is really $\tfrac{3}{8}\,wl$ at each of the end supports, and as the total load is $2\,wl$, there remains $\tfrac{10}{8}\,wl$ for the middle support. (See also *Example* on next page.)

Case 2.—A uniform and uniformly loaded beam rests on four equidistant supports, and the bending moments at these supports are A, B, C, D. Now, $\mathrm{A} = \mathrm{D} = 0$, and from symmetry $\mathrm{B} = \mathrm{C}$. Thus (11) gives us $0 + 5\,\mathrm{B} = \tfrac{1}{2}\,wl^2$ or $\mathrm{B} = \mathrm{C} = \tfrac{1}{10}\,wl^2$. If the span A B had been distinct, the first support would have had the load $\tfrac{1}{2}\,wl$; it now has $\tfrac{1}{2}\,wl - \tfrac{1}{10}\,wl$ or $\tfrac{4}{10}\,wl$. The supporting force at D is also $\tfrac{4}{10}\,wl$. The other two supports divide between them the remainder of the total load, which is altogether $3\,wl$, and so each receives $\frac{11}{10}wl$. The supporting forces are then $\frac{4}{10}wl$, $\frac{11}{10}wl$, $\frac{11}{10}wl$, and $\frac{4}{10}wl$.

Exercise.—A student ought to work at least one exercise like the following:—If a beam A B C has any kind of loading, and varies in section in any way, and is supported at three places, A and C, on the same level, B, on a level b inches below A or C, first find the diagram of bending moment and the deflection y everywhere of the beam, assuming the support B not to exist. Find, in particular, y_1, the deflection at the point B. Now consider a new problem—the same beam supported at two places, and with only an upward force at B. Find what the force B must be to cause a deflection upwards, $y_1 - b$, at B, and what the upward deflection z everywhere is. This force B is evidently the supporting force at B in the real problem, and the deflection in the real problem everywhere is $y - z$.

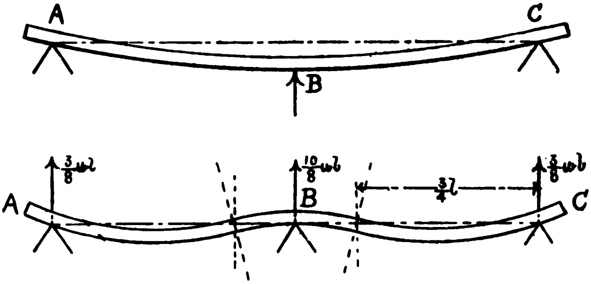


Fig. 260.

Example.—Uniform loading w per inch on A B and on B C, each of length l; beam of uniform section, the supports all on same level. (1) If prop B is absent, the deflection at B would be (Art. 339) $\frac{5}{384}\frac{2wl\,.\,(2l)^3}{\text{EI}}$. Call this δ. Each of the supporting forces is wl. (2) Beam of length $2l$, supported at the ends. If an upward force B produces an upward deflection, we know that $\delta = \frac{1}{48}\frac{\text{B}}{\text{EI}}(2l)^3$. Hence we have $\frac{1}{48}\frac{\text{B}\,(2l)^3}{\text{EI}} = \frac{5}{384}\frac{2wl\,.\,(2l)^3}{\text{EI}}$, and $\text{B} = \frac{5}{4}wl$—that is, when a uniform and uniformly loaded beam is on three equidistant supports, the supporting forces are $\frac{3}{8}wl$, $\frac{10}{8}wl$, and $\frac{3}{8}wl$.

CHAPTER XXI.

BENDING AND CRUSHING.

368. Stress over a Section.—When any portion of a column or beam or arch on one side of a section, B C, is acted upon by loads and supporting forces, we can generally find one force, representing the resultant of the stresses at the section, which will balance them all. If, instead of a force, we merely get a couple, then the section is exposed solely to bending moment, and we know now how to find the effect of this. If the force is parallel to the section, then we know that the section is either exposed to mere shearing strain or shearing and bending, as in a horizontal beam with vertical loads; but if the force is inclined to the section, there will usually be shearing and bending, and besides this a uniform distribution of compression or extension all over the section. In practice we generally find that compression and bending alone have to be considered. Thus, if B C (Fig. 261) is the edge view of the section of a structure, O being a line at right angles to the paper through its centre of gravity, and if F is the resultant of all the forces supposed in the plane of the paper which act on the structure to the right of the section, let P and S be the resolved parts of F normal to and tangential to the section; then S is balanced by an equal and opposite shearing force which must be exerted by the material to the left. P is a compressive load which is spread uniformly over the section, producing a compressive stress P/A if A is its area; but besides this we have in the section the varying compressive stress on the B side of O and the tensile stress on the C side of O, which the bending moment P . O D produces. In fact, the compressive stress at any place which is at the distance y from O on the compression side is $\frac{P}{A} + \frac{P \cdot OD}{I} y \ldots .$ (1).

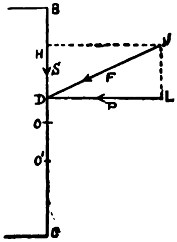


Fig. 261.

The student ought to draw a slice between two parallel cross-sections like B C near one another, and draw the change of shape, first making it uniformly thinner because of P/A, and then making it wedge-shaped. We have, in fact, the wedge-shape $c'e'f'd'$ of Fig. 207, where $H'J' = HJ$, and in addition we have to imagine $e'f'$ moved parallel to itself in the direction $c'd'$; but students must draw this for themselves. They will see that the result may be—compression everywhere, but much more at B than at C; or compression everywhere and just no stress at C; or compression at B, tension at C, and a neutral line somewhere between. Most people have the habit of calling the line through O the neutral line of the section, although it is the neutral line only when the component P is zero.

The proof of the above statement is this: Assume that a plane section remains plane, it follows, as it did in Art. 319, that there is a neutral line, say at O′, at right angles to the paper. Let any point H be at the distance z from O′; the compressive stress there is cz say, where c is some constant. The force on a small area a there, is cza. Then $P = c\Sigma za$ ... (2) and $P \cdot OD = c\Sigma za \cdot y$ ... (3) because, if $OH = y$, $za \cdot y$ is the moment of za about O. Now $z = y + OO'$, so that equation (2) is

$$P = c \cdot OO' \Sigma a + c\Sigma y \cdot a = c \cdot OO' \cdot A \ldots (4),$$

because $\Sigma ya = 0$; and $P \cdot OD = c\Sigma y^2 a + c\Sigma OO' \cdot y \cdot a = cI \ldots (5)$ if I is the moment of inertia of the area about O. We see that cy from (5) is $\frac{P \cdot OD}{I} y$ and $c \cdot OO'$ from (4) is $\frac{P}{A}$; but the sum of these is cz, the compressive stress, and so we have proved (1).

If the section is rectangular, the dimension at right angles to the paper being 1, and BC being d, then $I = 1 \times d^3/12$ and $A = 1 \times d$. The compressive stress is least at C where $y = -\frac{1}{2}d$, or by (1), the compressive stress at C is $\frac{P}{d} - \frac{6P \cdot OD}{d^2} \ldots (6)$. As OD gets greater, the compressive stress at C becomes less until there is a value of OD which just causes C to have no stress; a greater value of OD than this would create tensile stress at C. We usually take it that in a masonry joint there ought to be no tensile stress, and hence in a masonry joint the limiting value of OD is given by putting (6) equal to zero; that is, $6 \cdot OD = d$.

Hence D must fall within the middle third of the masonry joint, BC, if there is to be stability. This is the fundamental rule in the design of **arches and buttresses.** Another conditio of stability for a masonry joint is that S shall not exceed th frictional resistance, or the angle JDL must be less than t angle of repose (see Art. 96) for the materials.

The above rule is very generally useful in machine desig but we need not give many examples. Fig. 262 is a crane-hook. The section at B C is not usually elliptic; rather like an ellipse, with the end at B blunter than that at C. If O is a line at right angles to the paper through the centre of gravity of the section, and W is the load, the stress at any place is that due to a bending moment W . O D, together with a tensile stress due to W being spread uniformly over the section.

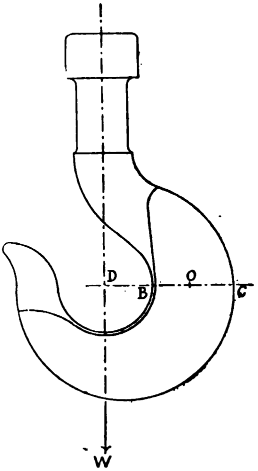


Fig. 262.

When a weight W hangs from a bracket as in Fig. 263 the strength at any section, such as O, is merely calculated from the bending moment W . D B, because when the distance D B is considerable the stresses due to the bending are usually many times greater than those due to W, divided by the area of cross-section.

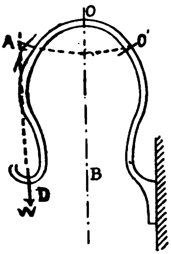


Fig. 263.

369. Shear Stress in Beams.—Let the distance measured from any section of a beam, say at O (Fig. 264), to the section at A be x, and let $OB = x + \delta x$. Let the bending moment at C′ A C be M, and at D′ B D be $M + \delta M$. Let A C be the compressive side of C C′. Let O A B (Fig. 264) and A A (Fig. 265) represent the neutral surface. We want to know the tangential or shear stress f_s at E on the plane C A C′. Now it is known that this is the same as the tangential stress in the direction E F on the plane E F, which is at right angles to the paper, and parallel to the neutral surface at A B. Consider the equilibrium of the piece of beam E C D F, shown

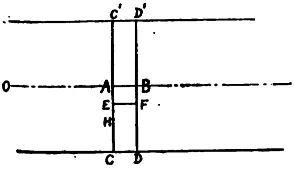


Fig. 264.

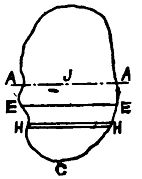


Fig. 265.

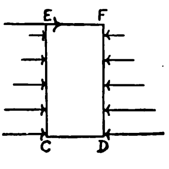


Fig. 266.

in Fig. 265 as E C E, and also shown magnified in Fig. 266. We have indicated only the forces which are parallel to the neutral surface, or at right angles to the sections. *The total pushing forces on* D F *are greater than the total pushing forces on* C E, *the tangential forces on* E F *making up for the difference.* We have only to state this mathematically, and we have solved our problem.

At a place like H in the plane C A C′, at a distance y from the neutral surface, the compressive stress is known to be $p = \frac{M}{I} y$; and if b is the breadth of the section there, shown as H H (Fig. 265), the total pushing force on the area E C E is

$$P = \int_{AE}^{AC} b \frac{M}{I} y \,.\, dy, \text{ or } P = \frac{M}{I} \int_{AE}^{AC} by \,.\, dy \;\ldots\ldots\ (1).$$

Observe that if b varies, we must know it as a function of y before we can integrate in (1). Suppose we call this total pushing force on E C by the name P, then the total pushing force on D F will be $P + \delta x \,.\, \frac{dP}{dx}$. The tangential force on E F is $f_s \times$ area of E F, or $f_s \,.\, \delta x \,.\, EE$, and hence

$$f_s \,.\, \delta x \,.\, EE = \delta x \,.\, \frac{dP}{dx}, \text{ or } f_s = \frac{1}{EE} \,.\, \frac{dP}{dx} \;\ldots\ldots\ (2).$$

Example.—**Beam of uniform rectangular section,** of constant breadth b and constant depth d. Then

$$P = \frac{12 M b}{b d^3} \int_{AE}^{\frac{1}{2} d} y \,.\, dy = \frac{12 M}{d^3} \left[\tfrac{1}{2} y^2 \right]_{AE}^{\frac{1}{2} d}, \quad P = \frac{6 M}{d^3} (\tfrac{1}{4} d^2 - AE^2),$$

and hence

$$f_s = \frac{6}{b\, d^3} (\tfrac{1}{4} d^2 - AE^2) \frac{dM}{dx} \;\ldots\ldots\ (3);$$

so that f_s is known as soon as M is known. As $\frac{dM}{dx}$ is the shearing force over the whole section, we may regard (3) as telling us what fraction of the total S there is on every square inch.

As to M, let us choose a case—say the case of **a beam supported at the ends, and loaded uniformly** with w lb. per unit length of the beam. We saw that in this case, x being distance from the middle, $-M = \frac{1}{8} w l^2 - \frac{1}{2} w x^2$. Hence $\frac{dM}{dx} = wx$, so that (3) is

$$f_s = \frac{6}{b d^3} (\tfrac{1}{4} d^2 - AE^2)\, wx \;\ldots\ldots\ (4).$$

If we like, we may now use the letter y for the distance A E; and we see that at any point of this beam x inches, measured horizontally from the middle, and y inches from the neutral axis, the shear stress is

$$f_s = \frac{6w}{bd^3}\left(\tfrac{1}{4}d^2 - y^2\right)x \ldots . \text{(5)}.$$

Observe that where $y = 0$ the shear stress is greater than at any other point of the section—that is, at points in the neutral axis. The shear stress is zero at C. Again, the end sections of the beam have greatest shear. A student has much food for thought in this result (5). It is interesting to find the directions and amounts of the principal stresses at every point of the beam—that is, the interfaces at right angles to one another at any point across which there is only compression or only tension without tangential stress.

We have been considering a rectangular section. The student ought to work exercises on other sections as soon as he is able to integrate by with regard to y in (1) where b is any function of y.

He will notice that $\int_{\mathrm{AE}}^{\mathrm{AC}} by \, . \, dy$ is equal to the area of EHCHE (Fig. 265) multiplied by the distance of its centre of gravity from A A. Taking a flanged section, the student will find that f_s is small in the flanges, and gets greater in the web. Even in a rectangular section f_s became rapidly smaller further out from the neutral axis; but now to obtain it we must divide by the breadth of the section, and this breadth is comparatively so great in the flanges that there is practically no shearing, the shear being confined to the web; whereas in the web itself f does not vary very much. The student already knows that it is our usual custom to calculate the areas of the flanges, or top and bottom booms of a girder, as if they merely resisted compressive and tensile forces, and the web or the diagonal bracing as if it merely resisted shearing. He will note that the shear in a section is great only where $\frac{d\mathrm{M}}{dx}$, or rather $\frac{d}{dx}\left(\frac{\mathrm{M}}{\mathrm{I}}\right)$, is great. But, inasmuch as in Art. 357 we saw that $\frac{d\mathrm{M}}{dx} = \mathrm{s}$, the total shearing force at the section, there is nothing very extraordinary in finding that the actual shear stress anywhere in the section depends upon its value.

Deflection of Beams due to Shear.—If a bending moment M acts at a section of a beam, the part of length δx gets the strain-energy $\frac{1}{2}\mathrm{M}^2\,\delta x/\mathrm{E\,I}$, because $\mathrm{M}\,\delta x/\mathrm{E\,I}$ is the angular change, and therefore the whole strain-energy in a beam due to bending moment is

$$\frac{1}{2\,\mathrm{E}}\int\frac{\mathrm{M}^2}{\mathrm{I}} . \, dx \ldots . \text{(6)}.$$

If f is a shear stress, the shear strain-energy per unit volume is $f^2/2\text{N}$ . . . . (7), and by adding we can therefore find its total amount for the whole beam.

By equating the strain-energy to the loads multiplied by half the displacements produced by them we obtain interesting relations. Thus in the case of a beam of length l, of rectangular section, fixed at one end and loaded at the other with a load W at the distance x from the end, $\text{M} = \text{W}x$, and the energy due to bending is

$$\frac{1}{2\text{E}}\int_0^l \frac{\text{W}^2x^2}{\text{I}}.dx = \text{W}^2 l^3/6\,\text{EI} \;.\;.\;.\;.\; (8).$$

The above expression (5) gives for the shearing stress

$$f = \frac{1}{b}\,\frac{6}{d^3}\left(\tfrac{1}{4}\,d^2 - y^2\right)\text{W} \;.\;.\;.\;.\; (9).$$

The shear strain-energy in the elementary volume $b.\,\delta x.\,\delta y$ is $b.\,\delta x.\,\delta y.\,f^2/2\,\text{N}$. Integrating this with regard to y from $-\frac{1}{2}d$ to $+\frac{1}{2}d$, we find the energy in the slice between two sections to be $3\,\text{W}^2l.\,\delta x/5\,\text{N}\,b\,d$, so that the shear strain-energy in the beam is $3\,\text{W}^2l/5\,\text{N}\,b\,d$ . . . . (10).

If now the load W produces the deflection z at the end of the beam, the work done is $\frac{1}{2}\,\text{W}z$ . . . . (11). Equating (11) to the sum of (8) and (10) we find

$$z = \frac{\text{W}\,l^3}{3\,\text{EI}} + \frac{6}{5}\cdot\frac{\text{W}\,l}{\text{N}\,b\,d} \;.\;.\;.\;.\; (12).$$

Note that the first part of this due to bending is the deflection as calculated in Art. 339, *Example* 1. We believe that the other part due to shearing has never before been calculated.

370. Springs which Bend.—We consider the bending in springs of regular shape, such as spiral springs, later, in Art. 521. But it seems natural to consider certain irregularly shaped springs here. Let Fig. 267 show the centre line of a spring *fixed* at A, loaded at B with a small load W in the direction shown. To find the amount of yielding at B, the load and the deflection are supposed to be very small. Consider the piece of spring bounded by cross-sections at P and Q. Let $\text{PQ} = \delta s$, the length of the spring between B and P being called s.

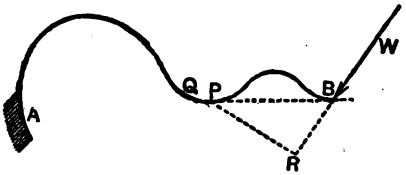


Fig. 267.

The bending moment at P is W . PR or W . x, if x is the length of the perpendicular from P upon the direction of W. Let BR be called y. Consider first that part of the motion of B which is due to the change of shape of QP *alone;* that is, imagine AQ to be perfectly rigid and PB a rigid pointer. The section at Q being fixed, the section at P gets an angular change equal to $\delta s \times$ the change of curvature there, or $\delta s \frac{M}{EI}$ or $-\frac{\delta s \, . \, Wx}{EI}$ . . . . (1), where E is Young's modulus and I is the moment of inertia of the cross-section. The motion of B due to this is just the same as if PB were a straight pointer; in fact, the pointer PB gets this angular motion, and the motion of B is this angle multiplied by the straight distance PB, or $\frac{\delta s \, . \, Wx}{EI} \, . \, PB$ . . . . (2). Now how much of B's motion is in the direction of W? It is its whole motion $\times \frac{PR}{PB}$ or $\times \frac{x}{PB}$, and hence B's motion in the direction of W is $\frac{\delta s \, . \, Wx^2}{EI}$ . . . (3). Similarly B's motion at right angles to the direction of W is $\frac{\delta s \, . \, W \, . \, xy}{EI}$ . . . . (4).

In the most general cases it is easy to work out the integrals of (3) and (4) numerically. We usually divide the whole length of the spring from B to A into a large number of equal parts so as to have all the values of δs the same, and then we may say (s being the whole length of the spring) that we have to multiply $\frac{s \, . \, W}{E}$ upon the average values of $\frac{x^2}{I}$ and $\frac{xy}{I}$ for each part. In a well-made spring, if b is the breadth of a strip at right angles to the paper and t its thickness, so that $I = \frac{1}{12} bt^3$, we usually have the spring equally ready to break everywhere, or $\frac{6 \, W x}{bt^2} = f$, a constant. When this is the case (3) and (4) become $\frac{2f \, . \, \delta s}{E} \, . \, \frac{x}{t}$ and $\frac{2f \, . \, \delta s}{E} \, . \, \frac{y}{t}$. And if the strip is constant in thickness, varying in breadth in proportion to x, then $\frac{2f \, . \, \delta s}{Et} \, . \, x$ is (3) and $\frac{2f \, . \, \delta s}{Et} \, . \, y$ is (4). If $\bar{x}$ and $\bar{y}$ are the x and y of the centre of gravity of the curve (see Art. 110), $\frac{2fs\bar{x}}{Et}$ is the total yielding parallel to W, and this is what we generally desire to know. $\frac{2fs \, . \, \bar{y}}{Et}$ is the total yielding at right angles to W.

371. Struts.—As we have already said, when a very short column is loaded (as it usually is) in such a way that the material is prevented from swelling laterally, it will withstand exceedingly great loads without fracture. It is only when the

length of a square or round prism is three or four tin
diameter that the material gets a chance of showing l
behaves under compressive stress; up to a length of abo
times its diameter its breaking load is now nearly the san
it remains the same for any length if slight lateral res
against bending are provided. In a tie-rod, want of unif
of stress causes yielding, and this is the same in a stru
in a tie the effect of yielding is to remedy the defect,
gets straighter, the stress better distributed, whereas in
local failure causes bending, instability, and a worse di
tion of stress. In Art. 243 we have discussed the beh
of material under uniform axial compressive stress merel
also in columns which are too short for mere axial comp
stress. We now know that if B C, Fig. 261, is a cross sec
a strut, and if the resultant load on the part of the st
one side of B C is F or P, not only have we the usually as
stress P/A over the section, but also the b
moment stress. What this may amount
pends upon O D; and this depends on two
—first, the want of accuracy in applying th
at the ends of the strut, a matter which can
taken into account in calculation unless we
how much error there is; secondly, the b
of the strut.

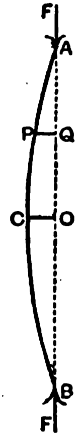


Fig. 268.

372. Bending of Struts.—Consider a str
fectly prismatic, of homogeneous material, i
weight neglected, the resultant force F at ea
passing through the centre of each end. Let
Fig. 268, show the centre line of the bent stru
$PQ = y$ be the deflection at P where $OQ = x$.
$= OB = l$; y is supposed everywhere small i
parison with the length $2l$ of the strut. L
Young's modulus for the material, and I th
moment of inertia of the cross-section every
about a line through the centre of gravity
section. Then the result of the following th
that the load F, which will produce bending, is $EI\pi^2/4$

Fy is the bending moment at P and $\frac{Fy}{EI}$ is the curvatur

Then as in Art. 339 the curvature being $-\frac{d^2y}{dx^2}$ we have

$$\frac{Fy}{EI} = -\frac{d^2y}{dx^2} \quad \ldots \text{ (1).}$$

Notice that when we choose to call $\frac{d^2y}{dx^2}$ the curvature of a curve, if the expression to which we put it equal is essentially positive, we must give such a sign to $\frac{d^2y}{dx^2}$ as will make it also positive. Now if the slope of the curve of Fig. 268 be studied as we studied the curve of Art. 338, we shall find that $\frac{d^2y}{dx^2}$ is negative from $x = 0$ to $x = 0\text{A}$, and as y is positive so that $\frac{\text{F}y}{\text{EI}}$ is positive, we must use $-\frac{d^2y}{dx^2}$ on the right-hand side.

If the student tries he will find that

$$y = a \cos. x \sqrt{\frac{\text{F}}{\text{EI}}} \quad . \; . \; . \; . \; (2)$$

satisfies (1) whatever value a may have. When $x = 0$ we see that $y = a$, so that the meaning of a is known to us; it is the deflection of the strut in the middle. The student is instructed to follow carefully the next step in our argument. When $x = l$, $y = 0$. Hence

$$a \cos. l \sqrt{\frac{\text{F}}{\text{EI}}} = 0 \quad . \; . \; . \; . \; (3).$$

How can this be true? Either $a = 0$, or the cosine is 0. Hence, *if bending occurs*, so that a has some value, *the cosine must be* 0. Now if the cosine of an angle is 0, the angle must be $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{5\pi}{2}$, etc. If we confine our attention to $\frac{\pi}{2}$, the condition that bending occurs is

$$l\sqrt{\frac{\text{F}}{\text{EI}}} = \frac{\pi}{2} \text{ or } \text{F} = \frac{\text{EI}\pi^2}{4l^2} \quad . \; . \; . \; . \; (4)$$

is the load which will produce bending. This is called **Euler's law of strength.** It is easy to see why we confine our attention to $\frac{\pi}{2}$ as it gives the least value of W. The meaning of the other cases is that y is assumed to be 0 one or more times between $x = 0$ and $x = l$, so that the strut has points of inflexion.

It will be found that the *complete* solution of any such equation as (1), which may be written $\frac{d^2y}{dx^2} + n^2y = 0$, is $y = a \cos. nx + b \sin. nx$ where a and b are arbitrary constants. a and b are chosen to suit the particular problem which is being solved. In the present case it is evident that as $y = 0$ when $x = l$ and also when $x = -l$ $0 = a \cos. nl + b \sin. nl$, $0 = a \cos. nl - b \sin. nl$, so that b is 0.

Now let us consider a strut **fixed at the ends.** The resultant force F at an end no longer acts through the centre of gravity of the end section. The solution is as before

$$y = a \cos. nx + b \sin. nx \; . \; . \; . \; . \; (5).$$

Differentiating we have

$$\frac{dy}{dx} = - na \sin. nx + nb \cos. nx \; . \; . \; . \; . \; (6).$$

Now $\frac{dy}{dx} = 0$ for $x = 0$, $x = l$, $x = - l$, and hence $0 = nb$, so that

$$b = 0 \text{ or } y = a \cos. nx \; . \; . \; . \; . \; (7).$$

Also $$0 = - na \sin. nl = na \sin. nl \; . \; . \; . \; . \; . \; (8).$$

Now (3) tells us that a is the deflection at c, where x is 0, and if a has any value, (4) tells us that sin. nl is 0—that is, $nl = \pi$, or

$$l \sqrt{F/EI} = \pi, \text{ or } F = \frac{EI\pi^2}{l^2} \; . \; . \; . \; . \; (9).$$

Hence if a strut is **fixed at both ends** the load which it will stand before bending is the same as for a strut of half the length hinged at the ends. In fact, if G C H is a strut fixed at the ends, its strength is like that of a strut K L hinged at K and L, or again like that of a strut G K L fixed at G and hinged at L. The strength to resist bending of a strut of length 2 l hinged at the ends may be calculated from (4); this is the strength of a strut of length 3 l, **hinged at one end and fixed at the other,** or of a strut of length 4 l fixed at both ends. It is only necessary, then, to remember the rules for struts hinged at the ends.

The load given by (4) will produce either very little or very much bending equally well, and we may take it that F given by (4) is the load which will break a strut if it breaks by bending. If f is the compressive stress which will produce rupture and A is the area of cross section, the load f A will break the strut by direct crushing, and we must take the smaller of the two answers. In fact, we see that f A is to be taken for short struts or for struts which are artificially protected from bending, and (4) is to be taken for long struts. Now, even when great care is taken we find that struts are neither quite straight nor homogeneous, nor is it easy to load them in the specified manner. Consequently, when loaded they deflect with even small loads,

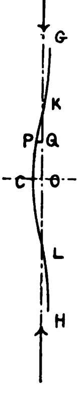


Fig. 269

and they break with loads less than either $f\text{A}$ or that given by (4). Curiously enough, however, when struts of the same section but of different lengths are tested, their breaking loads follow, with a rough approximation to accuracy, some rule as to length. Let us assume that as $\text{F} = f\text{A}$ for short struts and what is given in (4) for long struts, then the formula

$$\text{F} = \frac{f\text{A}}{1 + \frac{f\text{A}4l^2}{\text{EI}\pi^2}} \quad . \; . \; . \; . \; (10)$$

may be taken to be true for struts of all lengths because it is true both for short and for long ones. For if l is great we may neglect 1 in the denominator and our (10) is really (4); again, when l is small, we may regard the denominator as only 1, and so we have $\text{W} = f\text{A}$. We get in this way an empirical formula which is found to be fairly right for all struts. To put it in its usual form, let $\text{I} = \text{A}\,k^2$, k being the least radius of gyration of the section, then

$$\text{F} = \frac{f\text{A}}{1 + a\frac{l^2}{k^2}} \quad . \; . \; . \; . \; (11),$$

where a is $4f/\text{E}\,\pi^2$.

If F does not act truly at the centre of each end, but at the distance h from it, our end condition is that $y = h$ when $x = l$. This will be found to explain why struts not perfectly truly loaded break with a load less than what is given in (4). Students who wish to pursue the subject are referred to pages 464 and 513 of the *Engineer* for 1886, where initial want of straightness of struts is also taken account of.

When we consider, therefore, how the rule (11) has been arrived at, it is evident that it needs to be tested by practical experiment, the constants a for various materials and k^2 for various kinds of section being also determined by experiment. This has been done, and on the whole we feel fairly well satisfied when the rule is put in the following form:—For a *strut whose ends are hinged*, or a *column whose ends are not fixed*, the breaking load in pounds is equal to the breaking stress per square inch given in Table IX. multiplied by the area of cross section in square inches, and divided by $1 + n\text{B}$ where n is given in Table XI., and B is given in Table X.

Exercise for Advanced Students.—Test in the six cases of Table VIII. to what extent this rule agrees with (11).

TABLE IX.

	Breaking Stress, in pounds per square inch.
Cast Iron	80,000
Wrought Iron	36,000
Timber	7,200

TABLE X.

Value of B *for struts of the sections shown in Table XI. The first column gives the length of the strut divided by its least lateral dimension.*

Length divided by Lateral Dimension *d*.	B for Cast Iron.	B for Wrought Iron.	B for Strong Dry Timber.
10	0·748	0·132	1·0
15	1·68	0·300	3·6
20	3·00	0·532	6·4
25	4·64	0·832	10·0
30	6·76	1·200	14·4
35	9·20	1·632	19·6
40	12·00	2·132	25·6
45	18·72	3·332	40·0

EXERCISES.

1. Find the breaking load for a cylindrical strut of wrought iron 3 inches diameter and 10 feet long, supposing it to be not fixed at the ends. *Ans.*, 30·1 tons.

2. A hollow cast-iron pillar 10 feet long, and fixed at the ends, has an external diameter of 6 inches; what should be the thickness of the metal to carry a load of 30 tons, allowing a factor of safety of 8? *Ans.*, ·42 in.

3. What is the safe load for an angle iron, least breadth 3 inches and 7½ feet long, acting as a strut, firmly fixed at both ends? Factor of safety, 6. *Ans.*, 3·51 tons.

4. The diagonal brace of a Warren girder is 10 feet long, and is composed of two tee-bars 6″ × 3″ × ½″, placed back to back and riveted together. Find the maximum compressive working load which may be applied to it when the ends are firmly riveted to the boom. Use a factor of safety of 4. *Ans.*, 17·5 tons.

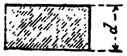
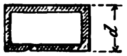
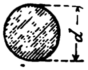
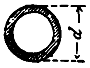
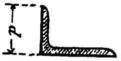
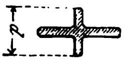
5. Section 4, Table VI., has flanges 4·02 inches broad, ·56 inch thick, depth over all 8 inches, web ·42 inch thick. The greatest moment of inertia about a line through its centre of gravity is 74·2, according to the rule of the table. Its least moment of inertia is about a line at right angles to the first, and is $\frac{\cdot 56 \times (4\cdot 02)^3}{12} \times 2 + \frac{6\cdot 88 \times (\cdot 42)^3}{12}$, or 6·1.

The area of the section is 7·39 square inch. The ultimate stress being taken as 45,000 lbs. per square inch, and the Young's modulus as 30×10^6 lbs. per square inch, the following are the breaking loads, according to Euler's theory, if it is the section of a strut fixed at the ends of the following lengths. For these loads to be withstood, it is necessary to carefully adjust the application of load at the ends so as to have absolutely no bending until the breaking loads are reached. $W = A f$ or $4 E I \pi^2/L^2$, if L is *whole* length of strut fixed at the ends; the lesser answer of the two to be taken.

L in inches	96	120	144	180	216	240	288
Breaking load in tons	148·5	148·5	148·5	99·5	69·1	56	38·9

TABLE XI.

Values of n for struts and pillars of the following sections :—

		Value of *n*.
Square of side *d*, or rectangle with smallest side *d* ...		1·00
Hollow rectangle, or square with thin sides		0·50
Circle, diameter *d*		1·33
Thin ring, external diameter *d*		0·66
Angle iron, smallest side *d*		2·00
Cruciform, smallest breadth *d*		2·00

If we want the breaking load for a strut whose ends are not hinged, it is necessary to find in what way it tends to bend, and to use the above rule regarding the strut as hinged at *two* points of contrary flexure. Thus in Fig. 270 the strut or column E is as strong as a strut hinged or rounded at both ends, whose length is only *a b*. The rule becomes—For *a strut fixed at both* *ends*, calculate by the above rule, but take n one fourth of what I have given in Table XI. For a strut one end of which is fixed and the other is only hinged; calculate the breaking load as if both its ends were hinged; then calculate as if both its ends were fixed, and take the mean value of the two answers.

373. Struts with Lateral Loads.—If the lateral loads are such that by themselves, and the necessary lateral supporting forces, they produce a bending moment which we shall call $\phi(x)$, then (1) Art. 372 becomes $Fy + \phi(x) = -EI\frac{d^2y}{dx^2}$. When we know the lateral loads we know $\phi(x)$, and it is quite easy to integrate. Thus let a strut be uniformly loaded laterally, as by centrifugal force or its own weight, and then $\phi(x) = \frac{1}{2}w(l-x)^2$ if w is the lateral load per unit length. We find it slightly more convenient to take $\phi(x) = \frac{1}{4}Wl\cos.\frac{\pi}{2l}x$, where W is the total lateral load. This is not a very different law. Hence

$$\frac{d^2y}{dx^2} + \frac{F}{EI}y + \frac{1}{4}\frac{Wl}{EI}\cos.\frac{\pi}{2l}x = 0 \;\ldots\ldots (1).$$

We find here that

$$y = -\frac{\frac{1}{4}Wl}{EI\frac{\pi^2}{4l^2} - F}\cos.\frac{\pi}{2l}x \;\ldots\ldots (2).$$

Observe that when $F = 0$ this gives the shape of the beam. The deflection in the middle is

$$y_1 = \frac{\frac{1}{4}Wl}{EI\frac{\pi^2}{4l^2} - F} \;\ldots\ldots (3),$$

and the greatest bending moment μ is $\mu = Fy_1 + \frac{1}{4}Wl$, or

$$\mu = \frac{1}{4}Wl \cdot \frac{EI\pi^2}{4l^2} \Big/ \left(\frac{EI\pi^2}{4l^2} - F\right) \ldots\ldots (4).$$

If $W = 0$, and if μ has any value whatever, the denominator of (4) must be 0. Putting it equal to 0, we have Euler's law for the strength of struts, which are so long that they bend before breaking. If Euler's value of F be called U, or $U = EI\pi^2/4l^2$, (4) becomes

$$\mu = \frac{1}{4}Wl\frac{U}{U - F} \;\ldots\ldots (5).$$

If z_c is the greatest distance of a point in the section from the neutral line on the compressive side, or if $I \div z_c = Z$, the least strength modulus of the section, and A is the area of cross-section, and if f is the maximum compressive stress to which any part of the strut is subjected, $\frac{\mu}{Z} + \frac{F}{A} = f$. Using this expression, if β stands for $\frac{U}{A}$ (that is, Euler's breaking load per square inch of section), and if w stands for $\frac{\mathrm{F}}{\mathrm{A}}$ (the true breaking load per inch of section), then

$$\left(1 - \frac{w}{f}\right)\left(1 - \frac{w}{\beta}\right) = \frac{\mathrm{w}\,l}{4 f \mathrm{z}} \dots . \ (6).$$

This formula is not difficult to remember. From it w may be found.

EXERCISES.

1. Every point in an iron or steel **coupling rod** of length $2b$ inches describes a radius of r inches. Its section is rectangular, d inches in the plane of the motion, and b at right angles to this. We may take $\mathrm{w} = lbdrn^2 \div 62{,}940$ in pounds where n = number of revolutions per minute. Take it as a strut hinged at both ends for both directions in which it may break. (1) For bending in the direction in which there is no centrifugal force where I is $\frac{db^3}{12}$. Euler's rule gives $\frac{\mathrm{E}\,db^3\pi^2}{48\,l^2}$. Now we shall take this as the endlong load, which will cause the strut to break in the other way of bending also, so as to have it equally ready to break both ways. (2) Bending in the direction in which bending is helped by centrifugal force. Our w is the above quantity divided by bd, or, taking $\mathrm{E} = 3 \times 10^7$, $w = 6{\cdot}17 \times \frac{b^2}{l^2} \times 10^6$. Taking the proof stress f for the steel used as 20,000 lbs. per square inch (remember to keep f low, because of reversals of stress), and recollecting the fact that I in this other direction is $\frac{bd^3}{12}$, we have (6) becoming

$$8{\cdot}4 \times 10^8\left(1 - 308\frac{b^2}{l^2}\right)\left(1 - \frac{b^2}{d^2}\right) = n^2l^2r \div d \dots . \ (7).$$

Thus, for example, if $b = 1$, $l = 30$, $r = 12$, the following depths d inches are right for the following speeds. It is well to assume d, and calculate n from (7).

d	1	1·5	2	2·5	3	4	6
n	0	205	277	327	368	437	545

2. A round bar of steel, 1 inch in diameter, 8 feet long, or $l = 48$ inches. Take $\mathrm{F} = 1{,}500$ lbs. Show that an endlong load only sufficient of itself to produce a stress of 1,910 lbs. per square inch, and a bending moment which by itself would only produce a stress of 816 lbs. per square inch, if both act together produce a stress of 23,190 lbs. per square inch.

Students will find that this subject will well repay further study. The effect of a small lateral load on a strut is sometimes very striking. Again, in tie-bars it is very important to consider the effect of a lateral load. The subject is treated more fully in a paper in the *Philosophical Magazine*, March, 1892.

374. Beams without Compression.—In Art. 368 we have seen that a tensile load applied to extend a beam may not only diminish the greatest compressive stress, but also the tensile stress. Again, there are many cases of beams or infinitely flat arches in which there is no tensile stress anywhere. In such a case, of course, the earth takes the necessary tensile load. When the pneumatic wheel tyre was invented, Professor Fitzgerald pointed out to me that columns to support loads and military bridges, easy to pack and unpack, might be made of inflated tubes, the solid material being everywhere in tension. I consider this a notion of great importance. In a thin straight tube of circular section if the greatest bending moment is M and R is the radius, t the small thickness of the material, the compressive stress anywhere due to bending is $\frac{M}{\pi R^3 t} y$ where y is the distance from the diameter which is the neutral line of the section on the compressive side. The greatest compressive stress is $M/\pi R^2 t$. Now imagine the tube to be subjected to internal fluid pressure P above that of the atmosphere; there is a tensile endlong stress $P \pi R^2 \div 2 \pi R t$ or $P R/2 t$, and hence the greatest compressive stress is $M/\pi R^2 t - P R/2 t$. This is just 0 when $P = 2 M/\pi R^3$. The greatest tensile endlong stress is then, of course, $P R/t$; but this is equal to the lateral tensile stress which the mere internal pressure produces. When, therefore, the internal pressure is just sufficient to remove all compressive stress in the material, the tensile stress, where it is greatest, is the same in all directions and is $2 M/\pi R^2 t$. We see, therefore, that great loads may be carried by inflated tubes of thin material if they are only large enough in diameter, or by a bundle of small tubes. Shearing forces might be taken up by side frameworks like lazy-tongs. I make no attempt here at exact theory. What I give is sufficiently correct to show the general value of the suggestion. One may go far in speculation on this idea—rigidity gained by using thin material and subjecting it to internal fluid pressure, so that there shall be no compressive stress. The great ships of the future may owe their stiffness and strength to the general use of fluid pressure in those parts of them where cargo is stored, and the same pressure which gives strength may serve to keep out the sea in case of a leak. It is the means by which the leaves of plants are made rigid. Similarly, large flat areas might be made of considerable size by fastening together two plane sheets by means of many connecting ties so that they may not balloon out, and then inflating them like an air cushion. Aeroplanes of sufficient size to support a man by Lilienthal's method can be made with comparatively small internal fluid pressures and are not liable to make splinters when they fall to the ground, these splinters being a cause of considerable risk with aeroplanes made with sticks as stiffeners. Kites much larger than those suggested for military purposes might be made, in which the whole kite might be like an air cushion; or thin tubes with compressed air might take the place of the present bamboo framework. The inflation might be maintained automatically.

375. Inflated Columns.—Again, a thin tube of radius R and thickness t has to act as a column carrying a load W, and this is the load which is carried when there is no axial tensile stress. The pressure of the fluid inside being P, we have $\pi R^2 P = W \ldots (1)$. Also the lateral tensile stress produced in the material is PR/t or $\frac{W}{\pi R t}$, so that great loads may be supported by inflated tubes of thin material if they are large enough in diameter. Thus, for example, I find that a tower of thin steel 1,000 feet high would have in it a lateral tensile stress of only 3 tons to the square inch, due to its own weight and the necessary fluid pressure. If large in diameter, the hemispherical top cap becomes of importance as a load; while if the diameter be not large, the structure is possibly in danger of instability as a pillar. Any moderate diameter like 20 feet would bear many tons on the top in addition to the weight of the structure itself. Thus 1,000 feet high and 20 feet in diameter and ·01 foot thick would itself weigh about 125 tons. Its hemispherical cap would weigh 6·3 tons, and it would support 325 tons on its top. The internal pressure would be 23 lbs. per square inch and the tensile stress 10 tons per square inch. There would be no compressive stress.

Neglecting lateral stiffness, whether we assume the adiabatic or constant temperature law for increasing pressure downwards in the fluid, we are led to rules as to the relationship of radius of column to thickness of metal if the column is to have the same stress in its material at all levels. In fact, Fitzgerald's idea gives rise to a number of easy and interesting problems which I have worked out, but for which I have no space.

In all probability it would be found cheaper to use long stays from the top, and possibly from several places at different heights to resist lateral motion due to wind pressure, etc., than to stiffen as with lazy tongs the sides of the tube itself. It is certain that the thing is practical.

376. Tapering Column of Circular Section.—If x is the distance downwards to any section where t is the thickness and r the radius, p then being the internal pressure due to fluid and q the external pressure due to the atmosphere; if f^1 is the compressive stress in the material and w^1 the weight per unit volume of the metal and w of the inside fluid,

$$\delta x \,.\, 2\pi r t \,.\, w^1 + \pi r^2 \,.\, \delta x \,.\, w + 2\pi r.\delta r.q = \delta x \,.\, \frac{d}{dx}(p\pi r^2 + 2\pi r t f^1).$$

It follows from this, if $w = cp^1/\gamma$, as w^1 is constant and q is a known function of x, that

$$\frac{dr}{dx} + \frac{r}{2}\left\{\frac{dp}{dx}(1 + a) + p\frac{2w^1}{f}\right\} / (q + pa) = 0,$$

if $a = 1 + 2f^1/f$, where f is the greatest stress in the material;

$$p = p_0(1 - bx)^s, \; q = q_0(1 - cx)^s,$$

where $s = \dfrac{\gamma}{\gamma - 1}$, taking the same fluid inside and outside. The result is, if q is taken to be small,

$$\frac{2w^1}{af}x + \frac{(1 + a)s}{a}\log.(1 - bx) + 2\log. r = \text{constant}.$$

If q is not small, the integration seems troublesome.

377. The limiting length of a vertical prism supporting its own weight.—h being measured downwards to any place from the centre of gravity of the uppermost section, let y be the deflection. Let the load per unit length be w, and let $\text{W} = \int w \,.\, dh$, the total load above any section. Considering bending moment M and $\text{M} + \delta\text{M}$ at the sections at h and $h + \delta h$, we see that $\text{W}.\delta y + \frac{1}{2}w.\delta h.\delta y = \delta\text{M}$, so that $\text{W} = \dfrac{d\text{M}}{dy} = \dfrac{d\text{M}}{dh}\,.\,\dfrac{dh}{dy}$. Also $\text{M} = -\text{EI}\dfrac{d^2y}{dx^2}$.

In the most general case, where w and I vary, and where there is a load F on the top,

$$\left(\int w \,.\, dh + \text{F}\right)\frac{dy}{dh} = -\text{E}\frac{d\text{I}}{dh}\frac{d^2y}{dh^2} - \text{EI}\frac{d^3y}{dh^3} \quad \ldots . (1).$$

Let I and w be constant, and let F be 0,

$$\frac{d^3y}{dh^3} + \frac{w}{\text{EI}}h\frac{dy}{dh} = 0 \quad \ldots . (2),$$

an equation whose solution is useful in other problems. If H is the total length of the column whose end is fixed in the ground, let $x = h/\text{H}$ and (2) becomes

$$\frac{d^3y}{dx^3} = -mx\frac{dy}{dx} \quad \ldots . (3),$$

when $m = \dfrac{w\text{H}^3}{\text{EI}}$.

The solution of (3) in series may be indicated by

$$y = \text{A}x\text{F}(x) + \text{B}x^2f(x) \quad \ldots . (4).$$

If we put $\dfrac{d^2y}{dx^2} = 0$ when $x = 0$ at top, we find $\text{B} = 0$, and the other part of (4) is

$$y = \text{A}\,x\left\{1 - \frac{mx^3}{2.3.4} + \frac{m^2x^6}{2.3.5.6.7} - \frac{m^3x^9}{2.3.5.6.8.9.10} + \text{etc.}\right\}$$

In this, if we put $\frac{dy}{dx} = 0$ when $x = 1$, we find

$$0 = 1 - \frac{m}{2.3} + \frac{m^2}{2.3.5.6} - \frac{m^3}{2.3.5.6.8.9} + \text{etc.},$$

and by trial we find that this is satisfied by $m = 7{\cdot}85$.* Hence if bending occurs, $w\,\text{H}^3/\text{E I} = 7{\cdot}85$; so that this gives us the limiting height H of a uniform column. Thus we find H in feet $= 8 \times 10^6/\text{L}^2$ for thin steel tubes, and $\text{H} = 4 \times 10^6/\text{L}^2$ for solid rods of steel, where L is the ratio of length to diameter. Thus, for example, the maximum height of a tube 8 feet in diameter is 800 feet.

Let $\text{A}\,k^2 = \text{I}$; $a =$ diameter in inches. Greenhill gives

$$x^2\frac{d^2p}{dx^2} + \frac{w}{\text{E}\,k^2}x^3p = 0 \;\ldots.\; (1).$$

p is $\frac{dy}{dx}$. First put $p = x^{\frac{1}{2}}z$, then put $x^3 = r^2$; we get

$$r^2\frac{d^2z}{dr^2} + r\frac{dz}{dr} + \left(\frac{4\,w}{9\,\text{E}\,k^2}r^2 - \tfrac{1}{9}\right)z = 0 \;\ldots.\; (2).$$

This is

$$r^2\frac{d^2z}{dr^2} + r\frac{dz}{dr} + (k^2r^2 - n^2)\,z = 0 \;\ldots.\; (3),$$

Bessel's equation, where $k^2 = \frac{4\,w}{9\,\text{E}\,k^2}$, $n^2 = \frac{1}{9}$. Solution of (3) is

$$z = \text{A}\,\text{J}_n\,(kr) + \text{B}\,\text{J}_{-n}\,(kr).$$

Consequently

$$p = \sqrt{x}\left\{\text{A}\,\text{J}_{\frac{1}{3}}\,(kx^{\frac{3}{2}}) + \text{B}\,\text{J}^{-\frac{1}{3}}\,(kx^{\frac{3}{2}})\right\}.$$

$\frac{dp}{dx} = 0$ when $x = 0$, makes $\text{A} = 0$;

$$\therefore\; p = \text{B}\sqrt{x}\,\text{J}_{-\frac{1}{3}}\,(kx^{\frac{3}{2}}) \;\ldots.\; (4).$$

Put $p = 0$ when $x = h$, the lowest point. $\text{J}_{-\frac{1}{3}}\,(kh^{\frac{3}{2}}) = 0$. If c is the least root of $\text{J}_{-\frac{1}{3}}\,(c) = 0$, then $c = kh^{\frac{3}{2}}$. Greenhill gives the expansion of $\text{J}_{-\frac{1}{3}}\,(c)$, and finds root by trial.

378. Stability of Shafts.—Rankine considered the effect of centrifugal force on shafting in 1869. Professor Greenhill has investigated the stability of a shaft between bearings (see his Paper to the Institute of Mechanical Engineers, 1883). He took into account an end thrust, F, a twisting moment, T, and centrifugal force. He found in practical examples of propeller shafts that the twisting moment was not nearly so important as the thrust in determining the maximum length of shaft for stability; in fact, that the shaft might be designed, merely like a strut, by Euler's formula. I consider, therefore, that such a shaft ought to be studied under the rules of Art. 372.

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* Prof. Maurice Fitzgerald published curves showing the values of y and $\frac{dy}{dx}$ when $x = 1$ for various values of m (Proc. Phys. Soc. of London, Oct., 1892). Prof. Greenhill had previously given the solution, not only for uniform solid cylinders, but also for cones and paraboloids of revolution (Proc. Camb. Phil. Soc., 1881).

379. Whirling Loaded Shaft.—If a uniform shaft, originally straight, is w lb. per unit length, and has an angular velocity of α radians per second; if B is its flexural rigidity, or E I; if the deflection from straightness is y at a point distant x from the centre, then the load due to centrifugal force is $\frac{w}{g}\alpha^2 y$ per unit length. If we also take into account the effect of gravity, there is one position in a revolution when the centrifugal force and the weight produce their greatest effects. We may approximately take it that the path of every point is a circle, and that the weight is a radial force always acting in the same direction as the centrifugal force. If we take into account also an endlong thrust, F, we are led to the equation

$$\frac{d^4y}{dx^4} + \frac{F}{B}\frac{d^2y}{dx^2} - \frac{w\alpha^2}{gB}y - \frac{w}{B} = 0 \dots (1),$$

and to the solution

$$y = A_1 \sin.\beta x + A_2 \cos.\beta x + A_3 e^{\gamma x} + A_4 e^{-\gamma x} - g/\alpha^2 \dots (2),$$

where

$$\left.\begin{aligned} \beta^2 &= \frac{F}{2B} + \sqrt{\frac{F^2}{4B^2} + \frac{w\alpha^2}{gB}} \\ \gamma^2 &= -\frac{F}{2B} + \sqrt{\frac{F^2}{4B^2} + \frac{w\alpha^2}{gB}} \end{aligned}\right\} \dots (3).$$

y is to have the same values for equal positive and negative values of x, so that $A_1 = 0$, $A_3 = A_4$. Putting $y = 0$ where $x = l$ and confining our attention to shafts whose bearings do not constrain them in direction at the ends, so that $d^2y/dx^2 = 0$ where $x = l$ we find $A_1 = 0$, $A_2 = \gamma^2 g/\alpha^2(\gamma^2 + \beta^2)\cos.\beta l$, $A_3 = A_4 = \beta^2 g/2\alpha^2(\gamma^2 + \beta^2)\cosh\gamma l$.

The bending moment anywhere is $EI\,d^2y/dx^2$, and hence the bending moment in the middle, where it is evidently greatest, is $M_0 = EI(-\beta^2 A_2 + 2A_3\gamma^2)$, or

$$B\beta^2\gamma^2 g\left(\frac{1}{\cosh\gamma l} - \frac{1}{\cos.\beta l}\right)/\alpha^2(\gamma^2 + \beta^2) \dots (4).$$

The greatest stress at the middle is

$$f = \frac{M_0}{Z} + \frac{F}{a} \dots (5),$$

where Z is the strength modulus of the section, and a is the area of the section; or if R is outside radius of a circular shaft we have

$$f = E w R\left(\frac{1}{\cosh\gamma l} - \frac{1}{\cos.\beta l}\right)/\sqrt{F^2 + 4w\alpha^2 B/g} + F/a \dots (6).$$

Exercise.—Show that in the case of a propeller shaft 13·4 inches diameter, of length 98 feet between its bearings, with endlong thrust 50,000 lbs., if $a = 2\pi$, or 60 revolutions per minute, the important terms in the above calculation are $F^2/4B^2 = 30 \times 10^{-14}$, and $w a^2/g B = 89 \times 10^{-12}$. So that the centrifugal force is 300 times as important as the endlong thrust.

If we altogether neglect twisting moment and endlong thrust and write n^4 for $w a^2/g B$, we have to solve

$$\frac{d^4 y}{dx^4} - n^4 y - \frac{w}{B} = 0 \ldots . (7).$$

Our old $\beta = \gamma = n$, and the greatest stress is

$$f = E w R \left(\frac{1}{\cosh nl} - \frac{1}{\cos. nl}\right) /2 B n^2 \ldots . (8).$$

This is infinite if cos. nl is 0, that is if $nl = \frac{\pi}{2}$; that is, if

$$2 l = \pi \left(\frac{g E I}{w a^2}\right)^{1/4} \ldots . (9).$$

This limiting length may be obtained very simply by leaving out the constant term in (1), as Rankine does.

Mr. Dunkerley (Phil. Trans. for 1894) discusses the stability under centrifugal force of shafts loaded with pulleys, and he has illustrated his results experimentally.

EXERCISES.

1. If the critical length of a shaft 13·4 inches in diameter, subjected to endlong thrust alone, is equal to the critical length when subjected to centrifugal force alone, show that $F = 68{,}500\, a$. Show that if the length is 98 feet, the critical F is 327,600, and the critical a is nearly 4·1 radians per second, or 46 revolutions per minute.

2. Professor Greenhill's result is that if F is the endlong force, and T the twisting moment, the limiting length of shaft being $2\,l$, we have

$$\frac{\pi^2}{4 l^2} = \frac{F}{E I} + \frac{T^2}{4 E^2 I^2}.$$

The propeller shaft of the Cunard steamer *Servia* is of wrought iron, 22½ inches diameter. The pitch of screw is 35½ feet. At 53 revolutions per minute the indicated horse-power was 10,350. Assuming that all this power is really utilised—an assumption which is, of course, quite wrong—prove that $F = 181{,}530$ lbs., and that $T = 12\cdot3 \times 10^6$ pound-inches. Take $E = 29 \times 10^6$, and show that the above formula becomes

$$\frac{\pi^2}{4 l^2} = 4\cdot98 \times 10^{-7} + 2\cdot8 \times 10^{-10},$$

so that the twisting moment term is quite inconsiderable compared with the thrust term. Show that $2\,l = 4{,}454$ inches. Consider this shaft of 22½ inches diameter, 4,454 inches long between bearings, subjected to no thrust or twisting moment, and not even to its own weight; prove that it cannot be rotated at a higher speed than $a = \cdot14$, or 1⅓ revolution per minute, without fracture by centrifugal force.

CHAPTER XXII.

METAL ARCHES.

380. The student must examine drawings and actual specimens of bridges to see how the weight of a roadway is brought to bear upon the arched ribs whose function is to carry weight, transmitting it with certain horizontal forces to the abutments. Our problem is, given the distribution of vertical loads on an arch-ring of which the shape of the centre line and the shape of cross-section everywhere are known, to find the stress everywhere. If K Z G is the centre line of the rib shown in Fig. 271, then any cross-section B C is supposed by our theory to remain plane. The resultant of all the loads acting to the right of B C, together with the resultant force R at the abutment, is a force whose direction is shown at D by the direction of the line of resistance there, and the force polygon shows its amount. We saw that in Fig. 261 this force, which we there called F, produces a bending moment M at the section whose amount is P . O D. But evidently this is the same as F . O J (Fig. 271) if O J is perpendicular to F. A much more important thing for our present purpose is to know that it is also equal to the horizontal component H of the force F multiplied by the vertical distance O L. This the student will easily prove for himself.

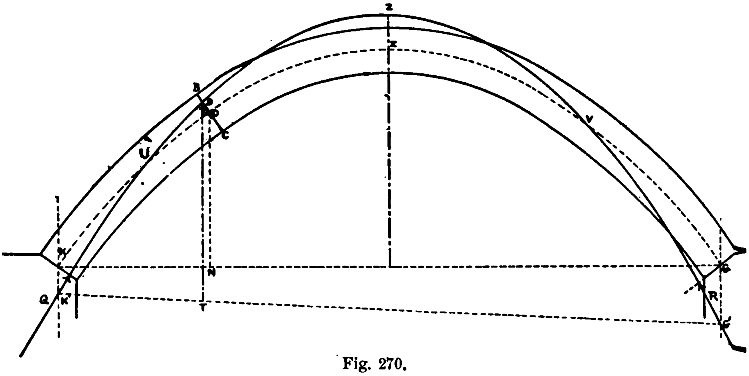

Fig. 270.

Now if all the loads are vertical, we know from Art. 349 that (1), the resultant force F at every joint has the same horizontal component H as at any other. This is represented on the force polygon (Fig. 236) by O H. It is, of course, also the horizontal thrust on each of the abutments, and is generally called the horizontal thrust of the arch. Of course the force polygon shows all the forces F at all the sections. (2) If two vertical lines be drawn through K and G, as in Fig. 271, and if any line of resistance be drawn whose ends K′ and G′ are in these lines, and if K′ and G′ are joined, the vertical height anywhere D T of this link polygon is inversely proportional to the O H of the force polygon by means of which it is drawn; so that if two such figures as K′ Z′ G′ are drawn, the vertical ordinates D T are all in the same proportion. We see, therefore, that since the bending moment at B C is the horizontal thrust multiplied by O L, the vertical distance from the centre line anywhere to the line of resistance there represents to some scale the bending moment at the section there. At V and U the lines cross. These are points of no bending. From K to U, and from G to V the bending moment tends to make the arch more convex upwards. From V to U the bending moment tends to make the arch less convex upwards. If we can only find the true line of resistance K′ Z′ G′, we know by Art. 129 the stress in every section. The true position of K′ Z′ G′ is determined by these conditions:—

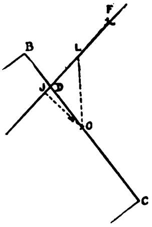

Fig. 271.

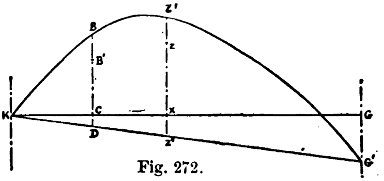

Fig. 272.

1. In **an arch hinged at the ends,** if K and G are the centres of the hinges, we know that the line of resistance must pass through them. We only need one other condition, and that is given by the statement: the yielding everywhere of the arch must be such that the distance K G remains constant.

2. In **an arch fixed at the ends,** if the fixing is perfect, we have the above condition that the distance K G remains constant, and also the condition that the inclinations of the centre line at K and G remain constant. Just as in the case of beams fixed at the ends, it is exceedingly difficult to fix the arch at the ends so perfectly that we can rely upon this condition being fulfilled; and hence, on account of our uncertainty, we prefer to use arches hinged at the ends.

3. The simplest case of all is that in which there are three **hinges** in the arch, one at each abutment K and G, and another at the crown. Even when the loads are not vertical it is easy to find the line of resistance, because two corners and a point in it are given, and we have merely the exercise of Art. 106.

Any system of oblique loads is given, acting upon an arch whose end hinges are K and G and whose crown hinge is L. **Find the resultant** of all the loads from K to L, and let it be A B

(Fig. 273). Find the resultant of all the loads from L to G, and l[illegible] it be B C. Draw A B and B C in the force polygon (Fig. 27[illegible] Choose the pole O. Join O A, O B, and O C. Draw K P, P Q, and Q[illegible] parallel to O A, O B, and O C. The resultant of A B and B C pass[illegible] through S. Now join G S and draw the new diagram, C O′ paral[illegible]

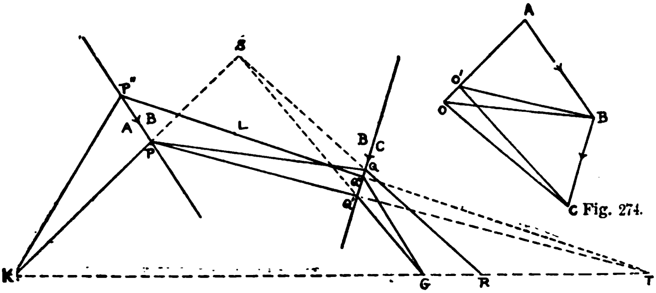

Fig. 274.

Fig. 273.

to G S, and join O′ B; so that K P, P Q′, Q′ G correspond to O′ A, O′ B, and O′ C. We have then a link polygon. But it does not pass through L. Produce P Q′ to meet K G in T. Join T L and let it meet the given forces in P″ and Q″. Join K P″ and Q″ G, and K P″ Q″ G is the link polygon or line of resistance required. I set the problem as an exercise for students, and one of them, Mr. Stansfield, gave me this solution.

If the loads are vertical, and K, G, and Z are the given hinges, draw any line of resistance K Z′ G′, its corners Z′ and G′ being in the verticals from Z and G. K Z″ G′ is a straight line. Draw a vertical Z′ Z X Z″ through Z. Construct a figure on the base K G such that if B′ is any point in it, B′ C is in the same proportion to B D that Z X is to Z′ Z″. This will evidently be the true line of resistance,

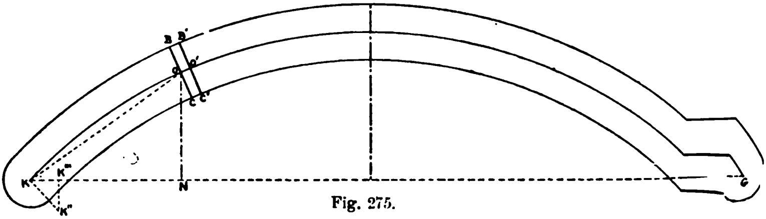

Fig. 275.

and the vertical ordinates between it and the centre lines of the portions of the arch will be the bending moments. One example of each of these must be worked out by each student.

381. Arch Hinged at the Ends.—Imagine the small slice of the arch between two sections BOC and B′O′C′, at the small distance δs asunder, to yield, and study the effect of the yielding of this portion alone, exactly as we did with our spring in Art. 370. That is, we imagine the part O′G (Fig. 275) to be absolutely rigid and fixed; the part OK to be rigid, but to move about O as centre, as if OK were merely a pointer, the motion KK″ being the angular motion of BC relatively to B′C′, multiplied by the straight distance OK. The angular change from O to O′, if OO′ is δs, is $\frac{M}{EI}\delta s$, if M is the bending moment, I the moment of inertia of BC about O through its centre of gravity, and E is Young's modulus; so that KK″ is $\frac{M}{EI}\delta s\,.\,OK$. The horizontal component of this is KK‴ or KK″ cos. K″KK‴. But this angle is the same as KON, whose cosine is ON/OK. Hence the horizontal motion of K due to the yielding of the little slice is $\frac{M\,.\,\delta s\,.\,OK}{EI}\cdot\frac{ON}{OK}$ or $\frac{M\,.\,ON\,.\,\delta s}{EI}$ . . . . (1).

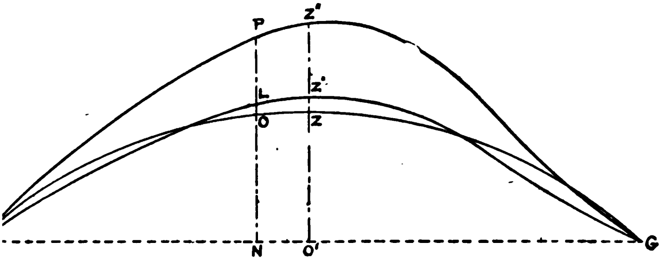

Fig. 276.

And as K does not yield at all, we make this sum zero. We beg to point out that we cannot in the same way state the vertical displacement of K. If ON is called y, and if KN is called x, then (1) is $\Sigma\frac{M}{EI}y\,.\,\delta s = 0$; and we might in the same way think that, as $\frac{M}{EI}x\,.\,\delta s$ is the horizontal displacement of K due to the yielding of the slice; therefore the sum of all these terms ought also to be 0. **If the end G is fixed,** this is true, and it enables us to calculate the fixing moment. But in truth $\Sigma\frac{M}{EI}x\,.\,\delta s$ is not 0 if G is hinged; it is equal to the angular movement at G multiplied by KG. Note that for the horizontal displacements the angular yielding at G produces no effect. We draw special attention to this, because it is usually not mentioned.

Since M is represented to scale by the distances O L, (1) becomes $\Sigma \frac{\mathrm{OL} \cdot \mathrm{ON} \cdot \delta s}{\mathrm{I}} = 0 \ldots . (2)$. In Fig. 276 K Z G is the centre line of the arch of which K and G are the hinges. K Z′ G is the line of resistance which we want to find. If a student wishes to keep (2) in his memory, let him imagine that the line K Z G has positive and negative density, represented by M/E I per unit length. Then (1) or (2) tells us that if the whole positive part of the linear mass is m_p, and if y_p is the distance of its centre of gravity from K G, and if the whole negative part of the linear mass is m_n, and if y_n is the distance of its centre of gravity from K G, then $m_p\, y_p = m_n\, y_n$. To find K Z′ G, let K Z″ G be any line of resistance through K G drawn at random. Then L N = k . P N, where k is an unknown constant; and if we know k, knowing P N, we can find L N. Divide the given centre line K O Z G into any number of equal parts, and draw an ordinate N O P at the middle of each of them. All values of δs are now equal, and (2) is $\Sigma \frac{\mathrm{ON} \cdot \mathrm{OL}}{\mathrm{I}} = 0 \ldots . . (3)$, or $\Sigma \frac{\mathrm{ON}\,(\mathrm{LN} - \mathrm{ON})}{\mathrm{I}} = 0$; so that, as L N is k . P N, we have $k \Sigma \frac{\mathrm{ON} \cdot \mathrm{PN}}{\mathrm{I}} = \Sigma \frac{\mathrm{ON}^2}{\mathrm{I}} \ldots . (4)$, and k is evidently $\Sigma \frac{\mathrm{ON}^2}{\mathrm{I}} \div \Sigma \frac{\mathrm{ON} \cdot \mathrm{PN}}{\mathrm{I}} \ldots . (5)$ This is easily calculated. Thus, for example, take K Z G, an arc of a circle, span 200 feet, rise 30 feet. For a given system of unsymmetrical loads, K P Z″ G was drawn. Dividing K Z G into sixteen equal parts and raising perpendiculars at the middles of the parts, we found the following values. The curves as drawn were measured in inches, no attention being paid to scale. The values of I are in inches to the fourth power. We see that k = ·03014 ÷ ·07447 or 0·405. Multiplying each value of P N by this, we find the values of L N. The horizontal thrust corresponding to this true line of resistance is greater than the one for K Z″ G in the ratio 1 : 0·405, and its force polygon may be drawn. The student needs no further hints for the completion of the calculation. The loads taken were as follows :—

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
ON	x	PN	I	$\frac{1}{I}$	$\frac{ON^2}{I}$	$\frac{ON . PN}{I}$	$\frac{ON}{I}$	$\frac{x}{I}$	$\frac{PN}{I}$	$\frac{ON . x}{I}$
·22	·34	·45	1200	·0008333	·000040	·0000825	·0001833	·009283	·000375	·0000623
·61	1·06	1·42	1154	·0008665	·000322	·000751	·0005286	·000918	·001230	·000560
·97	1·82	2·31	1111	·0009001	·000847	·00202	·0008731	·00164	·00208	·00159
1·26	2·59	2·96	1066	·0009381	·00149	·00350	·001183	·00243	·00285	·00306
1·50	3·39	3·55	1022	·0009785	·00220	·00521	·001468	·00332	·00347	·00497
1·68	4·20	4·01	981	·001019	·00288	·00687	·001711	·00428	·00409	·00719
1·80	5·00	4·35	940	·001064	·00345	·00833	·001915	·00532	·00463	·00958
1·86	5·83	4·55	900	·001111	·00384	·00940	·002067	·00648	·00505	·01205
1·86	6·66	4·64	900	·001111	·00384	·00959	·002067	·00740	·00515	·0138
1·80	7·50	4·54	940	·001064	·00345	·00870	·001915	·00798	·00483	·0144
1·68	8·31	4·26	981	·001019	·00288	·00730	·001711	·00847	·00434	·0142
1·50	9·12	3·86	1022	·0009875	·00220	·00567	·001468	·00892	·00378	·0134
1·26	9·90	3·28	1066	·0009381	·00149	·00388	·001183	·00929	·00308	·0117
·97	10·68	2·53	1111	·0009001	·000847	·00221	·0008731	·00961	·00228	·00933
·61	11·43	1·61	1154	·0008665	·000322	·00085	·0005286	·00990	·00139	·00604
·22	12·16	·56	1200	·0008333	·000403	·000103	·0001833	·01013	·000467	·00223
			Sums...	·01542	·03014	·07447	·01986	·09637	·04910	·1241

These columns refer to the arch fixed at the ends. (See later.)

382. Arch Ring Fixed at the Ends.—In the case of a trical loading, the end fixing moments being called m_1, w have this added on or subtracted everywhere to the bending considered in the last case; or, rather, the bending mom place O is OL − m_1, instead of being merely OL. (2) becoı

$$\Sigma \frac{(\text{OL} - m_1)\ \text{ON}\ .\ \delta s}{\text{I}} = 0 \ldots . (2),$$

(3) becomes

$$\Sigma \frac{\text{ON}\ (\text{LN} - \text{ON} - m_1)}{\text{I}} = 0 \ldots . (3),$$

and (4) becomes

$$k \Sigma \frac{\text{ON}\ .\ \text{PN}}{\text{I}} - \Sigma \frac{\text{ON}^2}{\text{I}} - m_1 \Sigma \frac{\text{ON}}{\text{I}} = 0 \ldots . (4).$$

But we have another condition to satisfy. The angular ch the length δs is $\frac{\text{M}}{\text{EI}}\,\delta s$, and the integral of this along the ceı from X to Y is 0. Hence $\Sigma \frac{\text{OL} - m_1}{\text{I}} = 0$, or

$$k \Sigma \frac{\text{PN}}{\text{I}} - \Sigma \frac{\text{ON}}{\text{I}} - m_1 \Sigma \frac{1}{\text{I}} = 0 \ldots . (5).$$

When we find the summations in (4) and (5), it is easy to c the two unknowns m_1 and k. When found, m_1 will be to t

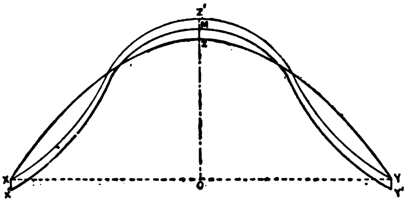

Fig. 277.

scale as that to which the distances OL represent bending ı and the curve of Fig. 277, XZ'Y, obtained when we know k displaced downwards everywhere by the constant displacer till its appearance is that shown in X'MY'. A thoughtful will see that the $\Sigma \frac{\text{M}}{\text{EI}}$ KN . δs summation is also 0 . . . . (6 case, but that (5) and (2) cannot both be true unless (6) is a because the loading is symmetrical.

In the case of an **unsymmetrical loading**, the bending everywhere is less than OL by an amount which is m_1 at th and is m_2 at the end Y, and which changes. A little consi will show that we must lower the curve KZ'G of Fig. amounts shown for each vertical line by the ordinates of diagram as K K′ G′ G of Fig. 271. If x is the *horizontal* distance of the O of any section from K, and l is the whole span, then we must subtract the amount $m_1 + x\frac{m_2 - m_1}{l}$ from the O I. of Fig. 276. Call this $m_1 + \text{P}\,x$, then (4) becomes

$$k\,\Sigma\frac{\text{ON}\,.\,\text{PN}}{\text{I}} - \Sigma\frac{\text{ON}^2}{\text{I}} - m_1\,\Sigma\frac{\text{ON}}{\text{I}} - \text{P}\,\Sigma\frac{\text{ON}\,.\,x}{\text{I}} = 0\ \ldots\ldots\ (4).$$

(5) becomes

$$k\,\Sigma\frac{\text{PN}}{\text{I}} - \Sigma\frac{\text{ON}}{\text{I}} - m_1\,\Sigma\frac{1}{\text{I}} - \text{P}\,\Sigma\frac{x}{\text{I}} = 0\ \ldots\ldots\ (5).$$

We also now find another condition by stating that the sum of the vertical displacements of K is 0 when G is held fixed—that is, $\Sigma\frac{\text{M}}{\text{EI}}\text{KN}\,.\,\delta s = 0\ \ldots\ldots\ (6)$, and this gives us

$$k\,\Sigma\frac{\text{PN}\,.\,\text{KN}}{\text{I}} - \Sigma\frac{\text{ON}\,.\,\text{KN}}{\text{I}} - m_1\,\Sigma\frac{x}{\text{I}} - \text{P}\,\Sigma\frac{x^2}{\text{I}} = 0\ \ldots\ldots\ (6);$$

and the summations indicated in (4), (5), and (6) being effected, we have no difficulty in solving for k, m_1, and P, and so finding the true line of resistance.

The table shows these summations. It is for the same arch ring, with same loads, as in Art. 381, and therefore some of the columns are the same.

The student will note that if y^1 is the ordinate of the centre line of the unloaded arch at any place, and if y is the ordinate of the loaded arch, and if $y^1 - y$, the downward motion of the point, be called z, then, unless in some case there is very great slope indeed, we may take $\frac{d^2z}{dx^2} \div \left\{1 + \left(\frac{dy^1}{dx}\right)^2\right\}^{\frac{3}{2}}$ as the change of curvature. The proof of this is easy. Equating this to M/E I, we see that we have only to imagine I at every place divided by $\left\{1 + \left(\frac{dy^1}{dx}\right)^2\right\}^{\frac{3}{2}}$ there, calling it by the new name I^1, and we take $\frac{d^2z}{dx^2} = \text{M}/\text{E}\,\text{I}^1$. Having a diagram showing everywhere the value of M/E I, it is easy to obtain the diagram of z by graphical statics, just as in Fig. 217 we obtained the shape of a beam from a diagram of M/E I.

CHAPTER XXIII.

FORCE OF A BLOW.

383. In Art. 46 we considered what occurs when a chisel is cutting metal. We shall now consider the same subject from a point of view which at first seems different. Consider how it is that a **blow of a hand hammer** will indent a steel surface, whilst a steady force applied to the same hammer-head would require to be very great to produce any indentation. The pressure between the hammer and the steel is very great, and it must be all the greater because the time of contact is very short. Indeed, if a hammer weighs 2 lbs., so that its mass is 2 ÷ 32·2 or ·0621, and if, just before touching the steel, its velocity was 10 feet per second, then we know from the principles of mechanics that its momentum was 10 × ·0621, or ·621. Now, if one-ten-thousandth of a second elapses from the time of actual contact until the hammer's motion there is destroyed —that is, until the elasticity of the steel is just about to send the hammer back again a little—the momentum ·621 is destroyed in ·0001 second; hence the average force (notice that it is a *time* average) acting between hammer and steel during this short time must have been ·621 ÷ ·0001, or 6,210 lbs. It is certain, however, that this average force is less than what the force actually was for some very small portion of the time. We observe, then, that we cannot tell the **average force of an impact** unless we know two things—the **momentum** and the **time in which it was destroyed.** Now the duration of an impact depends greatly upon the nature of the objects which strike one another, and we see that the average force of a blow is less as the time is greater. Sometimes, instead of a great force acting for a very short time, what we require is a smaller force acting for a longer time. For instance, when cutting wood we obtain this result by using a **wooden mallet** and a chisel with a long wooden handle, because the force required to make the chisel enter the wood is not very great, and we wish this force to act for some time, so that much wood may be cut at one blow. In **chipping,** we have the time short, because considerable force is required to cause the chisel to enter metal. The duration of an impact depends on the shapes of the bodies and their masses, and on the rigidity of their materials.

384. Why is it that in **driving a nail into wood** our blows seem to be of no effect unless the wood is thick and rigid, or unless it is backed up by a piece of metal or stone? It is because the wood yields quite readily, and so prevents the hammer losing its momentum rapidly. There are few subjects in which people are so apt to have erroneous ideas as in this one of impact. Thus a man will speak of the *force* produced by a weight falling through a height without having any idea of the *time* during which the motion of that weight is being stopped—in fact, without considering **what time** the weight is allowed for **delivering up its energy.** Now, a little consideration will show that the mean force of the blow will be quite different according as the weight falls on a long and yielding bar or on a short and more rigid one. If we could imagine bodies to be formed of perfectly unyielding materials, then the slightest jar of one against the other would produce an infinitely great pressure between them; and in the blow produced by a falling body there may be every gradation from exceedingly great pressures to very small ones, depending on the yielding power of the body that is struck. Everybody is acquainted with the sensation produced by **suddenly placing one's foot** on a level floor when one was preparing for a step downwards. The downward momentum of the body is suddenly destroyed, and great pressures have to act in all the bones of the body. **Carriages are hung on springs** for the purpose of preventing their losing or gaining momentum with too great rapidity when the carriage wheels pass over obstacles. When we are sitting on a hard seat in a third-class railway compartment, and the carriage gets a slight jerk upwards, momentum is given much too rapidly to our bodies for perfect comfort, and to sit on **cushioned seats** is preferable. A cannon-ball is safely, because comparatively slowly, stopped by **sandbags** or bales of cotton.

385. *Example.*—A **pile driver** of 300 lbs. falls through a height of 20 feet, and is stopped during 0·1 second. What average force does it exert upon the pile? A body which has fallen freely through a height of 20 feet has acquired a velocity equal to the square root of 64·4 × 20, or 35·89 feet per second. Its momentum is 35·89 × 300 ÷ 32·2, or 334·4, and this divided by 0·1 gives 3,344 lbs., which, together with the weight 300 lbs., is 3,644 lbs., the answer. From the instant when the motion of the driver ceases to diminish, the force exerted by it is its own weight. We have considered the **time average of the force.** The average force of friction in pounds between the pile and the ground, multiplied by the distance in feet through which the pile descends during the stroke, is equal to 300×20, or 6,000 foot-pounds, if we neglect the loss due to vibrations of the body and the energy carried off to the ground to be wasted in **earth vibration,** and if we also ignore the fact that the weight really descends a little farther than 20 feet. **The neglected energy may be the whole of the energy** if, for example, the blow does not make the pile move further into the ground. **We can only be certain about a time average force,** and even in this case we must assume that there is an instant at which one of the bodies has the same average velocity in all its parts.

385a. *Example.*—A column of water in a pipe 6 inches diameter, 30 feet long, was moving behind a piston at 15 feet per second. The piston's motion is stopped in 0·1 second. What is the time average of the pressure due to the stoppage?

Ans. The area of the circular section of the column of water being 28·274 square inches, the quantity of water in motion is $30 \times 28{\cdot}274 \div 144$ cubic feet or $62{\cdot}3 \times 30 \times 28{\cdot}274 \div 144$ lbs.; and, on the assumption that all the water has exactly the same motion, its momentum is $(62{\cdot}3 \times 30 \times 28{\cdot}274 \times 15) \div (144 \times 32{\cdot}2)$ units. The time average of the force required to stop the motion in 0·1 second is therefore $(62{\cdot}3 \times 30 \times 28{\cdot}274 \times 15) \div (144 \times 32{\cdot}2 \times 0{\cdot}1)$ lbs., which is equivalent to a pressure of 60·5 lbs. per square inch over the area of the piston.

386. Suppose a body A to strike another B, and that we can neglect the actions of outside bodies upon them both. I A loses momentum, B must gain the same amount becaus their mutual pressures are equal and opposite during the tim of impact. It is our knowledge of this fact that enables us t calculate the motions of bodies after they strike one anothe Again, for the same reason, if from any internal cause th parts of a body separate from one another, either violently o gently, the **total momentum** remains as it was; it is only th relative momentum which alters. Hence, **when a shell burst** in the air, some parts move in the same direction more **rapidl** *than* before, but others less rapidly; one part may double **it**

velocity and another may drop nearly vertically, its forward motion being stopped, but, on the whole, the total forward momentum is what it was originally.*

387. *Examples.*—If a cannon were perfectly free to move backward when the shot leaves it, the backward momentum of the cannon would be exactly equal to the forward momentum of the shot. Thus, if a shot of 20 lbs. leaves a cannon whose weight is 2,240 lbs. with a velocity of 1,000 feet per second, the velocity of the cannon backward would be 1,000 × 20 ÷ 2,240, or about 9 feet per second, neglecting the fact that the gases leave the gun also with a certain momentum. When a ship fires her broadside, each gun runs back, communicating, as it is stopped, its momentum to the ship, which heels over in consequence. A gun firing the above shot of 20 lbs. directly astern from a ship whose total weight is 600 tons gives to the ship (neglecting the momentum of water moving with the ship) so much momentum that its speed is increased (neglecting her friction with the water) 9 ÷ 600, or ·015 foot per second. We see, then, that a ship might propel herself by means of her guns. The steamship *Waterwitch* had powerful steam pumps, wherewith she brought a great quantity of water in nearly vertically, and sent it out backwards on the two sides below water level. The momentum given to the water backwards was equal to the momentum given in the other direction to the ship. It is on this principle that Hero's steam-engine and Barker's mill work, the momentum given to jets of fluid passing out of certain pipes being equal to the momentum given in an opposite direction to the vessel from which the fluid passed. In all such cases the propelling force in pounds is numerically equal to the momentum of the fluid which passes out in a second. Thus, if from a vessel moving with a velocity of 14 feet per second water comes through orifices of 4 square feet in area with a velocity (relative to the orifices) of 20 feet per second, then the quantity passing out in one second is 4 × 20, or 80 cubic feet—that is, 80 × 62·3, or 4,984 lbs. Now, recollecting that this water was first brought in and is now sent out, what is the velocity which we have really impressed upon it in the process? At the beginning it

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* Of course the kinetic energies of the parts of the shell added together are greater than they were before the shell burst; we are now merely speaking of the momentum. The total momentum of two equal bodies going in opposite directions with the same velocity is nothing, whereas their total kinetic energy is double that of one of them.

was motionless with respect to the sea; it now has a velocity of 20—14, or 6 feet per second with respect to the sea, so that the momentum given to it is its mass, 4,984÷32·2 multiplied by 6, or 928·7: hence, as this momentum is given every second, 928·7 lbs. is the propelling force exerted on the ship. In one second the ship moves through 14 feet, so that the useful mechanical work done is 14 × 928·7, or 13,002 foot-pounds. We have given to 4,984 lbs. of water a velocity of 6 feet per second, the kinetic energy of this water is wasted, and this kinetic energy is $\frac{1}{2}$ of 4,984 ÷ 32·2 × 6 × 6, or 2,786 foot-pounds. In fact, we have altogether spent 15,788 foot-pounds, and 13,002 of this have been usefully employed, so that the efficiency of the method is 13,002 ÷ 15,788, or ·824, or 82·4 per cent. As a matter of fact, however, the friction in pumps and pipes usually causes a third of the actual horse-power given out by the engine to be wasted, so that the true efficiency of this method of propulsion is two-thirds of the above, or 0·55, or 55 per cent., neglecting the friction of the engine itself. You will remember a fact which has come in casually here: if the water leaves any turbine, water-wheel, or any propeller of a vessel with a velocity relative to the still water into which it passes, or if it has any other form of energy, this energy has been wasted.

388. By calculation you will find that, **when two free and inelastic bodies strike**, the momentum communicated from one to the other is their relative velocity multiplied by the product of their masses and divided by the sum of their masses, and this quotient divided by the time of the impact gives the mean pressure. This pressure acts equally on both, of course, but it may not hurt both equally. If the bodies are surrounded by water, like ships, they can no longer be regarded as free bodies, and it is not easy to say in a few words how much mass we must add to the bodies to represent the mass of the water, which has also to undergo change of motion. In the case of a ship, the mass of water to be moved broadside on is much greater than when the ship is struck stem on.

389. A body falling into a liquid sets it in motion, and this motion appears at distant places more and more nearly *instantaneously* as the liquid becomes more and more incompressible. The nature of this motion is known to us if we know the velocity of yielding at the place of contact, and from this the total momentum given to the liquid. This

represents a very considerable pressure applied at the place of contact, and this pressure becomes greater as the velocity of the body, before it touches the liquid, increases. Hence a cannon-ball fired at sea rebounds from the water as from a rigid body. Hence also **a man diving unskilfully**, as he falls prone on the water, gets a very unpleasant shock, whereas a skilful diver enters in such a way as to make the momentum of the moving water as small as possible, and to make the creation of this momentum gradual.

390. If a body of inertia or mass M with velocity V overtakes a body of mass M^1 with velocity V^1, the motion being in the same direction, there is an instant when they move at the same speed, and then they usually separate. The equal and opposite forces equalise their velocities until they are both moving with the velocity v, such that

$$(M + M^1)\, v = M V + M^1 V^1 \ldots . (1),$$

or

$$v = \frac{M V + M^1 V^1}{M + M^1} \ldots . (2).$$

There has been a communication of the momentum, $M\ (V - v)$, or $M^1\ (v - V^1)$, and this is **the amount of the impact** or the time integral of the force from either on the other. We usually imagine the contact surfaces to be normal to the direction of motion.* During the impact the **total kinetic energy,**

$$E_1 = \tfrac{1}{2} M V^2 + \tfrac{1}{2} M^1 {V^1}^2 \ldots . (3)$$

becomes lessened to

$$E = \tfrac{1}{2} (M + M^1)\, v^2 \ldots . (4),$$

the amount $E_1 - E$ being stored as **strain energy**. In truth, much of it travels off and vibrations take place (see Art. 486), but let us speak of $E_1 - E$ as the stored energy. Assume that the energy of the bodies when free after collision

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* If not normal we may speculate on the connection between the ideas of force as rate of communication of momentum, the direction of the force being the same as that of the momentum communicated, and yet the direction of communication being different from both. Students who have leisure will find three quite neglected letters in *Nature* for 1878, by Prof. R. H. Smith, which will give them novel ideas on the subject—ideas likely to be of use to engineers.

$E_2 = \frac{1}{2}(M U^2 + M^1 U^{1^2})$ is less than E_1 by an amount which is a fraction of the stored energy, or

$$E_1 - E_2 = k\,(E_1 - E) \; \ldots . \; (5),$$

where k is a constant depending on the nature of the materials and the shapes of the bodies. We know also that

$$v = \frac{M U + M^1 U^1}{M + M^1} \; \ldots . \; (6).$$

From (2), (5), and (6) we can calculate U, U^1 and v in terms of the initial velocities, and we find that

$$U^1 - U = (V - V^1)\sqrt{1 - k}.$$

Or, "the relative velocity of separation is $\sqrt{1 - k}$ times the previous relative velocity of approach." **This was Newton's assumption.** It is usual to denote the ratio $\sqrt{1 - k}$ by the letter e, and to call e the "elasticity," but this is unscientific.

391. Newton found by experiment that $e = \frac{5}{9}$ for compressed wool, iron nearly the same, $\frac{15}{16}$ for glass. Hence for these substances our k has the values $\frac{7}{10}$ and $\frac{1}{8}$. That is, in the compressed wool or iron $\frac{7}{10}$ of the stored energy is wasted; in glass only $\frac{1}{8}$ of the stored energy is wasted. It is very difficult for us to imagine how this difference between iron and glass should occur; and there are no recent experiments. Note that $k = 1$ or $e = 0$ indicates the case of maximum loss of energy; in fact, there is maximum loss of energy when the two bodies continue to move together as the bullet and bob of a ballistic pendulum do. In this case the kinetic energy before collision is $\frac{1}{2} M V^2$, and after collision it is

$$\frac{1}{2}(M + M^1)\left(\frac{M V + M^1 V^1}{M + M^1}\right)^2, \text{ or } \frac{1}{2}(M V + M^1 V^1)^2/(M + M^1).$$

Suppose $V^1 = 0$, or we have M impinging on M^1 at rest, the energy remaining is $\frac{1}{2} M V^2 \Big/ \left(1 + \frac{M^1}{M}\right)$. Hence, the greater M^1 is in comparison with M the greater the loss. In fact,

$$\frac{\text{lost energy}}{\text{remaining energy}} = \frac{M_1}{M}$$

So that the larger the stationary body before collision, the greater the loss.

392. *Example.*—Thirty gallons of water per second enter a wheel in a direction $A B$ from a horizontal pipe 4 inches in diameter, the shortest distance from the axis of pipe to vertical axis of wheel being 1·3 feet. The water leaves the wheel in a horizontal direction, C D, with a velocity of 3 feet per second, the shortest distance between C D and the axis of wheel being 0·8 feet. What is the turning moment? And what is the power if the wheel makes 150 revolutions per minute?

Ans. The 30 gallons, or 300 lbs., or $\frac{300}{62\cdot3}$ cubic feet, have an initial velocity of $\frac{300}{62\cdot3}\cdot\frac{144}{4^2\times\cdot7854}$ feet per second, and mass $\frac{300}{32\cdot2}$. The product is the momentum per second, and is force. Multiply by 1·3, and we have the moment of the force due to the entering water, or 668·2 pound-feet. The same mass multiplied by 3 and by 0·8 gives 22·36 pound-feet; and subtracting this from the former, we have the resultant moment (or moment of momentum per second), 645·8 pound-feet. Multiplying this by $150\times2\pi$ radians per minute, and dividing by 33,000, we find 18·4, the horse-power.

393. *Example.*—A ball of 4 lbs., moving to the north on a smooth, level table with a velocity of 6 feet per second, strikes another ball, and after the collision is found to be moving at 5 feet per second to the east. What was the amount of the impulse? If the collision lasted 0·002 second, what was the average force of the blow?

Ans. Subtracting the second velocity (in the way vectors are subtracted) from the first, we find that the sudden gain of velocity was $\sqrt{61}$ feet per second in a direction making $\tan.^{-1}\frac{5}{6}$ east of south. The impulse given to the ball was therefore in this direction, and of the amount $\frac{4}{32\cdot2}\sqrt{61}$ pound-seconds. The force, therefore, was in this direction, and of the average amount $\frac{4}{32\cdot2}\sqrt{61}\div0\cdot002$ lb., or 485 lbs.

EXERCISES.

1. Compare the amounts of momentum in a pillow of 20 lbs. which has fallen from a height of 1 foot, and an ounce bullet moving at 200 feet per second. *Ans.*, 12·8 to 1.

2. A ball of 56 lbs. is projected with a velocity of 1,000 feet per second from a gun weighing 8 tons. What is the maximum velocity of recoil of the gun? *Ans.*, 3·1 feet per second.

3. There are two bodies whose masses are in the ratio of 2 to 3, and their velocities in the ratio of 21 to 16. What is the ratio of their momenta? If their momenta are due to constant forces acting on the bodies' respectives for times which are in the ratio of 3 to 4, what is the ratio of these forces? *Ans.*, 7 to 8; 7 to 6.

4. A body weighing 10 lbs. impinges on a fixed plane with a velocity of 20 feet per second. If the coefficient of restitution e is 0·5, find the velocity of rebound, and how many foot-pounds of energy are wasted in the collision. *Ans.*, 10 feet per second; 93·17.

5. A ball of 6 ounces strikes a bat with a velocity of 10 feet per second, and returns with a velocity of 30 feet per second. If the duration of the blow be $\frac{1}{20}$ second, find the average (time) force exerted by the striker. *Ans.*, 9·32 lbs.

6. A body weighing 50 lbs. moving at the rate of 10 feet per second overtakes another body of 25 lbs. moving at the rate of 6 feet per second. If both masses be perfectly elastic, find their velocities after the shock. *Ans.*, 7⅓ and 11⅓ feet per second.

7. A hammer-head of 2½ lbs., moving with a velocity of 50 feet per second, is stopped in 0·001 second. What is the average force of the blow? *Ans.*, 3,882 lbs.

8. A shell bursts into two fragments, whose weights are 12 and 20 lbs. The former travels onward with a velocity of 700 feet per second, and the latter with a velocity of 380 feet per second. What was the momentum of the shell when the explosion took place? *Ans.*, 496·8.

9. A shell weighing 20 lbs. explodes when in motion with a velocity of 600 feet per second. At the moment of explosion one-third of the shell is reduced to rest. Find the momentum of the other two-thirds. *Ans.*, 372·7.

10. Water is flowing through a service pipe at the rate of 60 feet per second. If the water be brought to rest uniformly in one-tenth second by closing the stop-valve, what will be the increase of the pressure of the water near the valve, the pipe being taken as 50 feet long, the resistance of the pipe and the compressibility of the water being neglected? *Ans.*, 403 lbs. per sq. in.

394. As for the way in which the vibrations take place in two colliding bodies, if mathematically treated it is generally difficult; but very good working notions for the engineer are derivable from the results of experimental study, the subject being taken up in books on physics under the head "Acoustics." The best mathematical treatise is Lord Rayleigh's "Sound." The time of an impact is known in several cases in which it has been calculated from theory. (See Art. 404.)

395. Let us consider what takes place when two ivory balls come together. There is a certain instant after they first touch when their centres move together just as if they were composed of soft clay—then they act on each other with their greatest pressure; they are in their most strained condition, and supposing no loss by internal friction, the strain in the balls represents an amount of stored-up energy (see Arts. 259, 267) equal to the kinetic energy which the bodies have lost. It is very important to remember this fact, that if bodies are to return to their old states after the collision, we must suppose that during the collision there is a storage of kinetic energy in the form of strain. All the kinetic energy will not be given out again, nor can we say that it is all stored, because there is a sort of **internal friction** causing part of the strain energy to be converted into heat when any change occurs. Now, if the whole of the stored-up energy is confined to one portion of the body, the strain may be too great. Thus, a **steel rod** 1 square inch in section, 1 foot long, will store up 167 foot-pounds of strain energy in its stretched condition before it breaks. For suppose breaking stress to be 100,000 lbs. per square inch. This will occur when there is a lengthening or shortening of ·0033 foot, so that the energy stored up is the work done by a force whose average amount is 50,000 lbs. acting through ·0033 foot, or 167 foot-pounds. If 2 feet of the same rod stored up this same amount of energy, there would only be 83 foot-pounds in each foot of its length; and it is easy to see that the stress is no longer the breaking stress of 100,000 lbs. per square inch, but only 70,700 lbs. per square inch. As we store the same amount of energy in smaller and smaller portions of a body, it is evident that we must approach a condition of fracture.

396. We see, then, that at the place where contact occurs, two bodies, A and B, are strained; but if A is of some very elastic material, such as tempered steel, the strain energy is conveyed very rapidly to every part of the body; whereas if B is a feebly elastic body, the strain accumulates at one place, leaving the rest of the body unstrained, whilst at this place the strain may produce fracture. This slowness to communicate strain to the rest of the body may also be produced by the shape of the body. For instance, **a rod struck sidewise** or a thin plate struck in the middle does not so immediately communicate its strain to the remote parts as a rod struck endwise. Again, the nature of the parts of A and B in contact may be such that not only does the strain energy leave this part of A rapidly, but immediately in the neighbourhood of the place of contact there is a greater capacity to bear strain energy without rupture than is the case with B. Thus, when ship A rams the broadside of ship B, the side of B is bent inwards and the strain energy produced is accumulated near the place of contact till fracture occurs; whereas, not only is A's stem able to transmit to all parts of A with great rapidity the strain energy which must be stored up in the whole mass, but at the stem itself the material of A is capable of withstanding greater stresses than the material of B's side. Suppose, however, that A's stem is not of steel, still B's iron or wooden side will be perforated if A has enough velocity; A's stem may also be damaged in the impact in such a case.

397. **A candle** may be fired, it is said, **through a thin deal board** with very little injury to its shape, and the usual explanation of this phenomenon given in books is that the candle has not time to get broken. This explanation is not satisfactory; it is a little too vague. If we had the board in rapid motion, and striking the candle in the same relative position, the candle having previously been at rest, would the candle perforate the board? There cannot be any doubt that it would. Hence it is not the body struck which must in every case get hurt; the pressure on one is equal to that on the other. Suppose ship A rushes at ship B when B is broadside on, and rams her, B will probably be sunk, even if she is a much larger and better ship than A. But suppose that B is able to meet her adversary stem to stem, if they are equally strong they will equally injure one another, and if B is the stronger A will suffer the most. This case differs very much from that of the candle, because we can assume greater strength even for slowly applied pressures from stem to stern of the ship A than from side to side of B; whereas the strength of the candle for slowly applied pressures cannot be compared with that of the wood which it punches from the board. What is meant by the usual explanation, "The candle has not *time* to get deformed"? Why has not the soft candle time to get broken, and yet the wood has time to get torn asunder? The fact is, the wood, if it were slowly pressed, would communicate its strain energy to every part of the board and its supports; but this communication takes an appreciable interval of time, however suddenly the pressure may be applied, or however great it may be. As the strain energy is rapidly produced it becomes accumulated near the place of contact to such an extent as to produce fracture of the *wood.* Now the point of the candle is subjected to the same

pressure as the wood, and begins to get spoiled in shape—that is, it is compressed—and this compression produces a lateral spreading. In the meantime, however, the compressive strain energy is communicated very rapidly backwards along the candle, and the spreading and spoiling goes on along its entire length, but is small at any point, since it is distributed over the whole mass. Practically, therefore, the spoiling occurs only at the point of the candle, since time is needed for fracture of the material.

398. An **earthquake,** when it acts on a house, usually tends to move it through a distance of probably a very small fraction of an inch, but it does this in a very short time—that is, the house gets a considerable velocity. The mass of the house multiplied by the greatest velocity, and divided by the short time during which the momentum is being communicated, gives the pressure which the foundations of the house are subjected to. Now, when the foundations are not very rigidly connected with the ground, **the time of communication of the momentum** is lengthened, and the pressure is consequently diminished. This is the usual Japanese plan of providing for earthquake effects. Unfortunately, the very means taken to diminish the pressure on the foundations also diminishes their capability of withstanding forces, and it has not yet been decided what sort of a house is best fitted to withstand destructive earthquakes. Want of rigidity, combined with strength or toughness in the materials, and especially the quality of internal friction in the materials, so that vibrations may rapidly die away—these are the qualities needed. They are found in steel, wrought iron, and wood, and especially in wicker-work, in a less degree in cast iron and in brick or stone set in cement, and less still in brick and stone set in bad mortar.

399. When you in some way understand the possibility of a candle perforating a board, you will be able to comprehend how **sand,** when blown in air against tempered steel, is able to abrade it; how the emery wheel and grindstone going at great velocities are able to cut into hard metals; and how in California a jet of water going with very great velocity is used for **mining purposes** instead of iron tools.

400. **Quasi-Rigidity Produced by Rapid Motion.**—A top when not spinning can with difficulty be balanced on its point, and if left to itself it almost instantly falls; whereas when it

is spinning the effect of slightly tilting it out of the perpendicular is not to make it fall, but to make it take a slow precessional motion.

There is a piece of apparatus called a **gyrostat**, which is, in a more or less perfect form, to be found in every mechanical laboratory, and the student ought to experiment for himself with this apparatus on the curious effects of quasi-rigidity which manifest themselves in **tops** and other spinning bodies. If he has a slight acquaintance with astronomy he will be interested in tracing the connection between the behaviour of a tilted top and the precession of the earth's equinoxes.

When a **circular sheet of drawing-paper** is mounted like a very thin grindstone on an axis, and is gradually made to rotate rapidly, it is found to have become quite rigid—that is, it greatly resists bending as if it were made of steel. In the same way a long loop of rope, hanging round a high pulley, which gives it a quick motion, takes a certain form which it is very difficult to alter, as may be shown by striking it with the hand or with a stick: it resembles more a rigid rod than a flexible rope.

Again, in the well-known lecture-experiments on smoke rings, we see that these little whirlpools of air have many properties in common with elastic solid bodies on account of the partial rigidity which is due to their rapid motion.

In objects which spin and rub on a level surface, like tops, we have the interesting general rule, "Positions which are stable when there is no spin, are unstable when there is spin, and *vice-versâ.*" Students of statics insist on a low centre of gravity in vehicles; students of dynamics sometimes insist on a high centre of gravity, as giving greater stability when there is rapid motion.

It would be beyond the scope of a book like this to explain these curious phenomena, and I merely direct the attention of students to these instances in order to incite them to make experiments, and to seek for the explanation of what they observe. My popular lecture on Spinning Tops may be worth reading.

401. Motion Produced by a Blow.—When a body subjected to a blow is quite free to move in any way, unless the blow acts through its centre of gravity, the body will not merely move as a whole, but it will revolve. When the blow acts in a direction through the centre of gravity there is no rotation produced. It is usual in such a case to consider the motion of the centre of gravity of the body, and the motion of the body about an axis through its centre of gravity, for it is known that any motion whatsoever of the body consists of a combination of two such motions. It is found that the kinetic energy communicated to a body by means of a blow is best calculated in the following way, if we know the nature of the motion:—*First*, find the kinetic energy, as if every portion of the body had the motion of the centre of gravity. *Secondly*, find the kinetic energy of rotation as if the axis of rotation through the centre of gravity were fixed in space. Add these two results together. We may regard from another point of view the instantaneous motion of a body when it is struck—namely, as a rotation about some axis which does not itself move. This is only the case for an instant; immediately afterwards we must regard the body as moving about another fixed axis. If a body is hinged so that it can only move about a fixed axis, it is always possible to find the point at which the body may be struck, and the direction of the blow, which will tend to produce an instantaneous rotation about this particular axis, and therefore to produce no pressure at the hinge. Thus the **ballistic pendulum** of Fig. 278 is always struck in such a way, and the point in which it is struck is called the "**centre of percussion.**" An easy way to find the centre of percussion is as follows:—Make the body vibrate like a pendulum, about its axis of suspension, under the action of gravity. Now find the *length of the equivalent simple pendulum.* This is the distance of the centre of percussion from

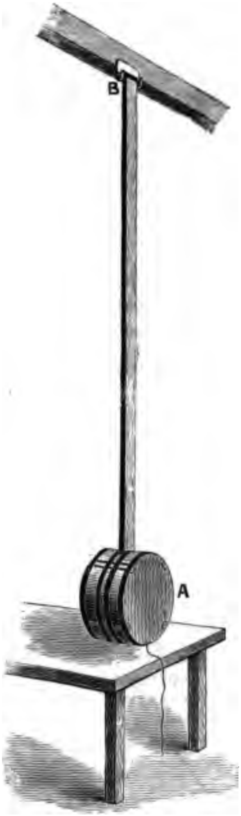

Fig. 278.

the axis. In a **tilt hammer** all blows ought to be delivered from this centre of percussion if we wish to have no pressure on the bearings. A **cricket-bat** or a rod of iron tingles the hand when we strike a blow with it, unless we happen to strike at the centre of percussion. For a rod of iron free to move about one end, the centre of percussion is at two-thirds of the way towards the other end. (See Art. 454.)

402. The **Ballistic Pendulum** of Fig. 278 is a contrivance which enables us to measure the velocity of a bullet. It consists of a mass of wood, A, forming part of a pendulum. The bullet is fired into it, and the wood swings backwards in consequence. The bullet is fired into that part of it which will cause no jar to be given to the pivot B. The momentum existing in the bullet before it enters the wood belongs now to the whole mass of which it becomes a part. C is a silk ribbon which is pulled through a moderately tight hole or over the edge of a table by the swing of the pendulum, and the length of ribbon pulled through is found to be proportional to the momentum of the bullet before entering the wood.

If w is the weight of the bullet, its horizontal velocity in the plane of swinging being v, its momentum is $\frac{w}{g} v$. The moment of momentum $x \frac{w}{g} v$ (if x is the perpendicular distance from B to the axis of the bullet) before impact is equal to the moment of momentum after impact; and if I is the moment of inertia of A and the bullet about B after collision, and a is its initial angular velocity, then $\mathrm{I}\, a = x \frac{w}{g} v \ldots.$ (1). A certain part of the whole kinetic energy has been converted into heat; the mechanical energy now in the system, $\frac{1}{2} \mathrm{I}\, a^2$, will be converted into potential energy. W being the total weight, the centre of gravity will be raised through the distance h such that $\mathrm{W} h = \frac{1}{2} \mathrm{I}\, a^2 \ldots.$ (2); so that, as a is known, h is known. Knowing h, we can either find θ, the total swing, or we can calculate or find experimentally the length of the ribbon which may be drawn up, and from the length of the ribbon we can calculate v.

In the short time when the bullet is being lodged it is exercising horizontal force at each instant, and there is horizontal force at B in the same direction. We may say that these external forces, acting at the centre of gravity, are balanced by the mass of the whole body multiplied by the acceleration of the centre of gravity at that instant, and hence their integral effects balance; that is, then, the impulse $\frac{w}{g} v + \mathrm{P}$ (where P is the horizontal impulse from the point of support in the same direction) $= \frac{\mathrm{W}}{g} a \,.\, \mathrm{BG}$ if G is the centre of gravity, because $a \,.\, \mathrm{BG}$ is the velocity produced in G, and W is the weight of the pendulum and bullet. Hence

$$\mathrm{P} = \frac{\mathrm{W}}{g} a \,.\, \mathrm{BG} - \frac{w}{g} v \;.\;.\;.\;. \;(3).$$

This is 0, therefore, if

$$\mathrm{W} \,.\, a \,.\, \mathrm{BG} = wv \;.\;.\;.\;. \;(4).$$

But by (1), (4) means that $\mathrm{W} \,.\, a \,.\, \mathrm{BG} = \mathrm{W} k^2 a/x$, or $x = k^2/\mathrm{BG}$. Now this is the rule (Art. 454) by which we find the length of the equivalent simple pendulum or the distance to the point P of oscillation (Art. 454). It also for this reason gets the name "point of percussion."

APPENDIX TO CHAPTER XXIII.

403. In the *Example*, Art. 385a, we had an example of stoppage of water in a pipe as if all the water had exactly the same motion. It was an interesting academic exercise. The following exercise is also interesting, and will give us more information.

Sudden Stoppage of Water in a Pipe of Uniform Section.—Suppose the pipe to be infinitely rigid. It will be found that the effects are independent of whether the pipe is vertical or horizontal, but we shall consider it to be vertical. It will be found on closer examination that if we know the pressure we need only consider, in our problem, p, the pressure *above* the normal pressure, and this is what we shall do. Let v be the axial velocity of a particle of water, w the weight per unit volume, so that w/g is its mass per unit volume, often called ρ; if K is the cubical elasticity (during quick changes of volume) and t is time; if x is distance to a point Q measured along the pipe in the direction of motion, and if the pipe is taken of unit cross-sectional area; if X is the total distance already travelled by a particle at Q, so that at $x + 1$, X becomes $\mathrm{X} + \frac{d\mathrm{X}}{dx}$, fluid which once occupied length 1 of the pipe now occupies length $1 + \frac{d\mathrm{X}}{dx}$, so that its increased volume is $\frac{d\mathrm{X}}{dx}$. Hence, as pressure-producing volumetric strain = K × compressive strain,

$$p = -\,\mathrm{K}\frac{d\mathrm{X}}{dx} \;.\;.\;.\;. \;(1).$$

Now velocity $v = \frac{d\mathrm{X}}{dt}$, and as the mass between x and $x + \delta x$ is $\rho \,.\, \delta x$ and the pressures are p and $p + \delta p$, we have the force $-\,\delta p$

causing acceleration $\frac{d^2 \mathrm{x}}{dt^2}$ in the mass $\rho \,.\, \delta x$, and hence

$$-\,\delta p = \rho \,.\, \delta x \,.\, \frac{d^2 \mathrm{x}}{dt^2} \quad \text{or} \quad -\frac{dp}{dx} = \rho \,.\, \frac{d^2 \mathrm{x}}{dt^2} \;\dots\dots (2),$$

or, from (1), $\kappa \frac{d^2 \mathrm{x}}{dx^2} = \rho \frac{d^2 \mathrm{x}}{dt^2}$. If κ/ρ or $\kappa g/w$ be called a^2, we have

$$a^2 \frac{d^2 \mathrm{x}}{dx^2} = \frac{d^2 \mathrm{x}}{dt^2} \;\dots\dots (3),$$

and we recognise this as the equation of wave propagation with the velocity a. Differentiate (3) with regard to t and use v for $d\mathrm{x}/dt$, and we have $a^2 \frac{d^2 v}{dx^2} = \frac{d^2 v}{dt^2}$.

Let our problem be this. When water is flowing with an uniform velocity $-\,v_0$, let an infinitely rigid, thin diaphragm suddenly produce a stoppage at $x = 0$; what will happen? Consider the pipe only for positive values of x. We have $v = -\,v_0$ everywhere at $t = 0$, and at $x = 0$ we have v such a function of the time that it was $-\,v_0$ till $t = 0$, and ever afterwards v there is 0.

To make things simpler, let v^1 be a new variable such that $v^1 = v + v_0$, and therefore the conditions are that $v^1 = 0$ everywhere at $t = 0$, and at $x = 0$ we have v^1 such a function of the time that it was 0 till $t = 0$, and ever afterwards $v^1 = v_0$. We see that

$$a^2 \frac{d^2 v^1}{dx^2} = \frac{d^2 v^1}{dt^2} \;\dots\dots (4).$$

If we let $\frac{d}{dt}$ be denoted by θ in the usual symbolical way, (4) is

$$a^2 \frac{d^2 v^1}{dx^2} = \theta^2 v^1.$$

Solving this as the simplest linear equation would be solved if θ were a quantity independent of x, we have

$$v^1 = \epsilon^{-x\theta/a}\,\mathrm{A} + \epsilon^{x\theta/a}\,\mathrm{B},$$

if A and B are functions of the time corresponding to $x = 0$. As x may be infinite, $\mathrm{B} = 0$; so that

$$v^1 = \epsilon^{-x\theta/a} f(t) \;\dots\dots (5),$$

where $f(t)$ is 0 till $t = 0$, and then is v_0, and remains of the value v_0; so that

$$v = -\,v_0 + f(t - x/a) \;\dots\dots (6).^*$$

We see now what occurs. Until $t - x/a = 0$ or $t = x/a$, $v = -\,v_0$ at any place; and after $t = x/a$, $v = 0$. Hence we have from $t = 0$ a state of no velocity spreading along the pipe with the

* The symbolic $\epsilon^{b\theta} f(t)$ means $f(t + b)$

velocity a to infinity. Since $v = \frac{d\,\mathrm{x}}{dt}$, if we integrate (6) in regard to time, we have

$$\mathrm{x} = -\,v_0 t + \mathrm{F}\,(t - x/a) \;\ldots\ldots\; (7),$$

where F is such that $\frac{d\,\mathrm{F}\,(y)}{dy} = f\,(y)$, whatever y may be. Hence, using (1), $p = \frac{\mathrm{K}}{a} f\,(t - x/a)$; or, since

$$\mathrm{K}/a = \mathrm{K}\,w^{\frac{1}{2}}/\mathrm{K}^{\frac{1}{2}}\,g^{\frac{1}{2}} = \mathrm{K}^{\frac{1}{2}}\,w^{\frac{1}{2}}/g^{\frac{1}{2}},$$

we have

$$p = \sqrt{\frac{\mathrm{K}\,w}{g}}\,f\,(t - x/a) \;\ldots\ldots\; (8).$$

This tells us that $p = 0$ till $t - x/a = 0$, and then is $v_0\sqrt{\frac{\mathrm{K}w}{g}}$, and remains of this value. This is the state of pressure produced everywhere with perfect rest accompanying it, at the velocity a from the place of the sudden stoppage if the pipe is infinitely rigid.

If at $x = l$ there is a place where the pressure is kept constant (we say $p = 0$ there), at zero, our wave is reflected as a wave of velocity $+\,v_0$ and no pressure, till on reaching $x = 0$ it is again reflected as a wave of no velocity, and so on. As the pipe is not infinitely rigid, and as there is friction, the wave really diminishes in its values of velocity and pressure as it travels, but we may take a pressure approaching the value $v_0\sqrt{\mathrm{K}w/g}$ as being instantly produced by a sudden stoppage. Experiments with suddenly closed valves give measured pressures in the laboratory considerably less, partly because the stoppage is not one of infinite suddenness, partly because of the inertia of the pressure-measuring apparatus. In the case of water the pressure in pounds per square inch is $20\,v_0$ if v_0 is the velocity in feet per second.

This great pressure produced by stoppage is taken advantage of in the **hydraulic ram.** If the bottom of the sea were smooth, and a sea with a translational velocity v_0 were suddenly stopped on its motion by a vertical wall, the pressure on the wall would be what we have given above, for a very short time, if the wall were perfectly rigid; what would occur subsequently I do not know, because there is atmospheric pressure at the surface of the sea. We know enough, however, to see the necessity for some springiness at the wall surface. It is for the same sort of reason that heavy seas produce so much damage sometimes when objects are struck by them on board ship. Curious stories are told by sailors of half-inch bars of iron being bent and broken by seas coming over the bulwarks, and it is just possible that they may be true, although it is more probable that such fractures are due to blows from passing wreckage. Anyone who has seen the ruined breakwater at Wick will believe in the greatness of the forces due to blows from ocean waves.

404. Students will find it an excellent easy mathematical exercise to assume that in an infinite length of pipe filled with water there is a piston whose displacement is a pure function of the time. The problem is identical with the simplest problem in Telephonic Signalling. If the pipe is supposed to yield and to be leaky, and if the viscosity of the liquid is considered, we have the same problem as that of the *Philosophical Magazine*, page 223, August, 1893.

When a prismatic bar of length l_1 and velocity v_1 in the direction of the common axis overtakes a bar of length l_2 (greater than l_1), moving with a velocity v_2, the bars being of the same material and cross section, Mr. Love says that the ends at the junction move with a common velocity $\frac{1}{2}(v_1 + v_2)$, and a compressive strain $\frac{1}{2}(v_1 - v_2)/a$ is produced. a is the velocity of a longitudinal wave of sound, or $E = a^2\rho$ where E is Young's modulus and ρ is the mass of the bar per unit volume, or our w/g above. Waves of compression run from the junction along both bars, and each element of either bar, as the wave passes over it, takes suddenly the velocity $\frac{1}{2}(v_1 + v_2)$ and the compression $\frac{1}{2}(v_1 - v_2)/a$.

When the wave reaches the free end of the shorter bar it is reflected as a wave of extension; each element of the bar as the wave passes over it takes suddenly the velocity which initially belonged to the longer bar and zero extension. After a time, equal to twice that required by a wave of compression to travel over the shorter bar, this bar has uniform velocity, equal to that which originally belonged to the longer bar, and no strain.

The impact now ceases, and there is in the longer bar a wave of compression of length equal to twice that of the shorter bar. The wave at this instant leaves the junction, and the junction end of the longer bar takes a velocity equal to that which it had before the impact. The ends, therefore, remain in contact without pressure. This state of things continues until the wave returns reflected from the further end of the longer bar. When the time from the beginning of the impact is equal to twice that required by a wave of compression to travel over the longer bar, the junction end of the latter suddenly acquires a velocity equal to that originally possessed by the shorter bar and the bars separate. The shorter bar rebounds without strain and with the velocity of the longer, and the longer bar rebounds vibrating.

CHAPTER XXIV.

FLUIDS IN MOTION.

405. WE tried in Art. 145 to give exact notions on the subject of pressure in fluids not in motion. When we supposed that the weight of the fluid itself was insignificant, we found that the pressure on each square inch of surface touched by the water, and on each square inch of interface separating two portions of the fluid, was everywhere the same.

One of the best methods of observing the pressure, anywhere, is by inserting a **pressure gauge**. The gauge indicates the pressure per square inch at the part of the liquid to which a communicating little gauge tube penetrates. It always does so when there is no motion of the fluid at the point in question; but unfortunately, when there is motion there, the introduction of the tube alters the motion there and more or less falsifies our measurement.

We are now about to consider pressure in fluids in motion, and we approach our subject by first speaking of pumps.

406. A **pump** is a machine which gives energy to water—that is, it can raise water to a height, giving it **potential energy**. It can force water into a vessel under great pressure, moving pistons--that is, it can give to water **pressure energy.** It can set water in motion—that is, give it **kinetic energy.**

Reciprocating-pumps are either lift-pumps, or force-pumps, or combinations of both.

In Art. 137 we described the force-pump used with hydraulic presses. In principle this is the same as all other force-pumps. The feed and other pumps of steam-engines, and the pumping parts of pumping engines, are described in detail in works on those subjects. As to **Lifting Pumps,** in Fig. 279 we have a diagrammatic representation of the common village or house pump. The rod R, usually worked by means of a handle or lever, pulls the bucket, B, up and down. The bucket has openings arranged with flaps or other valves in various ways, so that fluid may pass upwards but not downwards. At W there is another valve which opens upwards. First suppose there is only air between B and the well, J. Sometimes in the morning it is necessary to throw in some water on the top of B to make it air-tight. As B descends, much of the air between B and W passes through B. As B is lifted it produces a partial vacuum below it; air passes from H to the barrel, and water rises in H. As the pumping proceeds there is more of a vacuum until water fills H and the barrel, in case B is not nearly 34 feet above the level in the well. 25 feet is probably the maximum height in house pumps. After this it is water that passes through B in every down stroke, and is lifted in every up stroke.

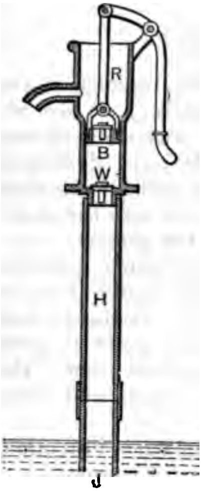

Fig. 279.

When a higher lift is needed, we sometimes place a valve in the upper part of the barrel, or in the delivery pipe, to prevent the return of the lifted water. Fig. 280 is a diagrammatic representation of a **force-pump.** Sometimes a piston is used instead of the plunger, P. A is an **air-vessel.** As more and more water enters at B, the pressure of the air becomes great enough to force the water up the delivery pipe, E. In this case the stream is not so intermittent. Unfortunately the air is apt to dissolve in the water and get carried off at a rate which depends upon how much air is already in the water. Fig. 281 shows a **double-acting force-pump.** When the piston B moves to the right, water passes through the valves, A, to the delivery pipe, D, and enters the barrel through D from the suction pipe, S, and well. When B moves to the left, C is the delivery valve, and E is the suction valve. Fig. 282 shows a combined force and lift pump whose action is very easy to understand. The delivery valve is not shown.

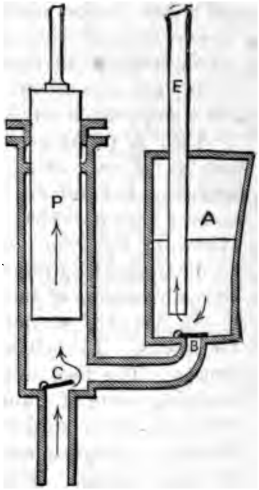

Fig. 280.

407. A pump must be efficient—that is, it must do nearly much work on water as the pump itself receives from an engine or labourers. But it must be remembered that the best pumps used for different purposes have very different efficiencies. Forty per cent. would be regarded as a reasonable efficiency for reciprocating pumps of low lift, whereas it would be regarded as rather poor for pumps of high lift.

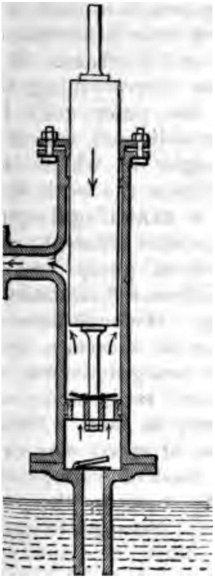

Fig. 281.

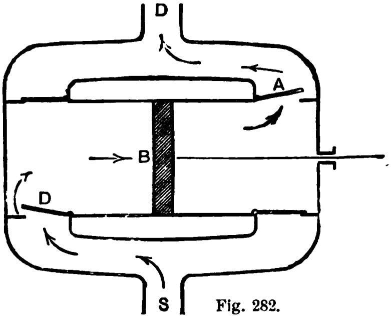

Fig. 282.

Example.—(1) Suppose that when a pump is delivering a certain quantity $Q = 1{,}000$ gallons per minute, there is a loss of 35 foot-pounds per pound of water in the pump and horizontal pipes, and 0·05 foot-pound per pound for each foot of vertical pipe. Calculate the total loss per pound in each of the following cases, the vertical height of pipes and of delivery being called h feet :—

h	Useful work per pound of water.	Lost energy.	Total energy.	Efficiency.
20	20	36	56	0·36
50	50	37·5	87·5	0·57
100	100	40	140	0·71
200	200	45	245	0·82
300	300	50	350	0·86
400	400	55	455	0·88

(2) If the loss of the pump is $20 + 15 \times 10^{-6} Q$ per pound of water, and of the vertical pipe is $5 \times 10^{-8} Q^2$ per pound, Q being

gallons per minute, work out two more tables like the above—one when the delivery is 500, and the other when it is 1,500 gallons per minute.

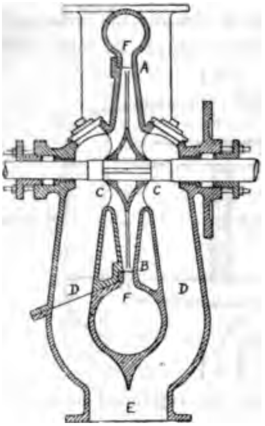

Fig. 283.

408. The peculiarity of reciprocating-pumps is that, when there is no **slip**, there is the same quantity of water passed through the pump at every stroke. If we know the size of everything, then the speed of the pump tells how much water is delivered, and if we know the height to which it delivered, we know the work done.

Now, in a **centrifugal-pump** things are somewhat different; with a given speed of pump we may have very different quantities of water passing. We may have the pump running at a certain speed, and no water being delivered, and very little work being done, only frictional work, in fact. Now a slight increase of speed may cause an abundant flow of water, and a tremendous increase in the work usefully done. Strange to say, if this speed be now diminished to what it was at first, it does not follow that the water will cease to flow. In any centrifugal-pump such as Fig. 283 we observe that there is a central wheel, A B, with **vanes**, which can be rotated very rapidly. Water can enter the wheel on both sides, C C, at its centre from two supply pipes, D D, which meet in one pipe, E, below. The water fills the wheel or space between the vanes, and, being whirled round with great velocity, tries to get away at the circumference of the wheel, because of the centrifugal force, and it flows out into the casing, F F, gradually becomes the discharge-pipe of the pump, as in the small scale drawing, Fig. 284. This is a

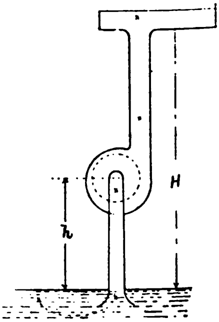

Fig. 284.

planation of what occurs, but we must examine the matter ore carefully.

409. What occurred when the pressure in the pump of the draulic press became greater than the pressure of fluid in e press? There occurred a flow of fluid. The fluid was set motion from pump to press. A **difference of pressure** tween two places which communicate with one another ually means a tendency to produce motion.

In the hydraulic press the flow is intermittent. Why? cause the pressure is intermittent. We may be sure that en we have a certain difference of pressure between two aces, and this is always the same, the flow of fluid is per-tly **steady**; and we saw in Art. 408 that work is then done the fluid with perfect uniformity. We also saw how to tell hat work was done. The work done on a fluid in a minute simply the difference of pressure per square foot which causes e flow, multiplied into the number of cubic feet of water hich flow per minute. This gives the answer in foot-pounds.

Suppose that with pumps, or in any other way, we establish difference of pressure between a place A and a place B, and ppose we know that our pumps and other arrangements will ot break down—in fact, that the difference of pressure etween A and B is really a fixed thing on which we can lepend—we know that the flow of water from A to B will be he same at all times, and that the same amount of work will e done upon it every minute. If, now, we leave out of our inds all consideration of how that constant difference of ressure has been produced—merely think of the two vessels and B—we know that this difference of pressure which has een established is really a **store of energy**. What enables us call it a store of energy? The fact that we know it will ot be **suddenly destroyed**.

Suppose we know that a man has a certain income paid, y, by Government, and suppose we are perfectly certain that is income is constant, we can regard the certainty of the an's income as a store. To say that a man makes a sovereign a day is not of much importance, but to say that the man as a regular income of one pound a day makes him a respect-ble member of society, and a store of social energy.

A man may be sitting in Parliament, but this in itself does ot make him a store of political energy; whereas if we know hat he is certain to sit there for a length of time—that a general election or general vote of the House is unlikely to unseat him—we can regard him as possessing a store of political energy.

Similarly, a pound of water in the vessel A, at rest, possesses more energy than a pound of water in the vessel B, at rest. (We must remember that there are some means of keeping the pressures in A and B what they were.) How much more energy has it? If the difference of pressure is p lbs. per square inch, then it has **2·3 p foot-pounds** of energy; not in virtue of its own intrinsic worth, but because it is where it is, and because we know that if it flows into the vessel B, nothing will alter in the pressure conditions of the two vessels till it gets into the vessel B. We understand, then, that a pound of water, subjected to a pressure of p lbs. per square inch, may be said to have a store of 2·3 p foot-pounds of energy, if we know that the motion which is occurring in the water is steady, and is not altering capriciously.

Thus a pound of water at the pressure of the atmosphere, 14·73 pounds per square inch, possesses, in virtue of this pressure, 2·3 × 14·73 or 34 foot-pounds of energy. It would possess the same energy if it were at the pressure 0, the pressure in a vacuum, but were 34 feet higher than it is in position.

Suppose that A is a closed box, and that it is filled with water, under great pressure. Now, suppose we open a valve, and let this water escape. Although there was a great pressure, p, just for an instant, and therefore a rapid flow of water just for an instant, this almost instantly dies away, because the pressure in A is almost instantly diminished. Every pound of the water did not then have a store of 2·3 p foot-pounds of energy, and yet it was at the pressure of p pounds per square inch. It is the certainty that the **state of pressure in A will continue constant** that gives to pressure its significance, and gives to us the liberty of regarding **pressure as a store of energy.**

410. Suppose that we have, anywhere, steady motion of water. Consider a pound of the water. What is its total store of energy?

1. It is h feet above some datum level. Then if one pound of water ever were allowed to fall to the datum level, it would do h foot-pounds of work in falling. The mechanical energy stored up in a miller's dam is simply the weight of the

ultiplied by the height through which it can fall. Of if any other volumetric force acts on the pound of his will constitute another store of potential energy. supposing that the weight of the water is the only ric force.

s the motion is steady, if the water is at a place ressure is p pounds per square inch, it possesses, in f the fact that the motion is of a steady character, the p foot-pounds of energy.

s the water is in motion, if v is its velocity in feet nd, as its mass (the mass of one pound) is 1÷32·2, we at its kinetic energy, or energy of motion, is the f the velocity divided by 64·4.

cise.—Calculate the numbers in the following table, ows the relative values of h and p and v, if we wish rt one of these forms of energy into another. Thus, gy due to a difference of level of 2·3 feet is equiva- hat due to a difference of pressure of 1 lb. per square to that due to a velocity of 12·18 feet per second.

of level.	Pressure.	Velocity.
eet.	1 lb. per square in.	12·18 feet per second.
,,	7 ,, ,,	32·2 ,, ,,
,,	14·73 ,, ,,	45·7 ,, ,,
,,	28 ,, ,,	64·4 ,, ,,

Now, we shall not suppose that the pound of water has r stores of energy than these. We know that, as it essible to some extent, it may have a store as the ng of a clock has. This store must always be taken unt when the fluid is air or any other gas. It may electrified or at a high temperature, or it may have res of energy which we are neglecting. Merely think three stores:—**Potential energy** due to height above level; **pressure energy** due to the unchangeableness ; **kinetic energy** due to its actual motion.

t we must remember carefully is the fact that this f water retains all of this energy except what it **friction.** Suppose that a man has capital in the gold, capital in the shape of shares which never alter in price, and capital in the shape of a pet manufactory, which wastes money just in proportion to the amount of capital invested in it. He may buy more shares or sell them out, invest more or less money in the pet manufactory, but all the time his only loss is the loss from the manufactory. He may have no gold, or no shares, or very little money in the factory, but all the time his total capital is unchanging, except that he loses in proportion to the value of his factory. It is only when water has part of its energy in the shape of kinetic energy, only when it is in motion, that it loses any part of its total store.

412. Consider a **lake of water** at rest. Consider a point A, and somewhat below its level another point B. A pound of water at A has the same store of energy as if it were at B. In neither case is there any energy of motion. The store of energy at A is merely due to height above some datum and the pressure per square inch at A. Now, if a pound of water gets to A from B, it loses potential energy h foot-pounds, if the difference of the level is h feet, and it ought to gain an equivalent of pressure energy, and the gain of pressure is, as we have already seen, simply $\frac{h}{2 \cdot 3}$ pounds per square inch. Thus, if h is 34 feet, and p is the gain of pressure, then there is a gain of pressure energy of 34 foot-pounds—that is, there is a pressure at B of 34÷2·3, or 14·73 lbs. per square inch greater than the pressure at A. This is an increase of pressure called one atmosphere. In still water there is an increase of pressure of one atmosphere for every 34 feet of descent. (See Art. 173.)

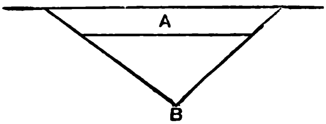

Fig. 285.

413. To further familiarise us with the idea, consider the **flow of water from an orifice.**

In Fig. 286 we see the stream lines along which water flows out of the orifice. The shapes of these stream lines will depend on the shape and position of the orifice.

Now, a pound of water at the upper still surface A is at atmospheric pressure, and when it reaches C it is also at atmospheric pressure, so that its pressure energy remains the same.

But at C it has fallen h feet; it has lost h foot-pounds of potential energy, and it must therefore have gained h foot-pounds f kinetic energy. If v is its velocity, then $v^2 \div 64{\cdot}4$ must be qual to h, so that v may be calculated; hence its velocity is just he same as that of a stone which had fallen freely from A to C.

To illustrate this, a high vessel may be used, from which at ifferent levels tubes come out, ending in **nozzles**, throwing ts vertically upwards. These ts do not rise to the same height, ecause there is a loss of energy ue to friction, and this friction ccurs principally at the nozzles. the jet reaches within a disnce h_1 of the level of still water side, and if the nozzle is at the pth h below still water level, en if we may say that all the iction occurs at the nozzles, $_1 \div h$ expresses the loss as a fraction of the whole kinetic energy at the nozzle.

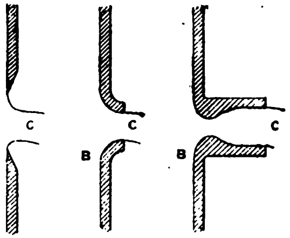

Fig. 286.

When the measurements and calculations are made for ifferent levels of the water in the vessel, we find nearly the ame result in every case, showing that the **frictional loss of nergy** seems to be **proportional to the kinetic energy** there.

There is very little friction in the case we are considering n Fig. 286, where the orifice is **sharp-edged**, and we have a ery simple statement of the velocity at C.

Can we say the same about the velocity at B? Certainly ot. If we knew the pressure energy at B, we could say how nuch of the lost potential energy has been invested in this hape, and therefore how much has been invested in the hape of kinetic energy; but without knowing the pressure at B, we cannot tell what is the velocity there.

414. In this subject of flowing water there are more **misleading hypotheses,** due to perverted ingenuity, than in almost any ther; and, unfortunately, the logical conclusions drawn from hese hypotheses, when known to be untrue, are said to be the tatements of *theory* as opposed to *practice.* We often hear he statement, "the theoretical velocity at an orifice is the elocity which the water would have acquired if it had fallen reely, as in a vacuum, from still water level;" whereas it is vident that we cannot tell the velocity at any point in the jet nless we know the pressure there, and we only know the ressure on the very outside of the jet.

Now, although we do not know the pressure or velocity a every point of water flowing from an orifice, the studies Professor James Thomson enable us to make certain important statements which agree with experiment. One these is this:—

When frictionless liquid flows from two similar vess through similar orifices similarly situated with regard to f water level, the lines of flow are exactly of similar shape; velocities at similar points are exactly as the square roots the dimensions of the vessels, and the total quantities of liq which flow are proportional to the square roots of the powers of the dimensions.

Thus, if we have water flowing similarly from three similar vessels, all made from the same drawings, but to different scales—say one 1 foot, another 4 feet, and another 9 feet deep —the velocities at similar points are as 1 to 2 to 3, the sections of stream tubes are as 1 to 16 to 81, and the quantities of liquid flowing from the three vessels are as 1 to 32 to 243; thus we are quite sure that 243 times as much liquid flows from the third vessel as from the first.

Hence, suppose we want to know **how much water is flowing** in a small stream, we dam the water up somewhere, and let it flow out of our dam through a **notch** like Fig. 287 (a right-angled isosceles notch) in a wooden board with sharp edges. Indeed, we prefer to have the edges of the notch made of metal, so that we shall be sure that they are straight and sharp. Measure the height of the still water in the dam above the lowest point of the notch (D). A graduated post, rising from the bottom of the dam some feet away, its zero being on the level of D, is very convenient for this. This one measurement tells us how much water is tumbling over. For we know that, suppose there is rather a drought one day, and *a* shows the appearance of the notch; and on another, a rainy day, A shows its appearance, we observe that the orifices through which the water is flowing are similar and similarly situated with regard to the still water level; and **Thomson's theory** enables us to say that, if on one day the height is 1 foot, and on another it

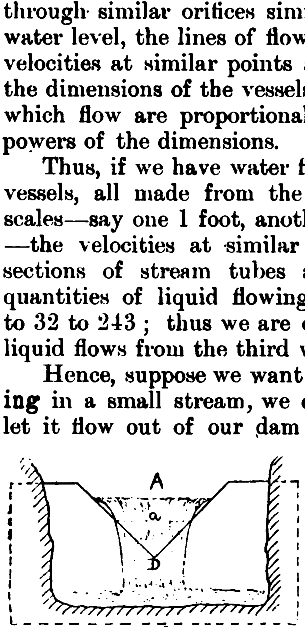

Fig. 287.

is 4 feet, then 32 times as much water comes over on the second day as on the first. The **flow of water** is exactly **proportional** to the **square root** of the **fifth power** of the height D A.

Now, Thomson measured very accurately how much water is flowing when the height is 1 foot, and he found it to be 2·635 cubic feet per second. Hence **we have the rule**: measure the vertical height D A, at any instant, in feet; raise this to the fifth power, and extract the square root, and multiply by 2·635, and we know how much water is flowing in cubic feet per second.

If we know the cubic feet of water flowing per second, we know the weight of the water, since a cubic foot of water weighs 62·3 lbs., and the weight of water, multiplied by the number of feet through which we can let it fall, tells us the foot-pounds per second—**the available power of the stream.** The foot-pounds per minute, divided by 33,000, is the **horse-power of the stream.**

415. Gauge notch observations, made from day to day on a stream, enable a person to make very exact calculations as to the power available for the **driving of mills** by **turbines**, for the working of hoists, cranes, and lifts, and for hundreds of other purposes.

How Thomson used his theory in proving that the famous **Lowell** empirical formula, for rectangular gauge notches, is really a rational one, will be found in Art. 436. Students ought to treasure anything published by Thomson on fluid motion.

416. Many rather abstruse-looking questions are easily answerable when we fully grasp the significance of the energy law.

The **fundamental fact** which makes any hydraulic problem clear to you is this. If we may neglect friction, then **a pound of water** at any place has its **total energy** in three shapes. It has h foot-pounds of energy, because it is h feet above a datum level. It has 2·3 p foot-pounds of energy, because its pressure is p pounds per square inch; and it has $v^2 \div 64{\cdot}4$ foot-pounds of energy, because its velocity is v feet per second.

To take another example:—

Suppose we have a **pipe** which is in the main horizontal, so that we may neglect differences of level. Then we have to remember that the pressure energy, plus the kinetic energy, of a pound of water does not alter. When water flows along a pipe which is full, there must be the same quantity flowing everywhere. We are sure, therefore, that there must be greater velocity wherever the pipe is contracted. But greater velocity means greater kinetic energy, and if this water invests more of its energy kinetically, it must have less in the shape of pressure energy. That is, the pressure of the water at B (Fig. 288) is less than at A or at C. The pressure at B may become very small indeed. It is easy, in this way, to reduce the pressure to much less than the atmospheric pressure—merely contracting the cross section of the pipe is sufficient. We cannot make the pressure as small as that of a vacuum, for before that limit is reached vapour forms.

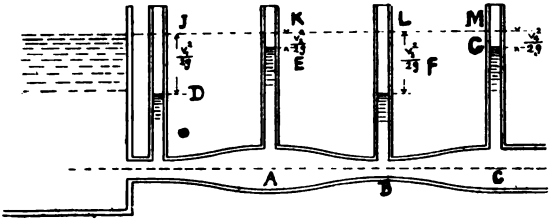

Fig. 288.

417. Suppose a **conduit** of this kind were carried over our fields, and that a quantity of water lay in our fields, at not too great a depth below the conduit. If we bring a pipe to the point B from the field-water, this becomes a suction-pipe, and we get our **fields drained** at the expense of the conduit owners. We spoil their water, if it is clean, but at all events we get our fields drained.

In this lies the theory of jet pumps, and much of the theory of injectors, etc. The **jet pump of Professor James**

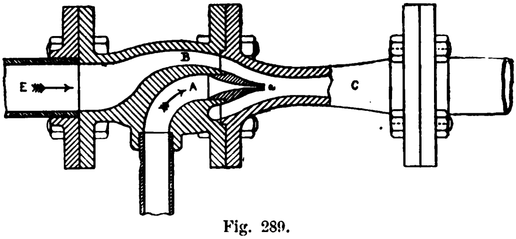

Fig. 289.

Thomson simply consists of a small pipe, A (Fig. 289), which ends in a nozzle. Through this, let us suppose, we have a small supply of water flowing from some pretty high reservoir. Suppose that this water flows into the atmosphere at C. Evidently the pressure at *a* is much less than the pressure at C. It is less then than the atmospheric pressure, and hence the neighbourhood of *a* is a partially vacuous space, so that the pipe B E becomes a suction-pipe, and water tends to flow from a point at E, to B, and on to C. Thus, if we have a small supply of water from a high reservoir, we are able to drain a marsh with it.

418. There is not space here to speak of the hundred other ways in which our principle comes in to simplify all sorts of puzzling phenomena. We may refer to **Mr. James Perry's syphon** for the discharge of flood-waters at the weirs in rivers. It has no moving parts in the water, being simply a wide, open pipe of varying sectional area, through which an object as large as a bullock might pass, without injury to the sluice. It has been found, by actual trial, that the quantity of water passing per minute through such a sluice is independent of the fall, so long as there is sufficient fall to balance the waste by friction in the pipe, a few inches being enough in the case of wide pipes; and a velocity of 45 feet per second, at the smallest section, may be calculated upon when the sluice is working full power. The cross-section of the syphon is like the letter **D** at the tail; the masonry, or concrete basin, is actually a portion of the syphon, and the horse-shoe shaped space between the lip of the basin and the iron edge of the syphon proper is the actual opening. The quantity of water passing at any time is regulated by the admission of air to, or its exclusion from the syphon. There is a throttle-valve arrangement by means of which the syphon may be adjusted at any time, so as to vary what may be called the normal water level in the reach above the sluice, and to vary the opening through which a constant stream of water falls on a ridge-shaped portion of the syphon. This stream of water is needed to exhaust the syphon of air, so that action may be set up at any time, even when the water is not passing over the top of the throttle-valve. The manipulation of such sluices is perfectly simple—they may be made self-acting; but it is proposed in important places to work all the sluices on a river from a single station electrically.

Example.—There is a circular sharp-edged orifice in a tank in which there is liquid kept standing to a certain height. A man is told that, without interfering with the actual edge of the hole, he is allowed to do what he pleases to increase the flow. How may he do so? Evidently by fitting on a tube which quickly but gradually gets to be of much larger diameter. He takes care that the tube shall run full.

Example.—A number of mill-owners receive water from the same lake. Each has a rectangular opening the depth of which below the lake and its breadth are supposed to fix his supply. Show that if a mill-owner is allowed to do anything he pleases on his own side of the opening, he may procure a very much greater supply of water.

419. We know that **loss of energy per pound by friction,** in water, is proportional to the square of the velocity, at such speeds as are common in pumps. In Art. 48 we gave the rules by which we are able to calculate the loss of energy which a pound of water experiences in going along pipes. But we wish to impress on you the fact that this loss always becomes very great when the flow of the water has to occur abnormally. In the pipe (Fig. 288) the wide and narrow parts gradually change into one another by continuous curves. It is practically impossible for a liquid to flow in a discontinuous curve. Suppose we try, then, to make it flow along the pipe shown in Fig. 290. What the water does is this: when it comes to the corner it produces for itself **wheels,** little **eddies** or **whirlpools,** as we might put rollers under a log of wood that we wanted to get along easily; and there is great loss of energy due to this, for the eddies have not only to be in the corners, but there are smaller eddies carried along by the water itself, maintained so long as they are needed. We have been speaking of actual discontinuity in the flow. But there is a fact which many **hydraulic engineers** seem to be **quite ignorant** of—namely, that a liquid cannot flow along a path which suddenly changes in curvature. A liquid cannot, for example, flow along this path (Fig. 291). At A it changes from a straight line suddenly to the arc of a circle, and, consequently, the water digresses at A, and creates little castors,

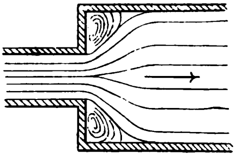

Fig. 290.

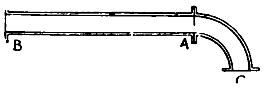

Fig. 291.

little eddies to carry it by a path of continuous change of curvature from B to C. Now this analogy can be shown to be true experimentally. Thus it has been found that if water flows along the bend (Fig. 291) it loses a certain amount of energy on account of the bend; but if we make the pipe bend as much again in the same direction as in A B (Fig. 293), we do not get again the same loss; indeed, there is comparatively very little loss at the second bend. But if we bend the pipe in the opposite direction, as in C D (Fig. 292), there is as much loss at the second bend as at the first.

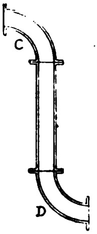

Fig. 292.

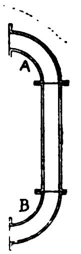

Fig. 293.

The little wheels, or castors, or eddies which the fluid creates for itself to carry it through the bend A are available when the water needs them again at B, whereas the wheels or eddies produced at C have to be destroyed, and new ones created, rotating just in the opposite direction to carry the fluid through the bend D.

We see now how necessary it is that all curved vanes or other surfaces along which water flows should be drawn, not with a pair of compasses, but rather with a batten, **a thin strip of wood**, which bends gradually.

420. We regard a pump as a contrivance which gives to every pound of water passing through it an additional store of energy. From the pond to the entrance to the pump, every pound of water has just the energy it has in the pond, barring frictional loss. From pump to cistern, every pound of water has an additional store of energy, and it is the pump which gives it this store.

Suppose we have a centrifugal pump going at a regular speed, and discharging a regular quantity of water. In the supply-pipe a pound of water has a certain total amount of energy which we know, if we know its height above datum, its pressure, and its velocity. But, in passing through the wheel, it receives a supply of energy. The total energy of a pound of water in the discharge-pipe is greater than what it is in the supply-pipe. We can make all sorts of calculations, if we know what is the clear gain, the clear gift of energy it gets in passing through the pump.

421. Suppose **a man jumps** into an American railway train anywhere, and after wandering about, fore and aft, jumps out again. Find the man's momentum in the direction of the train's motion just before he alights on the train. Find his momentum in the same direction when he has just sprung from the train; the difference of these is the total impulse with which he acts on the train. It is the momentum which he gives to the train. Suppose that a number of people could perform this acrobatic feat every second with the greatest regularity, then the momentum given in one second to the train could be calculated. But momentum given per second is what we call force; hence we have found the force acting on the train due to these **jumping individuals**, and this force, multiplied into the space passed through by the train in one second, gives the propelling work done upon the train per second. We have nothing to do with whether it is a propelling force or a retarding force. In the one case, the acrobats give momentum to the train; in the other, the train gives momentum to them. The loss of momentum per second in a regular stream of people, going on and off the train, is a force which is applied to the train. We only have to do with their momentum in the direction of the train's motion.

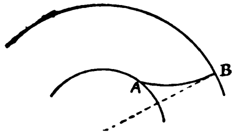

Fig. 294.

Now suppose that, instead of its being a train, it were a sort of circular turntable, or a merry-go-round, and that a regular stream of people jumped on and off. In this case, the place where a man jumps on may be going at a different speed from the place where a man jumps off; but our rule is not very different. Find how much is added per second to the momentum of the wheel at the point where people leap on, and regard this as a force. Multiply by the speed of the wheel there, and this is the **work done** by the mere leaping on, and staying on the wheel. Now, find how much per second is taken from the momentum of the wheel at the place where leaping off occurs, and regard this as a force opposing the motion. Multiplied into speed at the leaping-off place, we have the work taken by the stream of people from the wheel, per second, because they leap off. The difference in these two things is, of course, the work done in the jumping on and off.

Now, the water moves from the centre towards the vanes of the centrifugal pump, merely radially, and hence the water entering the vane cannot add to or diminish the momentum of the wheel just there. It had no momentum of its own previously in this direction. What occurs inside the wheel now we have nothing to do with, excepting that we know frictional loss occurs there. We are only concerned with how the water leaves the wheel. The way in which it is made to leave the wheel determines how much energy it takes from the wheel.

Take the simplest case. Suppose the vanes to be radial at B (Fig. 294). That is, besides moving outwards radially at B, the water leaves the vane with the same tangential velocity as the vane at B has. Suppose this tangential velocity to be v. Then W lb. of water leaving B per second leaves with a tangential momentum, $W v \div 32{\cdot}2$, and retards the wheel with a force of this amount acting at B. This force $\times\ v$ is the energy which it receives per second from wheel, or $W v^2 \div 32{\cdot}2$. One pound of water, therefore, receives the energy $v^2 \div 32{\cdot}2$ from the wheel in passing through it.

We understand, then, that a pound of water in the discharge-pipe of the centrifugal pump has this greater store of energy than a pound of water in the supply-pipe, except for frictional losses. If we make the water go out from the wheel as backward bent vanes make it go (Fig. 296), and as it would be dangerous for us to do from a railway train, less work has been done upon it by the wheel. If it goes out in the direction of motion relatively to the wheel, more work is done upon it than we are now supposing.

422. The wheel with radial vanes gives $v^2 \div 32{\cdot}2$ foot-pounds of energy to one pound of water. If we know how many pounds of water pass through the wheel, we know then the total amount of **work done by the wheel.**

The water gets this energy to squander or store as it pleases, and it does squander it in friction to a large extent. But suppose it squandered none of it, but converted it all into potential energy in lifting itself up to a cistern, it would lift itself $v^2 \div 32{\cdot}2$ feet high; that is, it would lift itself **above the pond to twice the height due to the velocity of the rim of the wheel.** Suppose the rim of the wheel has a velocity of 5·7 feet per second, a stone would have to fall freely 34 feet to acquire this velocity, and hence the total rise of water would be 68 feet, twice the height due to the velocity of the rim of the wheel. In this case we should say that the pu was perfect.

The wheel itself receives energy from the engine, else could not give energy to the water. It gives out all t energy that leaves the engine, except what is wasted in be ings everywhere, and what is wasted in friction with water.

The energy given out by the engine per pound of wa divided into $v^2 \div 32 \cdot 2$, is the **efficiency of the shafting, b ing, and wheel.** Again, the real height to which wate lifted by the pump, divided by the ideal height, $v^2 \div 32 \cdot 2$, the efficiency of the water passages from pond to wheel an from wheel to cistern.

The loss in these places is due to friction. Make the supply-pipe wide, **bell-mouthed** at the bottom, where water enters it, so that it may enter by gradual curves; make the approach to the wheel as gradual as possible; let the vanes of the wheel make the calculable angle with the central circle, which will reduce the shock there (see Art. 428); make the discharge-pipe wide, and let the velocity with which the water enters the upper cistern be as small as possible, and we greatly reduce the waste of energy. But there is one particular place where there is usually much greater waste than anywhere else, and that is the chamber outside the wheel.

423. Just when the water leaves the wheel a large portion of its energy is kinetic. It is in rapid motion. Now, in the large discharge-pipe there may be as little kinetic energy as we please. Hence, from the time the water leaves the wheel till it enters the discharge-pipe there ought to be great care taken in allowing the kinetic energy to become converted into pressure energy.

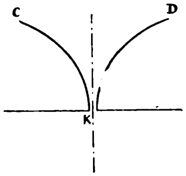

Fig. 295.

Professor James Thomson discovered here the efficiency of a **whirlpool chamber.** When we let water escape from a wash-basin, we know that the surface of the water takes a shape like this (Fig. 295).

The velocity of the water is greater the nearer it is to the centre. The pressure is greater the farther away from the centre. The spiral motion which we observe in this case is the only steady motion of water which allows a constant radial discharge without the water making objections, setting up little eddies of its own, and thus wasting energy in friction.

Hence, in Thomson's pump, after the water is discharged, it circulates in this cylindric whirlpool chamber, which he made of twice the diameter of his wheel. When a pound of water reaches this place, in consequence of its radial and circular motions, it retains more nearly the whole of its total energy than if we let it discharge in any other way. It has lost much of its velocity, but has gained in pressure. This whirlpool chamber of Thomson's, then, did for the water what gradual curves do for the water in a pipe; it enables the water to convert its kinetic into pressure energy, with a minimum of waste in friction.

It has, however, to be remembered that even here, at the outside of the whirlpool chamber, the water retains a very considerable amount of kinetic energy, even when the whirlpool chamber is made very large, and much of this is wasted afterwards. And, although no one believes more firmly in the reasoning of Thomson than I do, I feel that perhaps the whole problem admits of a better common-sense solution. Let the whirlpool chamber get wider or broader, as well as larger in diameter. Let the wheel have larger orifices on its outer circumference than on its inner circumference. The water will lose its kinetic energy far more rapidly as its passage widens more rapidly. The result of this will be that although in this rapid change there is more friction, yet when the water is only a short distance out, it is not moving much faster than it will do in the discharge-pipe, and there is much less loss in entering the discharge-pipe. We have always thought that since Thomson's chamber is expensively large, we ought to submit to a modification of his conditions even from the place where water leaves the wheel. But when we begin to consider a much smaller chamber, we see that possibly the vanes ought rather to slope backwards than to be radial. Let the radial velocity at M N be v_r. The velocity relatively (see Art. 36) to the vane is $v_r \div \sin. \theta$, if M N P is θ.

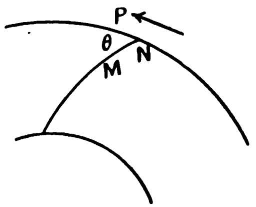

Fig. 296.

424. Let V be the tangential velocity of the wheel at N.

Using a very easily understood graphical method of working, let O N represent the radial velocity at N to scale. Let M N and N P represent the direction of the vane and rim at N. Let N P represent V, the tangential velocity, and if O M is drawn at right angles to O N, M N represents the velocity of the water *relatively to the vane.* Hence, as the resultant of M N and N P is M P, M P is the total velocity of the water leaving the wheel. It has the tangential component M Q, and the radial component Q P which it had in the wheel. Suppose we call M Q by the letter v. Now, every pound of water leaves with the *tangential* momentum $v \div 32{\cdot}2$, or say v/g. Every pound per second therefore represents a tangential retarding force, v/g, acting on the rim of the wheel, and this multiplied by V, or $V\, v/g$, is the work done usefully per pound of water. If we take it that in all cases there is a loss of the fraction s of the kinetic energy due to tangential motion, only $s\, v^2/g$ is wasted; $V\, v/g$ is total energy; $(V v - s\, v^2)/g$ is useful energy, or height to which the water is lifted. We may say roughly that the efficiency is $1 - s\frac{v}{V}$; s depends on the size of the chamber. It is therefore a question of minimum total waste of value, in interest on plant and waste of energy, etc., as to what the value of v, and therefore the angle θ, ought to be.

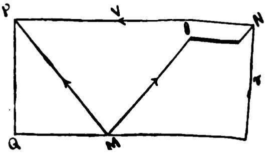

Fig. 297.

425. In the case of pumps we saw that each pound of water gets an increased store of energy, which may be in the shape of pressure energy, or kinetic energy, or both, but which mainly becomes potential.

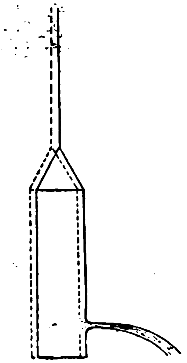
Fig. 298.

Now, in **water-wheels, turbines, water-pressure engines,** including hoists and lifts, we take part of the store of energy from each pound of water, giving it to machinery.

As a simple case of the abstraction of energy from water, and as an illustration of the acrobat and railway-train principle, consider the vessel (Fig. 298) from which the water is flowing. Water leaves this vessel horizontally from an orifice, taking away with it momentum. The quantity of momentum it takes away per second is simply the force acting on the vessel. You see that there is a force acting, for I have arranged the vessel as the bob of a pendulum.

426. If we let the water flow from an orifice through which it comes in parallel streams, it is easy to show that the force acting on the vessel is twice the total pressure which would act on this little sluice when it closes the orifice, and no water is flowing. For if a little area a (square feet) of the orifice is h feet below still water level, the pressure at a being atmospheric, the velocity $v = \sqrt{2 g h}$; the volumetric flow is $a\ v$ per second; or the mass per second is $w\ a\ v/g$, if w is 62·3 lbs. per cubic foot; the momentum per second is $\frac{w\ a\ v}{g} \times v$, or $2\ w\ a\ h$. When the orifice is closed the force due to pressure upon it is $w\ a\ h$. There have been no very sound writers on this subject except Thomson, but even the soundest imagine the force to be less when the vessel is moving. They forget in their calculation that the water leaving the vessel had at the beginning the motion of the vessel itself. Fig. 299 shows a vessel floating on a pond, and, moving under the action of its jet; with sufficiently delicate apparatus, it may be shown that the force on it is the same when it moves as when it is at rest. If such a vessel is kept supplied with water, it is easy to calculate the force due to the horizontal velocity of the supply water; in fact, we must consider that the acrobats enter the train (Art. 421) as well as leave it. Thus, in the **propulsion of a ship,** a large centrifugal pump draws water from beneath the ship, and propels it out at the sides and sternwards.

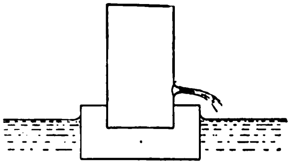

Fig. 299.

Suppose the water moves through the nozzles with the velocity of 30 feet per second, and that the ship is moving the other way at 20 feet per second, then it is evident that the water has a velocity relatively to the sea of 10 feet per second. The momentum, therefore, given to a pound of water is $\frac{1}{32} \times 10$, and this, multiplied by the velocity of the ship, gives $6\frac{1}{4}$ foot-pounds of energy, which each pound of pumped water imparts to the ship. Notice that all the kinetic energy is wasted, or $\frac{1}{2}\,\frac{1}{32} \times 10^2$, or 1·56 foot-pounds of energy per pound of water.

It is easy to see, if we had no friction in passages, that the greatest efficiency is arrived at by letting the water take with it only a very small amount of kinetic energy as it mingles with sea water; that is, by letting the backward nozzle velocity of the water be very little greater than the forward velocity of the ship. But of course this is not at all a practical solution of the problem to find the proper speed for maximum good result.

427. A **turbine**, water-wheel, or water-power engine takes energy from each pound of water, and gives it to machinery. Suppose, for example, that we have water in a tank or dam, and we have a clear fall of 60 feet. Now, when a pound of water is nearly motionless at the surface of the dam, it has just 60 foot-pounds more energy than when it is nearly motionless in the tail race at the bottom. A water-power engine of any kind is constructed to abstract this 60 foot-pounds of energy with as little waste in friction as possible. Instead of being at the same pressure in the dam and tail race, we may have the pressure energy much greater beforehand, as well as the potential energy; but in every case we try to take out of a pound of water the total difference of energy. Thus, suppose a pound of water to be motionless in a mill-dam 60 feet high above the tail race, we cannot take more from it than 60 foot-pounds of energy. Suppose a pound of water to be motionless 60 feet above the tail race, but that it is also inside an accumulator, where the pressure is 700 lbs. to the square inch; we can take from it $60 + 2{\cdot}3 \times 700$, or $60 + 1{,}610$, or 1,670 foot-pounds of work.

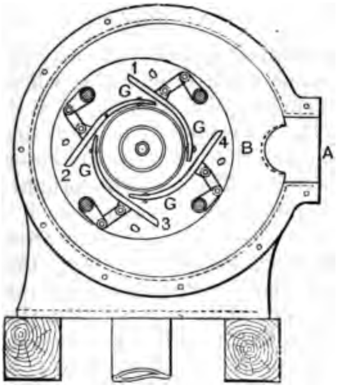

Fig. 300.

As we understand the action of the centrifugal pump, we have no difficulty in understanding the action of the turbine. It is because we have studied the centrifugal pump that we dwell upon the **Thomson turbine.** Water flows from a pen-trough through cast-iron pipes to A. These pipes must be bell-mouthed; they must open out gradually into the cistern; they must be as large in diameter as we can conveniently make them. In that case the velocity in the pipes will be small, and, therefore, the friction will be small. Fig. 300 shows a plan of the chamber, B, into which the water flows. The chamber is so large that the velocity there is small, and the water finds its way equally readily into the central space, whether it flows between the guide-blades 1 and 2, or 2 and 3, or 3 and 4, or 4 and 1. We are at last allowing the water to flow quickly, for the guide-blade chamber is narrow. When the water is just leaving the guide-blades it flows rapidly; of course it is flowing radially as well as tangentially to the rotating wheel, F, but the tangential motion ought to be equal to that of the wheel.

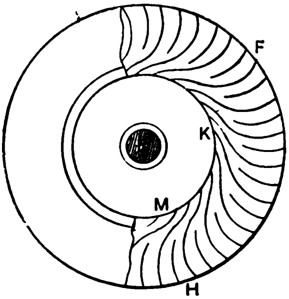

Fig. 301.

428. Suppose we want to enter a moving **railway train** or tram-car **without shock,** we try to get a velocity equal to that of the train, in the direction of the train's motion, before we venture to enter the train; hence the tangential velocity of the water must be equal to that of the end of the radial vane of the wheel, if the water is to enter it without shock. If the vane is inclined like Fig. 302, A, the tangential velocity of the water ought to be less than that of the wheel just here. If the vane is inclined like Fig. 302, B, the tangential velocity of the water is made greater than that of the vane. In fact, the relative velocity of water and vane must be in the direction of the vane, if there is

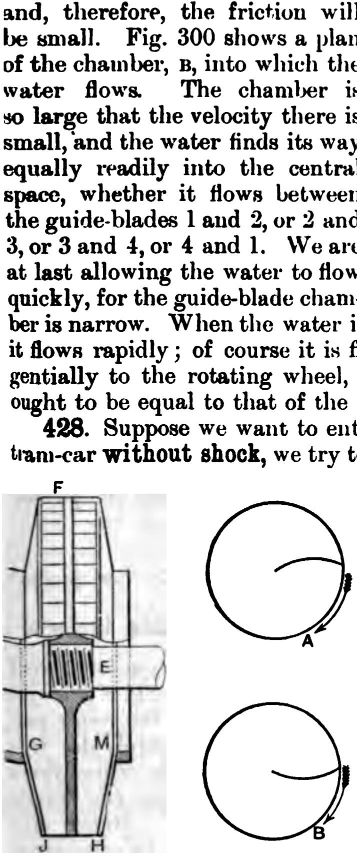

Fig. 302.

to be no shock. Usually the vane is shaped as we see it in Fig. 301, which is an enlarged section of the wheel, F; but we shall suppose it to be radial just at the outside, for simplicity of calculation. We must remember, then, that somehow or other we must try to get a **tangential velocity of water equal to the velocity of vanes** there. The water now flows through the wheel, which lets it escape at the centre. Here, again, we must remember that the water has to escape with **no velocity except a radial one.**

If we wanted to let a stone out of a railway carriage so that it would just fall to the ground vertically, so that it would possess no forward motion, we must shy it backwards, with respect to the train; give it a velocity backwards as much as it has forwards already. These vanes, then, at the centre, let the water out backwards, just because we want the water to have no forward velocity when it has left the wheel. The water has, of course, a radial velocity everywhere, which simply depends on the total quantity flowing per second, divided by the tangential areas of these orifices.

429. We want, now, to know how much store of energy each pound of water has lost in passing through the wheel, and we employ the rule already given. Find the tangential momentum of the water at F. If the velocity of the outside of the wheel is v, then $v \div 32{\cdot}2$ is the **forward momentum** of one pound of water. This, multiplied by v, is the **work done per pound** of water, or

$$v^2 \div 32{\cdot}2 \text{ foot-pounds,}$$

because it enters the wheel. Or W lb. of water per second means a momentum of $\frac{W}{32{\cdot}2}v$ per second, and **this is force**; force multiplied by velocity v gives work done per second, and this divided by W gives work per pound of water. It is evident that if we take **moment of momentum** lost by the water per second in passing through the wheel, and multiply by the **angular velocity**, we get the same answer. The wheel does no work on the water as it leaves at K, because the water leaves with no forward or backward momentum. Hence one pound of water, from the time it enters the wheel to the time it leaves, loses

$$v^2 \div 32{\cdot}2 \text{ foot-pounds,}$$

[illegible] its store of energy, and gives this store to the wheel.

If, then, it loses no energy by friction anywhere, when it enters the tail race it has just this much less energy than when it left the pen-trough. If h is the total height of the fall, evidently one pound of water really gives out h foot-pounds of energy, so that h is twice the height due to v. We know that, in practice, what the water gives to the wheel is less than h, and $\frac{v^2}{32} \div h$ is called the **hydraulic efficiency** of the turbine. It is the ratio of the energy given to the wheel to the total energy lost by the water in falling from one level to the other. If, then, there is no shock to the water in entering or leaving the wheel, its efficiency is twice the height due to the velocity of the rim divided by the real total fall of the water.

Of course all the energy given to the wheel is not utilised. There is friction between the wheel-covers and the wheel-case, friction at all the bearings, etc., of the shafting and gearing which transmit the power of the wheel to a mill. We are only speaking now of the efficiency of the passages, which is, however, the most important matter in connection with turbines.

Knowing the average amount of water passing through the wheel, and therefore the radial velocity at K, the angle of the vanes at K is determined if we know the average speed of the wheel. If the speed and quantity of water were exactly proportional to one another; that is, if the speed of the wheel were exactly proportional to the horse-power, the inner ends of the vanes once settled would remain right always. But if our wheel is to be **regulated** as a steam-engine, so that quickening speed causes less water to flow, then it is obvious that the inner ends of the vanes, although right for the calculated flow, are not properly shaped when the horse-power diminishes or increases. The loss of energy here is not, however, likely to be great in any case.

It is different at the entrance to the wheel F. Unless the **guide-blades** are directed so as to give a tangential velocity to the water equal to that of the wheel, there is a considerable loss by friction at F.

430. Suppose that less water flows through the turbine, the inclination of the guide-blades ought to alter, and this arrangement of links, which we see in the drawing, is for the purpose of making the guide-blades alter their inclinations to the wheel. Each guide-blade is pivoted at its extremity, K, and when one is shifted they are all shifted in position. Unless there is a great variation in the work which we require a turbine of this kind to do, it is not necessary to apply a governor which partially stops the water supply when the machinery runs a little too quickly, although such governors are very necessary for a great many water-wheels and turbines.

It is to be remembered that this turbine is really a centrifugal pump, through which the water is flowing negatively. Increased speed tends to stop the flow. If the wheel were at rest, the flow would be very much greater than it is. Hence, increasing the speed somewhat stops the flow, allows less water to pass through, and less work to be done. This action cannot be called a governor action, for it does not maintain a constant speed, but it may be called a **steadying action**, as it prevents any great change of speed, even for a considerable alteration in the work done.

Except at the speed for which the positions of the guide-blades are fixed, there is extra loss in friction, and the guide-blades are rearranged should any considerable change be meditated in the power to be given out.

431. In arranging a turbine, it is obvious that the great point to settle beforehand is this:—What ought to be the speed of the wheel for a given height of fall? If there were no loss in friction, we could say at once, if v is velocity of rim of wheel, $v^2 \div 32$, the total loss of energy by one pound of water, ought to be equal to h; that is, the velocity of the wheel ought to be that due to half the height of the total fall of the water. Thus, for a fall of 60 feet in height, half of this is 30 feet; and if a stone fell 30 feet, it would be falling with a velocity of 44 feet per second. The rim of the wheel ought to have a velocity of 44 feet, then, per second, and it is easy to show that, wherever the turbine may be placed, whether it has a long discharge pipe, or is submerged, the water may be made to flow tangentially into the wheel with the same velocity as the wheel itself has.

But we have usually to calculate on the assumption that a certain fraction of the energy of the water is wasted in the supply and discharge pipes, and the discharge chamber, and hence the velocity of the wheel is less than that due to half the height of the fall.

It is **usual to assume** that the **radial velocity** of the water through the wheel is **one-eighth of that due to the total fall.** Dividing this into the number of cubic feet of water flowing, we know the total tangential area of the space between the vanes everywhere in the wheel, assuming that it is the same everywhere and it usually is. It is usual to take the inner radius of the wheel, E M, equal to the depth, M G, of the passages in the wheel, so that both these dimensions are now fixed. The outer radius is generally twice the inner one, and we have already calculated the tangential velocity of the outside, so the number of revolutions per minute may be calculated. The horse-power given out is usually taken to be less than three-fourths of the true horse-power of the water. Thus, by rules, partly due to practical experience and partly due to imperfect theory, we are able to fix all the dimensions of a turbine of the kind we have been describing.

432. We think that, by entering thus fully into the theory and construction of Thomson's turbine, we can dispense with giving a catalogue of the constructions of turbines generally. This turbine is said to be one of "**inward radial flow.**" We see that, for a given quantity of water flowing, it can be made hydraulically perfect; that is, by proper construction of the guide-blades, there is no necessary loss of energy, any more than in the whirlpool chamber of Thomson's centrifugal pump.

In the same manner, we could discuss the action of water in the unsteady "**outward radial flow turbines,**" and, again, in the **axial-flow turbines** of Fourneyron and others. The principle of our stream of acrobats jumping on and off a merry-go-round will in every case tell us how much energy the water gives to the wheel of a turbine, whatever may be the nature of the flow. In much the same way, also, we consider the construction of the floats of **undershot water-wheels,** and all other wheels on which the water acts impulsively.

In the same way we might discuss a steam turbine, or the action of air in motion on windmills. Steam turbines are discussed in my book on the Steam Engine.

When the available fall is over 200 feet, it is not advisable to use a turbine water-wheel. In the turbine, as we saw, there is at least one part of the arrangement in which about half the total store of energy is in the shape of kinetic energy; and when the energy is in the shape of kinetic energy, there is a great waste by friction. The waste is proportional to the kinetic energy—that is, to the total energy—and hence turbines are at least not more economical on very high falls than on moderately low ones.

433. Steady Motion in Fluids.—The mathematical expression of the law Art. 410 is true whether the motion is steady or not, but we give to the mathematical expression the *energy* meaning when the motion is steady. Motion of a fluid is steady when every particle at any point moves in exactly the same way as all its predecessors there. The successive positions of a particle mark out a stream line, and a bundle of stream lines lie in a stream tube. Let AB be a very thin stream tube. At C let the pressure be p, the velocity in the direction towards B be v, and let C be h feet vertically above some datum level. Let the tangent to the centre line CD make an angle α with the vertical. Let $p + \delta p$, $v + \delta v$, $h + \delta h$ be these quantities at D, and let D be very near C. Let the section at C or at D be a. Consider the dynamic condition of the portion of fluid in the tube between C and D in its motion along the tube. The force urging it towards B is

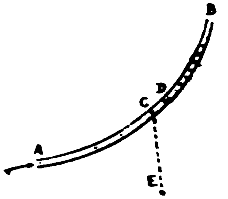

Fig. 383.

$$pa - (p + \delta p)\,a - w\,.\,\mathrm{CD}\,.\,a\,.\,\cos.\,\alpha$$

if w is its weight per unit volume, because $w\,.\,\mathrm{CD}\,.\,a$ is the weight of the element, and we resolve this weight in the direction of the inclined line. We must state that this force is equal to the mass multiplied by its acceleration. The mass is $w\,.\,\mathrm{CD}\,.\,a \div g$. If δt is the time which a particle would take in going through the distance CD, its velocity is $\mathrm{CD} \div \delta t$, and its acceleration is $\delta v \div \delta t$. Hence we may say

$$pa - (p + \delta p)\,a - w\,.\,\mathrm{CD}\,.\,a\,.\,\cos.\,\alpha = \frac{w\,.\,\mathrm{CD}\,.\,a}{g}\,.\,\frac{\delta v}{\delta t}.$$

But $\mathrm{CD}\,.\,\frac{\delta v}{\delta t} = \frac{\mathrm{CD}}{\delta t}\,.\,\delta v = v\,.\,\delta v.$

The above statements are true only when CD is thought to be smaller and smaller without limit. Dividing by a, and recollecting that $\mathrm{CD}\,.\,\cos.\,\alpha = \delta h$, we find

$$-\delta p - w\,.\,\delta h = \frac{w}{g}v\,.\,\delta v,$$

or

$$\frac{v}{g}\delta v + \frac{\delta p}{w} + \delta h = 0,$$

or, as we may write it, with the idea that the values of δv, etc., are smaller and smaller without limit,

$$\frac{v\,.\,dv}{g} + \frac{dp}{w} + dh = 0 \;\ldots\ldots\; (1).$$

For liquids, w is constant, and hence, integrating,

$$\frac{v^2}{2g} + \frac{p}{w} + h = \text{constant} \quad \ldots . \ (2).$$

If w is not constant, we cannot integrate until we know how w varies as a function of p. Until we know this, we can only write

$$\frac{v^2}{2g} + \int \frac{dp}{w} + h = \text{constant} \ (3).$$

We note in (2) that $v^2/2g$ is the kinetic energy of a pound of water. h is its potential energy. And, be it because of Art. 410, or merely as a help to the memory, we mean to call $\int dp/w$, or p/w, the pressure energy per pound of fluid. **The total energy of a pound of fluid remains constant, except in so far as friction may diminish it.**

Example.—In Fig. 306 we see some stream lines of a fluid leaving a vessel by an orifice. We assume no friction. I shall consider the various orifices shown on an enlarged scale in Fig. 286.

We wish to make the statement that (2) is the same at A as at C. Observe that at A the pressure is atmospheric. But what is the pressure at C? If C is a stream line touching the atmosphere, of course the pressure is known; but inside the stream at C the pressure is *not* known with certainty. We assume it to be atmospheric. Now a pound of water in coming from A to C has had no change in its pressure energy. It has lost potential energy H feet, if H is the vertical depth of C below A. It had no kinetic energy at A; and if v is its velocity at C, its kinetic energy there is $v^2/2g$. Therefore we say that the loss H is equal to the gain $v^2/2g$, or $v = \sqrt{2gH}$.

It will be noticed that we choose C in Fig. 286 at places where it cannot be very wrong to assume atmospheric pressure, and where the stream lines are all presumably normal to the cross-section a. Hence the total quantity of water flowing per second is $Q = a\sqrt{2gH} \ldots . (4)$ cubic feet. Our two difficulties, what is a? what loss is there by friction? are solved by experiment. It is interesting to know that the most careful measurements of a and Q, when orifices are sharp-edged and water is flowing, show no perceptible loss by friction. This is one reason why sharp-edged orifices are preferred in the measurement of water and other fluids. Another reason for this is the accuracy with which we can verify the shape of an orifice. Hence in square, or round, or triangular-shaped orifices we have taken great pains to find such a place as C, and to find a there. In the case of a round orifice a equals the area of the hole multiplied by 0·62 very nearly. The part C is called the "**contracted vein**" often referred to by writers. In the case of Fig. 305, a is half the area A of the small hole M N. This is the only case in which we can make any easy attempt to calculate the area of the contracted vein. Note that with this orifice, whatever be the shape of the vessel, the total pressure of the fluid upon it in the direction opposite to the arrow at C is shown to be, being A w H if H is the depth of the centre of A below water-level, because the velocities, and therefore the pressures, everywhere, except near MN, are the same as if the orifice were closed, and near MN there are no pressure-forces parallel to the arrow at C. Hence A w H is the momentum leaving the vessel per second. It is only when the orifice is small that we can be sure that the average velocity at C is $\sqrt{2gH}$, and the volume flowing per second is $a\sqrt{2gH}$. The mass per second is this multiplied by w/g, and the momentum per second is this multiplied by $\sqrt{2gH}$; and hence A w H = 2 $a w$ H, or A = 2 a.

434. For all other **sharp-edged orifices** than that shown in Fig. 305, we rely upon experiment. Q the quantity in cubic feet per second flowing from a sharp-edged orifice of area A square feet, the centre of the orifice being H feet below still water level, $Q = kA\sqrt{2gH}$. In Fig. 305 k is $\frac{1}{2}$, as we have seen. For circular orifices, k is 0·62 very nearly, even when the upper edge of the orifice is comparatively near the upper level, so long as the orifice keeps filled. Again, for square and rectangular orifices K is very nearly 0·62. We shall not give the great table of numbers, varying from 0·61 to 0·63, which have been experimentally determined for various sizes and positions of rectangular orifices, because we do not think it more accurate than the statement that 0·62 is very nearly correct for all.

435. **The shape of the stream** coming from a rectangular orifice is very interesting; and a student must meditate, when looking at such a stream, upon the way in which its component parts collide to cause the curious palpitating change of shape

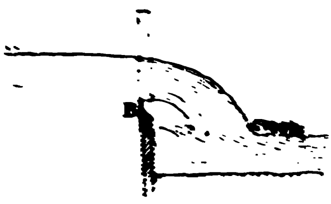

Fig. 304.

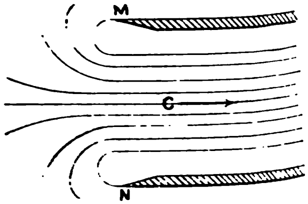

Fig. 305.

and section which is going on. Fig. 306 gives some notion of these changes.

436. Experiment has shown that in the case of sharp-edged orifices there is no practical difference in the actual flowing of

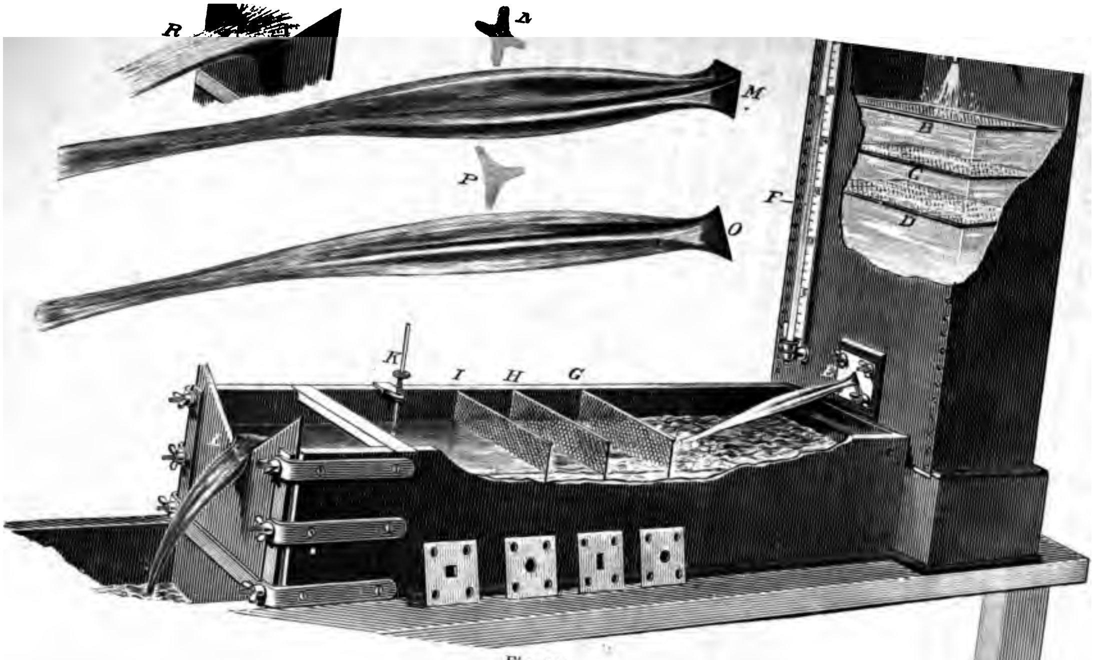

Fig. 306.

water from what the flow would be if the liquid were frictionless. It can be shown that when liquid is frictionless the stream lines from similar and similarly placed orifices in similar vessels with the same kind of liquid at similar heights are similar, the corresponding velocities being proportional to the square roots of the dimensions, and therefore the volumes

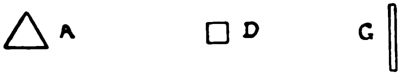

Fig. 307.

flowing being proportional to the two and a half powers of the dimensions. If, then, water flows from a pond over a **sharp-edged notch shaped** like a **right angled isosceles triangle,** each of the edges making 45° with the horizontal, as in Fig. 287, and if the difference of level from B to A is H, the quantity Q flowing in cubic feet per second is proportional to $H^{\frac{5}{2}}$. Prof. James Thomson gave us the above principle and this method of measuring water. By careful measurement, he found that, H being in feet, $Q = 2·635\,H^{\frac{5}{2}}$ . . . . (1). The **rectangular notch** is more convenient. Professor Jas. Thomson showed that the empirical formula of **Mr. Francis, of Lowell,** arrived at with great care and at great expense, is a rational one.

If L is the length of the notch in feet, H being vertical height in feet from sill B to still-water level, for a given H there is a certain value of L beyond which increase in L means that the increase in Q is proportional to the increase in L. In fact, we distinguish the flow through the two ends of length mH at one side and mH at the other, and the flow through the middle part L − 2mH, where all the lines of flow may be regarded as in vertical planes. We have good reason to take m to be constant. Imagine an orifice of length 2mH. The flow through it is $b_1 H^{\frac{5}{2}}$, where b_1 is some constant. The flow through a square orifice of height H, the lines of flow being in vertical planes, is $b_2 H^{\frac{5}{2}}$, where b_2 is some constant, and therefore the middle flow is [illegible] 2mH, H times this or

$$Q = b_1 H^{\frac{5}{2}} + b_2 \frac{L - 2mH}{H} H^{\frac{5}{2}}.$$

This may be shown to reduce to $Q = b(L - cH)H^{\frac{3}{2}}$. If there is only one end contraction, c is evidently halved; and if there are no end contractions c is 0.

The experiments of Mr. Francis give us the values of b and c, so that

$$Q = 3·33\,(L - ·2\,H)\,H^{\frac{3}{2}} \;. \;. \;. \;. \;(2).$$

where n is 2 or 1 or 0, according as we have all the edges sharp or we have the edge B C a smooth vertical guiding plane, or both B C and D C smooth vertical guiding planes.

437. In a gas, we have $w \propto p$, where p is the pressure in pounds per square foot, if the temperature could be kept constant, or we have the rule for adiabatic flow $w \propto p^{\frac{1}{\gamma}}$, where γ is the well-known ratio of the specific heats. In either of these cases it is easy to find $\int \frac{dp}{w}$ and write out the law. This law is of universal use in all cases where viscosity may be neglected, and is a **great guide** to the hydraulic engineer. Thus in the case of adiabatic flow, $w = cp^{\frac{1}{\gamma}}$, the integral of $\frac{dp}{w}$ is

$$\int \frac{dp}{cp^{1/\gamma}} \text{ or } \frac{1}{c}\int p^{-\frac{1}{\gamma}} . dp \text{ or } \frac{1}{c}\,\frac{\gamma}{\gamma - 1}\,p^{1-\frac{1}{\gamma}},$$

and hence $h + \frac{v^2}{2g} + \frac{1}{c}\,\frac{\gamma}{\gamma - 1}\,p^{1-\frac{1}{\gamma}} = \text{constant} \;.\;.\;.\;.\; (4).$

In a great many problems, changes of level are insignificant, and we often use $v^2 + \frac{2g}{c}\,\frac{\gamma}{\gamma - 1}\,p^{\frac{\gamma-1}{\gamma}} = \text{constant} \;.\;.\;.\;.\; (4)$ for gases. Thus, if p_0 is the pressure and w_0 the weight of a cubic foot of gas inside a vessel at places where there is no velocity, and if outside an orifice the pressure is p, the constant in (4) is evidently $0 + \frac{2g}{c}\,\frac{\gamma}{\gamma - 1}\,p_0^{\frac{\gamma-1}{\gamma}}$, and hence outside the orifice

$$v^2 = \frac{2g}{c}\,\frac{\gamma}{\gamma - 1}\left(p_0^{\frac{\gamma-1}{\gamma}} - p^{\frac{\gamma-1}{\gamma}}\right) \;.\;.\;.\;.\; (5):$$

and as c is $w_0 \div p_0^{\frac{1}{\gamma}}$, it is easy to make all sorts of calculations on the quantity of gas flowing per second. Observe that if p is very little less than p_0 and if we use the approximation $(1 + a)^n = 1 + na$, when a is small, we find $v^2 = \frac{2g}{w_0}(p_0 - p) \;.\;.\;.\;.\; (6)$, a simple rule which it is well to remember in fan and windmill problems. In a Thomson water turbine the velocity of the rim of the wheel is the velocity due to half the total available pressure; so in an air turbine, when there is no great difference of pressure, the velocity of the rim of the wheel is the velocity due to half the pressure difference. Thus, if p_0 of the supply is 7,000 lbs. per square foot, and if p of the exhaust is 6,800 lbs. per square foot, and if we take $w_0 = 0{\cdot}28$ lb. per cubic foot, the velocity of the rim V is, since the difference of pressure is 200 lbs. per square foot,

$$\sqrt{\frac{2g}{\cdot 28}(100)} = 151 \text{ feet per second.}$$

Returning to (6): Neglecting friction, if there is an orifice of area A near which the flow is guided so that the streams of air are parallel, Q, the volume flowing per second, is $Q = v\,A$; and if the pressure is p, the weight of stuff flowing per second is $W = v\,A\,w$ or $v\,A\,c p^{\frac{1}{\gamma}}$. Using v from (5), and letting p/p_0 be called α, we have after simplification

$$W = A\,\alpha^{\frac{1}{\gamma}}\,p_0 \sqrt{\frac{2g\gamma}{\gamma-1}\frac{w_0}{p_0}\left(1-\alpha^{\frac{\gamma-1}{\gamma}}\right)}.$$

Problem.—Find p, the outside pressure, so that for a given inside pressure there may be the maximum flow.

It is obvious that as p is diminished more and more, v, the velocity, increases more and more, and so does Q. But a large Q does not necessarily mean a large quantity of gas. We want W to be large. When is W a maximum? That is, what value of α in all will make

$$\alpha^{\frac{2}{\gamma}}\left(1-\alpha^{\frac{\gamma-1}{\gamma}}\right) \text{ or } \alpha^{\frac{2}{\gamma}} - \alpha^{1+\frac{1}{\gamma}} \text{ a maximum?}$$

Differentiating with regard to α, and equating to 0, we have

$$\frac{2}{\gamma}\,\alpha^{\frac{2}{\gamma}-1} - \left(1+\frac{1}{\gamma}\right)\alpha^{\frac{1}{\gamma}} = 0.$$

Dividing by $\alpha^{\frac{1}{\gamma}}$, we find $\alpha = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{1-\gamma}}$,

or
$$p = p_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}.$$

In the case of air $\gamma = 1{\cdot}41$, and we find $p = {\cdot}527\,p_0$. That is, there is a maximum quantity leaving the vessel per second when the outside pressure is a little greater than half the inside pressure.

Problem.—When p is indefinitely diminished what is v?

$$\textit{Answer}: \; v = \sqrt{\frac{2g\gamma}{\gamma-1}\frac{p_0}{w_0}}.$$

This is greater than the velocity of sound in the ratio $\sqrt{\frac{2}{\gamma-1}}$, being 2·21 for air. That is, the limiting velocity in the case of air is 2,413 feet per second × $\sqrt{\frac{t}{273}}$, where t is the absolute temperature inside the vessel, and there is a vacuum outside. Students ought to work out as an example the velocity of flow into the atmosphere.

Returning to equations (3) and (4), we assumed h to be of little importance in many gaseous problems of the mechanical engineer. But there are many **physical problems** in which it is necessary to take account of changes in level. For example, if (3) is integrated on the assumption of constant temperature, and we assume v to keep constant, we find that p diminishes as h increases, according to the **compound interest law.** Again, under the same condition as to v, but with the adiabatic law for w, we find that p diminishes with h according to a law which may be changed into "the rate of diminution of temperature with h is constant."

A great number of interesting examples of the use of (2) might be given. It enables us to understand the flow of fluid from orifices, the action of jet-pumps, the attraction of light bodies caused by vibrating tuning-forks, why some valves are actually sucked up more against their seats instead of being forced away by the issuing stream of fluid, and many other phenomena which are thought to be very curious.

438. *Example.*—Particles of **water in a basin,** flowing very slowly towards a hole in the centre, move in nearly circular paths, so that the velocity v is inversely proportional to the distance from the centre. Take $v = \frac{a}{x}$, where a is some constant and x is the radius or distance from the axis. Then (3) becomes $h + \frac{a^2}{2gx^2} + \frac{p}{w} = c$. Now at the surface of the water p is constant, being the pressure of the atmosphere; so that there $h = -\frac{a^2}{2gx^2}$, and this gives us the shape of the curved surface. Assume c and a any values, and it is easy to calculate h for any value of x and so plot the curve. This curve rotated about the axis gives the shape of the surface, which is a surface of revolution.

A student who depends upon a text-book to give him complete information is not learning to become an engineer. Meditation when looking at water flowing from a basin ought to greatly add to what he will obtain from such a book as this. Perhaps he will begin to notice that it is centrifugal force due to whirling motion which maintains pressure at a place. What will be the effect of friction at the solid surface of the basin? It will diminish velocity and diminish pressure. Water will flow, therefore, down the surface of the basin and towards the hole. If this is well understood, the student will understand how it is that at a bend of a river the earth from the outer bank is dragged along the bottom and deposited on the inner bank, and hence that a river through an alluvial plain is always tending to get more crooked, until at length it cuts off a bend and so straightens itself in a new channel. In the basin problem we have also James Thomson's explanation of the phenomena of great forest fires, and also of the prevailing wind system of the earth. (See *Phil. Mag.*)

439. If **water flowing spirally** in a horizontal plane follows the law $v = \frac{b}{r}$, where r is distance from a central point: note that $p = c - \frac{1}{2}\frac{w}{g}\frac{b^2}{r^2}$. The ingenious student ought to study how p and v vary at right angles to stream lines. He has only to consider the equilibrium of an elementary portion of fluid, as

(Fig. 303), subjected to pressures, centrifugal force, and its own weight in a direction normal to the stream. He will find that if $\frac{dp}{dn}$ means the rate at which p varies in the direction of the radius of curvature away from the centre of curvature, and if α is the angle D C E (Fig. 303), the stream being in the plane of the paper, which is vertical, and if r is the radius of curvature,

$$\frac{dp}{dn} = \frac{w}{g}\frac{v^2}{r} - w \sin. \alpha \quad \ldots . (1).$$

If the stream lines are all in horizontal planes,

$$\frac{dp}{dn} = \frac{w}{g}\frac{v^2}{r} \quad \ldots . (2).$$

Stream lines all circular and in horizontal planes in a liquid, so that h is constant. If $v = \frac{b}{r}$, where b is a constant,

$$\frac{dp}{dr} = \frac{w}{g} \cdot \frac{b^2}{r^3},$$

$$p = -\tfrac{1}{2}\frac{w}{g}\frac{b^2}{r^2} + \text{constant} \quad \ldots . (3).$$

We see, therefore, that the fall of pressure as we go outward is exactly the same as in the exercise at the end of last article.

Example.—Liquid rotates about an axis as if it were a rigid body, so that $v = br$, then $\frac{dp}{dr} = \frac{w}{g} b^2 r$, $p = \frac{1}{2}\frac{w}{g} b^2 r^2 + c$. This approximately shows the law of increase of pressure in the wheel of a centrifugal pump when full, but when delivering no water. It is the answer already obtained (Art. 175).

EXERCISES.

1. The pressure at the inside of the wheel of a centrifugal pump is 2,116 lbs. per square foot; the inside radius is 0·5 foot; the outside radius 1 foot. The angular velocity of the wheel is $b = 30$ radians per second. Draw a curve showing the law of p and r from inside to outside when very little water is being delivered. If the water leaves the wheel by a spiral path, the velocity everywhere outside being inversely proportional to r, draw also the curve showing the law of p in the whirlpool chamber outside.

2. The expression

$$\frac{v^2}{2g} + \frac{1}{w}p + h = \text{E} \quad \ldots . (4),$$

which remains constant all along a stream line, may be called the total store of energy of 1 lb. of water in the stream if the motion is steady. Now $\frac{d\text{E}}{dn} = \frac{1}{g}v\frac{dv}{dn} + \frac{1}{w}\frac{dp}{dn} + \frac{dh}{dn}$ becomes from equation (1),

$$\frac{d\text{E}}{dn} = \frac{2v}{g} \times \frac{1}{2}\left(\frac{v}{r} + \frac{dv}{dn}\right) \quad \ldots . (5),$$

n being in a normal direction away from the centre of curvature and r the radius of curvature. This expression $\frac{1}{2}\left(\frac{v}{r} + \frac{dv}{dn}\right)$ is called the "average angular velocity" or "*the rotation*" of the liquid. Hence

$$\frac{d\text{E}}{dn} = \frac{2v}{g} \times \text{rotation} \quad \ldots \ldots (6).$$

3. Show that the law (1) for a gas under adiabatic conditions, being 4 of Art. 437, the above laws (5) and (6) hold for a gas as well as a liquid.

4. **Circular Stream Lines.**—What is v as a function of r, the radius, if the flow is to be irrotational—that is, if a pound of water or of gas has the same energy in one stream line as in another? Here $v/r + dv/dr = 0$, or $dr/r + dv/v = 0$, or $\log. r + \log. v = c$, a constant, or $v \propto 1/r$.

5. In a gas flowing irrotationally in circular streams, if $v = \frac{b}{r}$, find p everywhere. Inserting $v = b$ in (4) of Art. 437, we find

$$p/p_0 = \left\{1 + \frac{\gamma - 1}{\gamma}\,\frac{b^2 w_0}{2gp_0}\left(\frac{1}{r_0^{\,2}} - \frac{1}{r^2}\right)\right\}^{\gamma/\gamma(-1)}.$$

6. A centrifugal fan makes 3,000 revolutions per minute. It is 2 feet in inside diameter, 4 feet outside, and there is a whirlpool chamber outside the fan. No air is being delivered from the chamber. The pressure just inside the wheel is 1 atmosphere, 2,116 lbs. per square foot, and 0° C. Draw curves showing the pressure at any distance from the centre of the wheel, both in the wheel and the whirlpool chamber. The air follows the adiabatic law. If the speed is 40 revolutions per minute, and the fluid is water, draw the curves.

440. When liquid flows by gravity from a small orifice in a large vessel, where at a distance inside the orifice the liquid may be supposed at rest, it is obvious the E is the same in all stream lines, so that $\frac{d\text{E}}{dr}$ is 0, and there is no "rotation" anywhere. It can be proved that if a portion of frictionless fluid possesses "rotation," this can never be destroyed. Nor can rotation be created. But the student must refer to books on hydrodynamics for further information. When unstable states of motion set in, the mathematics get beyond the author's powers, even in straight pipes. The papers of Professor Reynolds may be consulted. As for what occurs in viscous fluid at a bend in a pipe, nobody has done more than guess as yet. At small velocities, we have a fairly good knowledge of what happens. Suppose very viscous liquid is flowing along a straight pipe, the velocity everywhere is such that the loss of energy by viscosity is a minimum. If the pipe has a curved centre line, the distribution of velocity everywhere is different from what it is in the centre line.

When fluid flows in circular cylindric stream lines, a stream of internal and external radii r and $r + \delta r$, of unit breadth, is acted on by tangential force on its internal cylindric surface $\mu\left(\frac{dv}{dr} - \frac{v}{r}\right)$ per unit area, if v is the velocity. These forces act on the area $2\pi r$ at a distance r from the axis, so that their total moment

$$\text{M} = 2\pi\mu r^2\left(\frac{dv}{dr} - \frac{v}{r}\right) \quad \ldots \ldots (1).$$

Let the moment of the forces on the outside surface be $M + \delta M$. If there is no alteration of pressure along a stream line, and only balanced forces in planes at right angles to the axis, then δM is the only cause of increase of moment of momentum of the stream. The state into which the system settles is one in which the motion is steady. In fact, $\frac{dM}{dr}$ is 0, or

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0 \quad \ldots . \; (2).$$

The general solution of this is

$$v = Ar + \frac{B}{r} \quad \ldots . \; (3).$$

If at a cylindric surface $r = r_0$, $v = v_0$, and if at another $r = r_1$, $v = 0$, then

$$\left.\begin{aligned} v_0 &= A r_0 + \frac{B}{r_0} \\ 0 &= A r_1 + \frac{B}{r_1} \end{aligned}\right\} \quad \ldots . \; (4).$$

So that $B = -Ar_1^2$, $v_0 = A\left(r_0 - \frac{r_1^2}{r_0}\right)$, $A = \frac{r_0v_0}{r_0^2 - r_1^2}$, and hence (3) becomes $v = \frac{r_0v_0}{r_0^2 - r_1^2}\left(r - \frac{r_1^2}{r}\right)$. If, then, viscous fluid escapes from the rim of the wheel of a centrifugal pump with little radial velocity into a chamber bounded by parallel frictionless faces, if $r = r_1$ is the outer radius of the chamber,

$$v = \frac{r_0v_0}{r_1^2 - r_0^2}\left(\frac{r_1^2}{r} - r\right) \quad \ldots . \; (5).$$

And if $r_1 = \infty$, then A must be 0 in (3), and $r_0 = \frac{B}{r_0}$, so that (3) becomes

$$v = \frac{r_0v_0}{r} \quad \ldots . \; (6).$$

The loss of energy per second is $-\mu v\left(\frac{dv}{dr} - \frac{v}{r}\right)$ per unit volume, or $-\mu\frac{r_0v_0}{r}\left(-2\frac{r_0v_0}{r^2}\right) = 2\mu\frac{r_0^2v_0^2}{r^3}$. The volume of the ring between r and $r + \delta r$ is $2\pi r \cdot \delta r$, so that $\int_{r_0}^{\infty} 4\pi\mu r_0^2v_0^2 \cdot \frac{dr}{r^2}$ is the total loss, or $4\pi\mu r_0v_0^2$. If V is the small radial velocity of fluid leaving a wheel, the weight of fluid per second leaving unit breadth of the wheel is $2\pi r_0 V w$. The kinetic energy of it due to its tangential velocity only is $2\pi r_0\frac{Vw}{g}v_0^2$. Its kinetic energy per pound is $\frac{v_0^2}{2g}$. $\frac{\text{Total kinetic energy}}{\text{wasted energy}} = \frac{Vw}{2g\mu}$. The kinetic energy due to the radial velocity only is $2\pi r_0$. If the law were $v = m\left(\frac{r_1^2}{r} - r\right)$ $- \mu m r\left(-\frac{r_1^2}{r^2} - 1 - \frac{r_1^2}{r^2} + 1\right) = + \mu m r\left(\frac{2r_1^2}{r^2}\right)$. This is loss of energy per second per unit volume, or $2r_1^2m^2\mu\left(\frac{r_1^2}{r^3} - \frac{1}{r}\right)$. Total

$$\text{loss} = \int_{r_0}^{r_1} 4\pi r_1^2m^2\mu\left(\frac{r_1^2}{r^2} - 1\right)dr\left(-\frac{r_1^2}{r} - r\right)$$

$$= 4\pi r_1^2\mu\left(-2r_1 + \frac{r_1^2}{r_0} + r_0\right)\frac{r_0^2v_0^2}{(r_1^2 - r_0^2)^2}.$$

MISCELLANEOUS EXERCISES.

1. In a force pump used for feeding a boiler the ram has a diameter of 2 inches and a stroke of 24 inches. How many gallons of water (neglecting leakages) would be forced into the boiler for each 1,000 double strokes (one forward and one backward) of the pump?

Ans., 272 gallons.

2. How many gallons of water will be delivered per hour by a single-acting pump (diameter of plunger, 4 inches; length of stroke, 12 inches) making 24 strokes per minute, the slip being 15 per cent.? How long would it take this pump to fill a tank 10 feet by 6 feet to a depth of 3 feet? *Ans.*, 9,982 gallons.

3. Certain machinery worked from an accumulator requires 20-horse power for a quarter of an hour every hour. The pressure in the accumulator is 700 lbs. per square inch. If 25 per cent. be allowed for frictional losses, find the size of a single-acting pump which, driven for thirty-five minutes every hour by a donkey engine, making 50 strokes a minute, will just have the ram at the top of its stroke at the beginning of each hour. Assume a slip in the pump of 25 per cent.

Ans., ·094 cubic foot.

4. In a double-acting force-pump the diameter of piston is 12 inches and the stroke 2 feet 6 inches. The distance from the pump to the well is 15 feet, and from the pump to the place of delivery is 35 feet. Find the horse-power required to work the pump if 30 per cent. is wasted in friction and the number of strokes be 40 per minute. *Ans.*, 21 16.

5. The diameter of a pump bucket being 6 inches, and the vertical lift from the well to the point of delivery being 40 feet, find the load on the bucket. What horse-power will be necessary if the stroke be 15 inches and there are 20 double strokes per minute? Allow 30 per cent. for all losses. *Ans.*, 489 lbs.; ·53.

6. The discharge from a pipe is 12 gallons a second. At a point 110 feet above datum level the diameter is 5 inches, and the pressure 2,050 lbs. per square foot. Find the total energy of each pound of water. If at a point where the pipe is $2\frac{1}{2}$ inches diameter and 10 feet above datum the pressure is 1,000 lbs. per square foot, find the loss of energy between the two points mentioned. *Ans.*, 146 ft. lbs.; 107·6 ft. lbs.

7. The volume of water passing along a pipe running full is 10 cubic feet per second. At one section the area is 2 square feet; at another

place, 12 feet below the level of the former, the area of the cross-section is $1\frac{1}{4}$ square feet. Find the difference of pressure at the two sections, friction being neglected. *Ans.*, 710 lbs. per sq. ft.

8. At a certain point, 15 feet above datum, in a stream flowing from a reservoir, the still surface of which is 150 feet above datum, the velocity is 20 feet per second. What is the pressure at this point, assuming no loss by friction? *Ans.*, 70·4 lbs. per sq. in.

9. Find the power of a waterfall where 2,000 cubic feet of water pass per minute, the height of the fall being 30 feet.

In a waterfall, 20 tons of water fall from a height of 36 feet in each minute, and are employed to turn a turbine which transforms six-tenths of the energy of the water into useful work. Find the horse-power of the turbine. *Ans.*, 113·3 h.p.; 29·32.

10. The vertical distance from still-water level to the lip of a rectangular notch was observed to be ·3 feet during an interval of six hours, ·65 feet during the next twelve hours, and ·4 feet during the next six hours, the width of the notch being $2\frac{1}{2}$ feet. Find the number of gallons passing through during the twenty-four hours.
Ans., 1,567,000 gallons.

11. The flow of water in a certain stream is measured by employing a Thomson's V-shaped weir gauge. The vertical distance from still-water level to the lowest point of the notch is observed to be 1·4 feet. The water which passes through has a fall of 20 feet, and is employed to drive a water-wheel having an efficiency of 60 per cent. Find the horse-power which may be obtained from the wheel. *Ans.*, 8·33.

12. A waterfall is to be utilised for electric lighting. The engineer sent to inspect the place finds out the following data:—The water at one place flows in a straight rectangular channel $4\frac{1}{2}$ feet wide, $2\frac{1}{2}$ feet deep, with an average velocity of 3 feet per second. The available fall is 20 feet, and the water-wheels to be used have an efficiency of 62 per cent., the dynamo efficiency being 80 per cent. Neglecting all other losses of energy, find approximately how many 60-watt incandescent lamps may be supplied. *Ans.*, 471.

13. The rim of the wheel of a centrifugal pump goes at 30 feet per second. Water flows radially at 5 feet per second. The vanes are inclined backwards at an angle of 35° to the rim. What is the absolute velocity of the water? What is the component of this parallel to the rim? If 120 cubic feet of water leave the rim every minute, find the tangential retarding force at the rim. What is the work done usefully per pound of water?

Ans., 29·12 ft. per sec. at 12°·3 with direction of rim; 22·86 ft. per sec.; 88·45 lbs.; 21·3 ft. lbs.

14. Suppose water to flow in a steady stream with a constant total head of 100 feet, reckoned from the datum plane and from zero pressure. Determine the discharge into the atmosphere in gallons per minute from a pipe 2 inches diameter at a point 5 feet above the datum. *Ans.*, 638.

15. An orifice 1 square inch area is made in the side of a large tank, at a depth of 4 feet below the surface of the water, and the issuing jet is *horizontal*. If the jet falls vertically through $1\frac{2}{3}$ feet in a horizontal

motion of 5 feet, and the discharge be 16 gallons per minute, find the coefficients of velocity, contraction, and discharge. *Ans.*, ·97; ·64; ·62.

16. Find the discharge in gallons per minute from an orifice 2 inches in diameter in the side of a tank under a constant head of 6 feet, measured from the centre of the orifice. The coefficient of discharge may be taken at ·6. *Ans.*, 96.

17. In an inward flow wheel the velocity of flow is 8 feet per second, the internal diameter 9 inches, and the revolutions 10 per second. Find the angle of the vanes at exit, so that the water may leave the wheel radially. *Ans.*, 18°·75.

18. Determine the velocity with which the water enters an inward flow turbine under a head of 36 feet, the speed of the periphery of the wheel being 32 feet per second. The vanes of the wheel are radial at entrance, the velocity of flow is constant, and the water leaves the wheel with no tangential velocity. *Ans.*, 32·56 ft. per sec.

19. What horse-power is required to drive a radial-vaned pump of 15 feet diameter at 50 revolutions per minute when delivering 15,000 gallons of water per minute? What is the efficiency if the lift is 22 feet? *Ans.*, 21·8; ·46.

20. A stream of water, the volume of flow of which is 3,000 gallons per minute, has a velocity of 20 feet per second. It impinges on a succession of curved vanes moving with a velocity of 8 feet per second in a direction inclined at 45° to the direction of the stream. Determine the direction of the tangent to the vane at entrance, so that the water may impinge without shock. If the vanes are circular arcs of 90°, find the resultant pressure on the vanes, and the component force in the direction of motion. *Ans.*, 21°·5 with direction of jet.

21. Find the horse-power developed in a Thomson turbine which is supplied with 15 tons of water per minute, with a forward tangential velocity of 40 feet per second, equal to the speed of the periphery of the wheel, the diameter of which is 2 feet. The water leaves the turbine at a radius of 6 inches, with a backward tangential velocity of 10 feet per second. *Ans.*, 4·1.

22. The diameters of the inner and outer circumferences of an inward flow turbine are 2 feet and 4 feet respectively. The direction of the vanes at their outer ends is radial. Determine the angle at which the inner ends are arranged, supposing that velocity of flow through the turbine is one-eighth the velocity due to the total head, and that of the outer ends is that due to half the head. *Ans.*, 19°·6 with rim.

23. A turbine with radial vanes receives 50 gallons per minute with an effective head of 28 feet. Find what should be the total area of the inlet passages, and the velocity of the lips of the vanes for maximum efficiency. *Ans.*, 1·51 sq. ft.; 28 ft. per sec.

24. The wheel of a centrifugal pump is ·6 feet in diameter; the turning moment on the spindle is 12 pound-feet. If 160 gallons of water are raised per minute, find the mean velocity with which the water leaves the wheel, assuming that on entering it has no velocity of whirl. *Ans.*, 24·1 ft. per sec.

CHAPTER XXV.

PERIODIC MOTION.

441. When, after a certain interval of time, a body is found to have returned to an old position, and to be there moving in exactly the same way as it did before, the motion is said to be **periodic**, and the interval of time that has elapsed is said to be the **periodic time** of the motion. Thus, if a body moves uniformly round in a circle, the time which it takes to make one complete revolution is called its periodic time.

442. When a body moves uniformly in a circle, as, for instance, the bob of a *conical pendulum*, if we look at it from a point in the plane of its circle, it seems merely to swing backwards and forwards in a straight line. Thus, it is known that Jupiter's satellites go round the planet in paths which are nearly circular, but a person on our earth sees them move backwards and forwards almost in straight lines. Now, if we were a very great distance away from the bob of a conical pendulum in the plane of its motion, we should imagine it to be moving in a straight line, and the motion which it would appear to have—slow at the ends of its path, quick in the middle—would be a **simple harmonic motion.** To get an exact idea of the nature of this motion—in fact, to define what I mean by simple harmonic motion—draw a circle, A O′ L O″ (Fig. 308), and divide its circumference into any even number of equal parts. Draw the perpendiculars B′ B, C′ C, etc., to

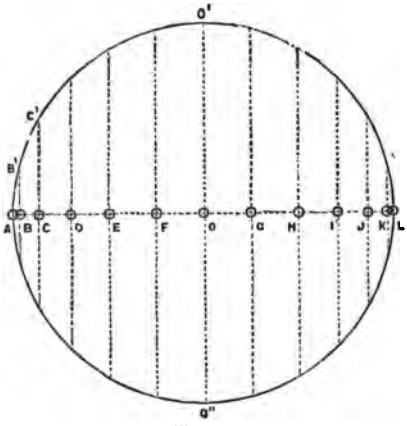

Fig. 308.

any diameter. Now, if we suppose a body to go backwards and forwards along A O L, and if it takes just the same time to go from A to B as from B to C, or from any point to the next, then its motion is said to be a simple harmonic motion. This sort of motion is nearly what we observe in Jupiter's satellites; it is almost exactly the motion of the bob of any long pendulum or the cross-head of a steam-engine; it is the motion of a point in a tuning-fork, or a stretched fiddle-string when it is plucked aside and set free; of the weight hung from a spring balance when it is vibrating; of the up and down motion of a cork floating on the waves in water; and of the free end of a rod of metal when the other end is fixed in a vice and the rod is set in vibration; it tells us in all these cases the nature of the motion, when such motion is of its simplest kind. Thus, for example, a cork floating on water may really have a very complicated motion, but if the wave in the water is of its simplest kind, the cork goes up and down with a simple harmonic motion. If you study the figure which you have drawn, and then watch the vibration of a very long pendulum, you will learn about this kind of motion what cannot be learnt by reading.

443. Now let me suppose that the body takes one second to go from A to B, or from B to C, or from any point to the next in Fig. 308. Then the *length of* A B *in inches represents the* average velocity between the points A and B, and in the same way we get the average velocity anywhere else. Thus, in the figure from which the woodcut is drawn I find

Velocity from	A to B	B to C	C to D	D to E	E to F	F to O	O to G	G to H	H to I	I to J	J to K	K to L
Is in inches per second	0·34	1·00	1·59	2·07	2·41	2·59	2·59	2·41	2·07	1·59	1·00	0·34

We observe that the velocity increases as the body approaches the middle of the path, and diminishes again as it goes away from the middle. Now the increase in the velocity of a body every second is called its acceleration, and so we can observe what is the acceleration at every place. You see that the velocity changes from ·34 to 1·00 near B in one second—that is, the acceleration near B is ·66 inch per second per second. Similarly subtracting 1·00 from 1·59 we find the acceleration at C to be 0·59, and so on. Make a table of these values, and place opposite them the distances of the points B, C, etc., from the centre. In this way we find from the figure the following Table of Values:—

Distance from O to	Acceleration at	Displacement divided by Acceleration.
B is 9·66	B is 0·66	14·6
C is 8·66	C is 0·59	14·7
D is 7·07	D is 0·48	14·7
E is 5·00	E is 0·34	14·4
F is 2·59	F is 0·18	14·4
O is O	O is O	...
G is 2·59	G is 0·18	14·4
H is 5·00	H is 0·34	14·4
I is 7·07	I is 0·48	14·7
J is 8·66	J is 0·59	14·7
K is 9·66	K is 0·66	14·6

From this it is evident that when the distance of a point from the centre is divided by the acceleration at the point, we get about 14·6 in every case—that is, if we worked more exactly we should have the exact law that the **acceleration** at a place is **proportional to** the **distance from the centre**. This curious property is characteristic of the kind of motion which we are describing. If, again, we draw a number of figures, such as Fig. 308, and divide the circles into very different numbers of equal parts, we shall find that in every case the **following law is true**:—The periodic time of a simple harmonic motion—that is, the time which elapses from the moment when the body is in a certain condition until it gets into exactly the same condition again—is equal to 6·2832 multiplied by the square root of the ratio of displacement to acceleration given in the third column of the above Table. Thus, in the Table we find the mean value of the ratio (adding all the quotients and dividing by their number we get 14·56) to be, let us say, 14·6. Now the square root of 14·6 is 3·82, and this multiplied by 6·2832 is 24 seconds, which we see by inspection is the periodic time in Fig. 308.

444. We see, then, that if the force acting on a body and causing it to move is always proportional to the distance of

e body from a certain point, and acts towards that point, e body gets a simple harmonic motion, and we have a rule r finding the periodic time.

The acceleration is always towards the middle point—that is, whilst a body is leaving the middle its velocity is being lessened; when it is approaching the middle its velocity is being increased. The velocity at the middle is equal to the uniform velocity in the circle from which we imagine the harmonic motion to be derived—that is, the velocity in the middle is equal to 3·1416 times the distance A L divided by the periodic time.

Suppose the body to be at Q, Fig. 309, moving with a harmonic motion in the path A O L. Describe the circle, draw Q P perpendicular to A L, then P is the position of a body which has corresponding uniform circular motion.

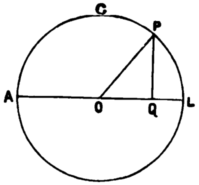

Fig. 309.

Let the time t seconds have elapsed since the point Q was at O. The corresponding point P was then at C. Let the uniform speed of P be v feet per second; then C P $= vt$. Let O P $= r$; let the angle C O P $= vt/r$, so that the radius O P moves with the angular velocity v/r. The time T of revolution of P or of complete oscillation of Q is $2\pi r \div v$, or $2\pi \div$ the angular velocity, which we shall call p. O Q = O P . sin. O P Q = O P . sin. C O P, or, if we call O Q $= x$, $x = r \sin. pt$. By Art. 19,

$$\text{velocity of Q} = \frac{dx}{dt} = rp \cos. pt,$$

$$\text{acceleration of Q} = \frac{d^2x}{dt^2} = -rp^2 \sin. pt.$$

Hence the displacement x, divided by the value of the acceleration, if we neglect the $-$ sign, is $1/p^2$ or $\frac{4\pi^2}{\text{T}^2}$ (since $\text{T} = 2\pi/p$).

Notice that the acceleration $-rp^2 \sin. pt$ is the resolved part, in the direction x of the acceleration of P. The speed of P is v, a constant. If, then, it has an acceleration, it cannot be, nor any part of it, in the direction of its path; it is, then, in the direction of the radius. Let the acceleration of P in the direction O P be a, then $a \cos.$ P O L = acceleration of Q, or $a \sin. pt = -rp^2 \sin. pt$, or $a = -rp^2$. The acceleration of P is, then, centripetal from P towards O, and its amount is rp^2. This can be put in other shapes. Thus $p = \frac{v}{r}$, so that $a = -v^2/r$, the ordinary formula for centrifugal acceleration.

Here we have obtained a knowledge of the centripetal acceleration mathematically from knowing what is meant by linear acceleration. Conversely, suppose we know that the centripetal acceleration of P in the direction P O is v^2/O P, then the acceleration of Q towards the centre is this multiplied by cos. P O L; that is, by

OQ/OP, and it is therefore equal to $\frac{v^2}{OP} \cdot \frac{OQ}{OP}$ or $OQ \,.\, v^2/OP^2$, so that it is proportional to OQ. Also, OQ divided by the acceleration (towards the centre) is OP^2/v^2 or $4\pi^2/T^2$ (since $2\pi \,.\, OP/v = T$). We are therefore led in many ways to the rule

$$\text{Periodic time} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

(*See also* Art. 19.)

445. *Example.*—In Fig. 310, A is a ball of lead weighing 20 lbs. carried by means of a **spiral spring** whose own weight may be neglected, let us suppose. Find by experiment how much the spring lengthens when we add 1 lb. to the weight of A, or shortens when we subtract 1 lb. from the weight of A. Let it lengthen or shorten 0·01 foot. Evidently, if ever A is 0·01 foot upwards or downwards from its position of rest, it is being acted upon by a force of 1 lb. tending to bring it to its position of rest. We know also that if A is 0·02 foot or 0·03 foot above or below its place of rest, there is a force of 2 or 3 lbs. trying to bring it back. We see, then, that the up and down motion of A must be **simple harmonic**. When the displacement is, say 0·02 foot, the force acting on A is 2 lbs., and the acceleration of A is force 2 ÷ mass of A; and as the mass of A is 20 ÷ 32·2, or 0·621, the acceleration of A is 3·22 feet per second per second when it is displaced 0·02 foot from its middle position. Now, employing the rule given above, divide 0·02 by 3·22 and extract the square root, then multiply by 6·2832, and we get 0·495 second, or about half a second as the **periodic time of the swinging ball.** When we make experiments we find that, unless the coils of the spring are flat, and the rigid support of A exactly in the axis, the ball has a tendency to turn and to vibrate laterally, which somewhat disturbs our observations if we make careful measurements of the length of swing.

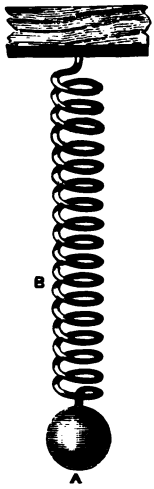

Fig. 310.

446. *Example.*—**The Simple Pendulum.**—A simple pendulum consists in an exceedingly small but heavy body suspended by means of a long inextensible thread, whose

weight may be neglected, capable of swinging backwards and forwards in short arcs. If the arcs are not too long, the time of one swing is always the same. Thus, in Fig. 311, S is the point of suspension, S P a silk thread, P a small ball of lead. P will move backward and forward along the path A O L with a motion which is simple harmonic, provided the thread is so long and A L so short that the force acting on the ball at any time in the direction of its motion is proportional to the distance of the ball from O. To show that this is so, resolve the vertically acting weight of the ball in the direction of its motion along A O. We find that it is not quite proportional to A O unless A O is very short, but if this slight discrepancy is neglected the force urging the ball towards O is the weight of the ball multiplied by O A and divided by S A, the distance from the point of support to the centre of gravity of the ball. As a matter of fact, the nature of the vibration does not depend on the weight of the ball; but, to fix our ideas, let us suppose that the weight is 2 lbs., then the mass of the ball is 2 ÷ 32·2, and acceleration along A O is the force ÷ mass, or $\frac{2 \times \text{AO}}{\text{SA}}$ the force, divided by $\frac{2}{32{\cdot}2}$ the mass, so that the acceleration is $32{\cdot}2 \times \text{AO} \div \text{SA}$.

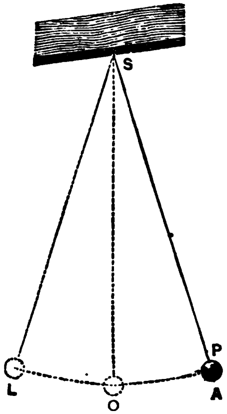

Fig. 311.

Now, **our rule** is to divide A O by the acceleration at A, and this gives $\frac{\text{SA}}{32{\cdot}2}$; extract the square root, and multiply by 6·2832 for the periodic time of oscillation of the pendulum. **The general rule for a simple pendulum** swinging in short arcs is then :

$$\textit{Time of a complete oscillation} = 6{\cdot}2832\sqrt{\frac{\textit{length of pendulum}}{32{\cdot}2}}.$$

The mass or inertia is 2 ÷ 32·2 at all places, but the weight may be more or less than 2 lbs. at different parts of the earth in the proportion of g to 32·2, and hence we find

$$\text{T} = 2\pi\sqrt{l \div g}$$

as our general rule.

The time of *one swing* is half this. The number 32·2 expresses the effect of the force of gravity at London. At any other place on the earth's surface it would be different—that is, at different places on the earth a given pendulum has different times of oscillation. For instance, a pendulum taking 2 seconds for a complete oscillation at Paris—that is, taking 1 second for one swing, called a seconds pendulum—if swung at Spitzbergen would gain 94 seconds per day; and if swung in New York would lose 30 seconds per day, provided the pendulum did not alter in length in being taken from one place to the other. Evidently when a pendulum gets longer it oscillates more slowly; hence in summer, when the pendulum of a common house-clock expands with heat, it goes more slowly, and in winter it goes more quickly, unless the position of the bob is adjusted. A pendulum which is self-adjusting—that is, which is so constructed that it remains of the same length whatever be the temperature—is called a *compensation pendulum.*

447. *Example.*—In Fig. 312, B represents a **strip of steel** fixed firmly in a vice at C, with a heavy ball A fastened at its free extremity. Find the force in pounds which will increase the deflection of A by 0·01 foot; say that it is 1 lb. We know that a force of 2 or 3 lbs. will cause an increased deflection of twice or three times this amount; and as the force acting on the ball in any position is proportional to its distance from its position of rest, the ball will swing with a simple harmonic motion. If we can neglect the weight of the strip of steel, and if the ball is small in comparison with the length of the strip, its time of vibration may be calculated in exactly the same way as that of the ball in Fig. 311. Experiments with this enable **Young's modulus** E to be determined.

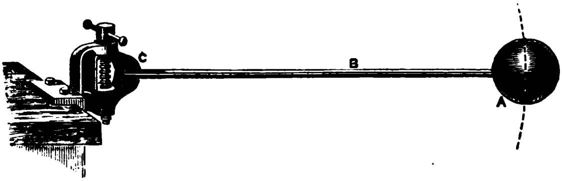

Fig. 312.

448. *Example.*—Suppose that B C (Fig. 313) is a **bent glass tube of uniform section** containing a liquid which can move without friction in the tube. If the liquid be disturbed so that the level is higher in B than in C, it will continue to

swing about its position of equilibrium—that is, the position in which the liquid is at the same level in both limbs of the tube. Thus, if C is ·01 foot below the proper level, and B is ·01 foot above this level, the force which tends to cause the liquid to return to its proper level is twice the weight of the liquid O B. Suppose the weight of the liquid is 10 lbs. per foot length of the tube, then the force acting on the liquid is ·02 × 10, or ·2 lbs. If the whole length of tube filled with liquid is 6 feet, then the weight of liquid which has to be set in motion is 60 lbs., and its mass is 60 ÷ 32·2, or 1·863; hence the acceleration is ·2 ÷ 1·863, or 0·107 foot per second per second. The displacement is ·01, and, working by our old rule, displacement divided by acceleration is ·0935. The square root of this is ·3058, and multiplying by 6·2832 we get 1·92 second as the periodic time of the oscillation.

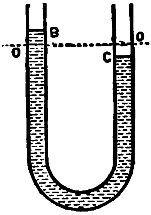

Fig. 313.

We find it easy to prove that the liquid swings in the same time as a simple pendulum whose length is half the total length of the liquid in the tube, and that it is the same whatever be the density of the liquid—that is, whether it is mercury or water.

If w lbs. is the weight of liquid per foot in length of the tube, if x is the displacement O B or O C, the force causing motion is $2xw$. (This force is multiplied by $g \div 32{\cdot}2$ if the observation is not made in London.) If a is the total length of liquid in the tube, the weight of liquid moved is aw, and its mass is $aw \div 32{\cdot}2$. Hence the acceleration is $2xw \div \frac{aw}{g}$ or $\frac{2xg}{a}$, and the displacement divided by acceleration is $x \div \frac{2xg}{a}$ or $\frac{a}{2g}$; so that the periodic time is $2\pi\sqrt{\frac{1}{2}\cdot\frac{a}{g}}$.

449. It will be observed that in all these cases of vibration of bodies there is a **continual conversion going on of one kind of energy into another.** At each end of a swing the body has no motion; all the energy is therefore potential, whether it is the potential energy of a lifted weight or the potential energy of strained material. In the middle of the swing the body is going at its greatest speed, and its energy is kinetic. At any intermediate place the energy is partly potential and partly kinetic, but the sum of the two remains always the same, excepting in so far as friction is wasting the total store. Now, in time-keepers the **office of the mainspring** is to give just such supplies of energy to the balance as are necessary to replace the loss by friction; and we have to ask the question—At what part of the swing of a pendulum or balance can we give to it **an impulse** which shall increase its store of energy without disturbing its time of oscillation? The answer is this. If a blow is given to the bob of a pendulum when it is just at its lowest point, energy is given to the pendulum; we give it power to make a greater swing, but the time which it will take to make this greater swing is just the same as the time it would have taken for a smaller swing. This **middle point** is the only point at which we can give an impulse to the bob without altering the time of its swing. In the lever escapement, and in other detached **escapements** of watches, the impulse is always given just at the middle of the swing.

EXERCISES.

1. A point describes a S.H. motion of 1½ foot amplitude in a period of $\frac{1}{7}$th of a second. Find its maximum velocity and maximum acceleration. *Ans.*, 21 π feet per second; 294 π^2 feet per second.

2. A piston with rod and crosshead weigh 350 lbs. If they have a S.H. motion with amplitude 1·1 foot, and if the maximum accelerating force is equal to that produced by a pressure of 15 lbs. per square inch on a piston 14 inches diameter, what is the periodic time? *Ans.*, 0·37 sec.

OTHER EXAMPLES OF PERIODIC MOTION.

450. When the **periodic motion** of a body is not simple harmonic, we find that by imagining the body to have two or more kinds of simple harmonic motion at the same time we can get the same result. Thus, it is known that a float, employed to measure the **rise and fall of the tide** by marking on a moving sheet of paper with a pencil, has a motion which is **periodic** and not simple harmonic. If horizontal distances represent the motion of the paper (unwound from a barrel by means of clockwork), and therefore represent time, and if vertical distances mean the rise or fall of water-level in feet, we get such a curve as is shown in Fig. 314. Now this is not a simple harmonic motion. The *difference* becomes evident if you plot on squared paper the

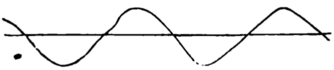

Fig. 314.

distances O A, O B, O C, O D, O E, O F, etc. (Fig. 308), for equal intervals of time, for you will get a curve like Fig. 315, which is easily recognised, and is called a **curve of sines or cosines.** But it has been found that if we take certain curves of sines whose periodic times are—1, the semi-lunar day; 2, the semi-solar day, and some others, their amplitudes and epochs being properly chosen, and draw them on squared paper, and add their ordinates together, we get the curve which shows the real rise and fall of the tide. In the very same way we can **combine simple harmonic motions** to arrive at any periodic motion. A good way of combining simple harmonic motions experimentally is to let a body hang from a string which passes over two or more movable, and the same number of fixed, pulleys. These pulleys are pivoted on crank pins, and their pivots are made to revolve at any desired relative speeds, and each gives to the body a purely simple harmonic motion by its action on the string. The body gets a motion compounded of the motions of the pulleys, and if it is an ink-bottle or pencil pressing on the paper on a revolving paper roller, we get a time curve of the periodic motion. This is the principle of the construction of **Lord Kelvin's Tide-Predicting Machine.**

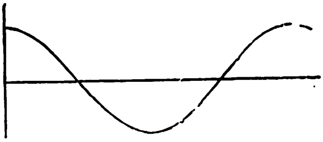

Fig. 315.

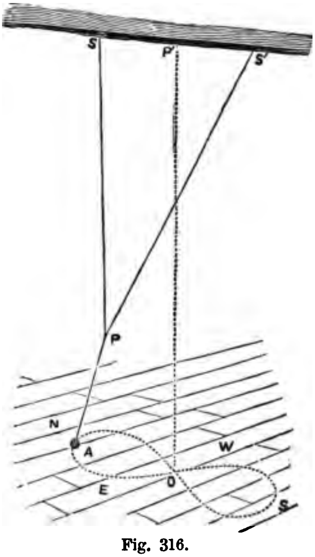

Fig. 316.

451. When a body can swing east and west under the influence of forces which have no tendency to move it except in a direction due east and west, and if forces acting due north and south can make it swing in their direction, then both sets of forces acting together on the body will give it a **motion**

compounded of the two simpler motions. Thus, a ball A (Fig. 316) is suspended by a string, P A, which is knotted at P to two other strings, P S and P S′, equal in length, and fastened at S and S′. The ball may swing in the direction E O W as if it were the bob of a pendulum hung directly from the ceiling at P′, but it may also swing in the direction N O S at right angles to E O W, and if it does so it swings as if the point P were the fixed end of the pendulum A P. When it swings under the influence of the two sets of forces tending to make it move both ways at once, the motion of A is compounded of the other two simpler motions. If P A is one-quarter of the length O P′, then the east and west swing takes twice as long as the north and south swing. If P A is one-ninth of O P′, then the east and west swing takes three times as long as the north and south swing. The motion of A is sometimes very beautiful, and the experiment is easily arranged.

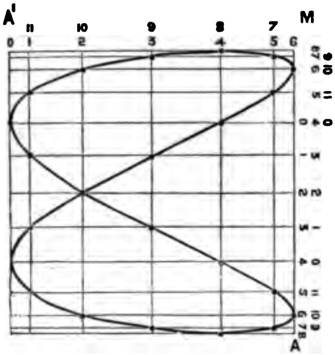

Fig. 317.

452. The motion is quite easily represented on paper. Thus, in Fig. 317, A′ M is the north and south direction, and A M, at right angles to it, is the east and west direction. Let the points 0, 1, 2, etc., in each of these lines be found as in Fig. 308. Let the bob be supposed to go from 0 to 1 in A′ M in the same time as it goes from 0 to 1 in A M. Notice that we have twice as many points in A M as in A′ M, showing a slower oscillation in the direction A M. We can begin to number our points anywhere, remembering that when the bob completes its range it comes back again in the opposite direction. Now put marks where the east and west lines meet the north and south ones, drawn through corresponding points. It is evident that the curve drawn through these successive marks is the real path traced out by the ball when acted upon simultaneously by the two sets of forces urging it in a north and south, and an east and west direction.

If we have the same number of points in A′ M as in A M, we get a circle, ellipse, or straight line, as in C, B, A, Fig. 318. This represents the motion of a conical pendulum free to swing in every direction. Again, D, E, F, and many other curves that might be drawn, represent the case which we took up in Fig. 317, where one vibration is twice as quick as the other. If the time of vibration in A M is to the time of vibration in A′ M as 2 to 3, we get curved paths like G, I, J, and so on. In experimenting with the pendulum, Fig. 316. it will usually be found that slight inaccuracies in the lengths of the cords will cause a continual change to go on in the shape of the path traced out by the ball.

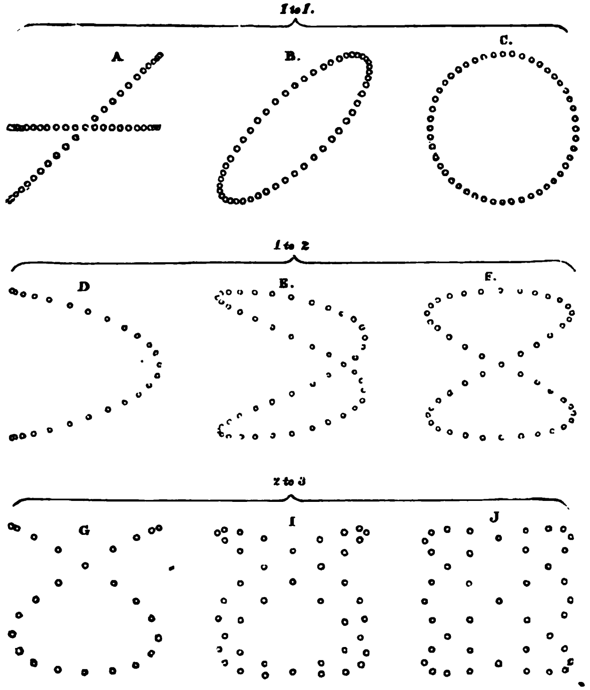

Fig. 318.

We can produce these motions by spiral springs, and in other ways. Thus, for example, if we use instead of the strip of steel, in Fig. 312, a combination of two strips, B and B′, as in Fig. 319, so that the heavy bright bead A is capable of vibrating in two directions at the same time, we get the same combinations of simple harmonic motions, depending on the point at which B is held in the vice C.

453. When a body has a **periodic rotational motion** about an axis like the balance of a watch or a rigid pendulum, we must no longer speak of the force causing motion, and the mass of the body, and the distance of displacement; but if we substitute for these terms, moments of forces, moment of inertia of the body and angle of displacement, we have exactly the same rule for finding the periodic time of oscillation. The periodic time is 6·2832 times the square root of the angular displacement of the body at any instant, divided by the angular acceleration at that instant. And we know that angular acceleration may be calculated by dividing the turning moment acting on a body by the moment of inertia of the body. A point in the **balance of a watch** swings in circular arcs, but if we only take account of the distances which it passes through, and suppose it moved in a straight line instead of in the arc of a circle, the motion is very nearly simple harmonic. If there were no friction or other forces acting on the balance except the turning moment of the balance spring (*see* Arts. 521-2-3), and if the moment of the spring were always exactly **proportional to the angular displacement of the balance**, the motion would be simple harmonic.

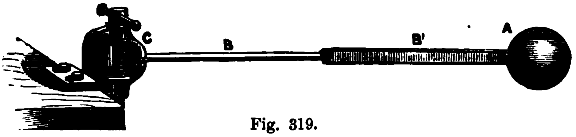

Fig. 319.

We shall see in Art. 522 that the turning moment due to the spring is $\frac{\mathrm{E}\,bt^3}{12\,l}\,\theta$, if E is the modulus of elasticity of the spring, b its breadth, t its thickness, and l its length, and if θ is the angular displacement in radians. Angular acceleration is this moment divided by moment of inertia I of the balance, or $\frac{\mathrm{E}\,bt^3}{12\,l}\,\frac{\theta}{\mathrm{I}}$. Hence angular displacement θ divided by angular acceleration is $\frac{12\,l\,\mathrm{I}}{\mathrm{E}\,bt^3}$, so that the **periodic time of the balance** is $\mathrm{T} = 6{\cdot}2832\sqrt{\frac{12\,l\,\mathrm{I}}{\mathrm{E}\,bt^3}}$.

Increasing the moment of inertia of the balance or the length of the spring makes the vibration slow. Increasing the breadth and, what is still more important, increasing the thickness of the spring makes the vibration quick. As we shall see in Arts. 521-3 that our calculation of the turning moment of the spring is not quite right, that the dimensions of the balance and spring **alter with temperature**, and that, above all, the elasticity of the steel alters with temperature, and with its own state of **fatigue**, the rule is not perfectly true, nor can any balance be regarded as taking exactly the same time for its oscillation in different lengths of arc. At the same time it is of great help to the **watchmaker** to know that with considerable, although not with perfect accuracy, the time of vibration of a balance is proportional to the square root of the length of the spring, and so on. For example, suppose the spring is 3 inches long, and the balance makes one swing in 0·251 second, now if he wishes it to make a swing in 0·25 second, he must shorten it in the ratio of ·251 × ·251 to ·25 × ·25, or in the ratio ·063001 to ·0625, so that the length of his spring ought to be 3 × ·0625 ÷ ·063001, or 2·976 inches—that is, it ought to be shortened ·024 inch. In the same way he can calculate the effect of adding little masses at any distances from the centre of the balance, so that its moment of inertia may be increased, and the balance made slower in its swing. The same law tells him how he can **compensate the balance**, so that when in summer the steel of the spring loses its elasticity, some of the mass of the balance will come nearer the centre, in order that the moment of inertia may diminish in the same proportion.

454. Compound Pendulum.—The simple pendulum described in Art. 446 is not like the pendulums used in practice. In these the bob is not so small that we can consider it as a point; the long part is not a thread but a stiff rod of metal or wood, and there is usually a **knife-edge** for support, about which it can turn with little friction. In common clocks, however, the top end of the pendulum is a **thin strip of steel** held firmly in the chops, but the easy bending of this strip is such that we may imagine an equivalent pendulum to move freely about an axis. Employing our general rule of Art. 453, we find how to calculate the time of vibration. This compound pendulum **vibrates in the same time** as a certain simple pendulum, called the *equivalent simple pendulum*, whose length we find by experiment.

In Fig. 320 let S be the *axis of suspension*, G the centre of gravity, and P a point in the continuation of the line SG such that SP is the length of the equivalent simple pendulum. Then P is called the *centre of oscillation*, and it is also known to be the *centre of percussion* of the pendulum (*see* Art. 402). It can be proved that if the pendulum be inverted and made to vibrate about a parallel axis through P, it will vibrate in exactly the same time as it does about S; and it was in this way, by inverting a pendulum which had two knife-edges, and adjusting these until the pendulum took the same time to vibrate about one as about the other, and then measuring the distance between them, that Captain Kater found the length of the simple pendulum which vibrates in a given time. This method is still employed in gravitation experiments everywhere to find the value of g, which is 32·2 feet per second per second at London. It is an excellent laboratory exercise.

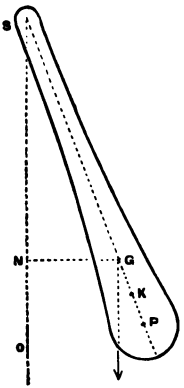

Fig. 320.

If S is the axis of suspension, G the centre of gravity, W the weight of the pendulum, then the moment with which gravity urges the pendulum to return to its position of rest is W × GN; but if the angle GSO be measured in radians, and if it is very small, this moment is almost exactly equal to W × SG × angle GSO. The angular acceleration is obtained by dividing this by I, the moment of inertia of the pendulum about S, and our rule becomes

$$\mathrm{T} = 6{\cdot}2832\sqrt{\frac{\text{angle GSN}}{\mathrm{W}\,.\,\mathrm{SG}\,.\,\text{angle GSN} \div \mathrm{I}}},$$

or

$$\mathrm{T} = 6{\cdot}2832\sqrt{\frac{\mathrm{I}}{\mathrm{W}\,.\,\mathrm{SG}}}\;\dots\dots\;(1).$$

When we examine this formula we see that it may be put in another form. Find a point K such that if all the mass of the pendulum were gathered there its moment of inertia about S would be the same as at present; in fact, such that $\frac{\mathrm{W}}{32{\cdot}2}$, the mass of the pendulum × SK^2, would be equal to I. The distance SK is called the **radius of gyration** of the pendulum (*see* Art. 112), and our rule now becomes

$$\mathrm{T} = 6{\cdot}2832\sqrt{\frac{\mathrm{SK}^2}{g\,.\,\mathrm{SG}}}\;\dots\dots\;(2),$$

where g is 32·2. In the simple pendulum SK and SG are equal, and (2) gives the same rule which is given in Art. 446. However, in an ordinary pendulum, SK and SG are not equal, but $SK^2 \div SG$ is equal to some length such as SP, and our rule becomes

$$T = 6{\cdot}2832\sqrt{\frac{SP}{g}} \;\ldots\ldots\; (3).$$

Evidently SP *is the length of the imaginary simple pendulum which would vibrate in the same time as our real pendulum.* The imaginary point P has been called the centre of oscillation, because when the pendulum is inverted and made to vibrate about an axis through P it vibrates in the same time as before.

To prove this it is necessary to return to equation (1) We know that I is equal to the moment of inertia of the body calculated as if all its mass existed at G, together with the moment of inertia of the body as it is at present, but calculated about an axis through G parallel to the present axis; that is,

$$I = \frac{W}{g}\,SG^2 + \frac{W}{g}\,k^2,$$

where k is some length unknown to us just now, being the radius of gyration about the axis through the centre of gravity. Rule (1) becomes

$$T = 6{\cdot}2832\sqrt{\frac{\frac{W}{g}\,SG^2 + \frac{W}{g}\,k^2}{W\,.\,SG}},$$

or

$$T = 6{\cdot}2832\sqrt{\frac{SG + \frac{k^2}{SG}}{g}}.$$

That is, the length of the simple pendulum which will vibrate in the same time is $SG + \frac{k^2}{SG}$, and we have already found it to be SP in equation (3); so that $GP = \frac{k^2}{SG}$, or $GP \times SG = k^2$. But in the very same way, if we considered the pendulum as vibrating about P, we should find the length of the equivalent simple pendulum to be greater than GP by an amount equal to $\frac{k^2}{GP}$, and we know that SG is equal to this amount; so that SP would, as before, be the length of the equivalent simple pendulum. *The axes of oscillation and suspension are therefore interchangeable.*

455. *Examples.*—The **bar** of Fig. 321 with **two adjustable masses** may be fixed to one end of a wire, the other end of which is fixed to the ceiling. By twisting and untwisting the wire the bar will oscillate with a motion which is much more nearly simple harmonic than that of the balance of a watch. Students who experiment with such a bar can adjust the weights A and B at any distance from the axis (there ought to be an engraved scale on the bar), so that the moment of inertia can be varied. They can fasten the bar at the end of a wire, or they can use it as in Fig. 321, with a flat spiral spring, or as in Fig. 322, with a cylindric spiral spring; and the rate of its vibration gives one of the best ways of investigating the twisting moments of wires and such springs when strained through given angles.

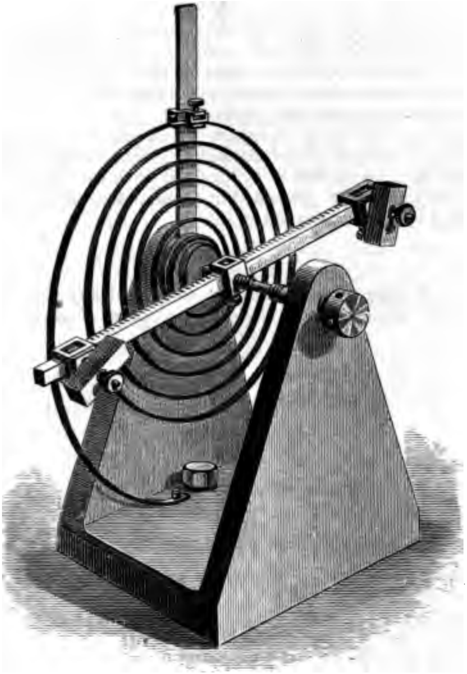

Fig. 321.

In the case of a wire the twist always tends to bring the bar to its position of rest with *a moment which is proportional to the angle of displacement from this position*—it is this property which causes the motion to be simple harmonic. *This moment is also proportional to the fourth power of the* *diameter of the wire, and it becomes less as the length of the wire is increased.* By means of a circular scale and a pointer we can measure the extent of each swing, and this is found to decrease gradually, due to friction with the air and the internal friction or *viscosity* of the metal. The **amount of diminution of swing** gives us a means of determining the viscosity, and the apparatus can so easily be fitted up that no person who wishes to understand the properties of materials can be excused from making these experiments. This is a common method of finding N, the **modulus of rigidity** of a material.

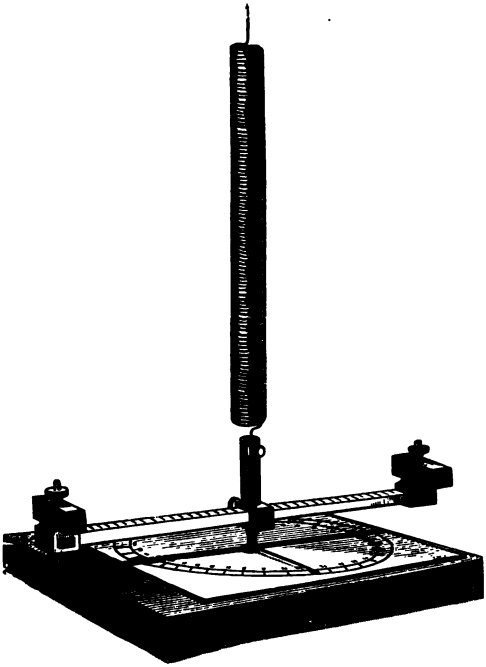

Fig. 322.

If the length of the wire is l inches, its diameter d, and if N is its modulus of rigidity (*see* Table XX.), then from Art. 295 we see that the moment with which the wire acts on the bar, when its angle is θ from the position of rest, is $\frac{\pi}{32} \mathrm{N} \frac{\theta}{l} d^4$. If the moment of inertia of the bar is I (we are neglecting the fact that the wire itself has some mass which has to be set in motion), then the moment, divided by I, is the angular acceleration; and using this quotient as denominator, and θ as numerator, extracting the square root, and multiplying by 6·2832, or 2π, by the general rule of Art. 453, we find the square of **the period of a complete oscillation** to be $\mathrm{T}^2 = \frac{128\pi \mathrm{I} l}{\mathrm{N} d^4}$. If we are in doubt as to our calculated I, we can find it experimentally; and this is very necessary in many magnetic experiments. We add a *known* moment of inertia i (say two equal small masses at equal distances from the axis), and find the new T (call it T_1). Then $\mathrm{T}_1^2/\mathrm{T}^2 = (\mathrm{I} + i)/\mathrm{I}$, and I may be found.

When motion is slow, the friction in fluids is proportional to the velocity, and any friction which follows this law is called fluid friction. A great many vibrating bodies tend to come to rest by the action of such friction as this; and it is found that if the friction is numerically f times the angular velocity, then the logarithm of the ratio of the length of one complete swing to the next is nearly equal to k times the periodic time. Hence this logarithmic decrement, as it is called, is proportional to the friction co-efficient. If we observe twenty-one elongations on one side of the middle position, then one-twentieth of the logarithm of the first elongation divided by the last is k times the periodic time of oscillation.

EXERCISES.

1. **Bifilar Suspension.**—In many measuring instruments a body is suspended by two thin wires nearly vertical. If the vertical length of each of these is l, the distance between their ends at the top a, and at the bottom b, and the weight of the body w, it is easy to show that for a small angular displacement θ, the moment tending to bring the body to its position of rest is very nearly (neglecting torsion of the wires themselves) $\frac{1}{4} \frac{ab}{l} w\theta$. Find the time of vibration of such a body when its moment of inertia is known. In truth, the constraining moment is proportional to sin. θ.

2. **A magnet**, turning on a frictionless pivot at its centre of gravity, is subjected to a turning moment H sin. θ, or very nearly $\mathrm{H}\,\theta$, due to the earth's magnetic action, if it makes only a small angle θ with its position of rest. Find the time of a vibration if the moment of inertia is known, and show that the square of the time of vibration of the magnet in different places is inversely proportional to H. Find what is the effect of *adding* a known moment of inertia, and show that the observations on the

two times of vibration enable us to calculate the original moment of inertia if it was unknown.

3. Prove that the time of complete oscillation of a ship is $2\pi k/\sqrt{gd}$, where d is the distance from the centre of gravity to the metacentre.

456. Stilling of Vibrations.—When a simple harmonic motion is represented on paper in the manner described in Art. 450, we have a curve of sines. The curve may be obtained by producing the lines B B′, C C′, etc., of Fig. 308, cutting them at right angles by equidistant horizontal lines, and joining the successive points of intersection so found. It may also be drawn by finding from a book of tables the sines of 0°, 10°, 20°, etc., and plotting 0 and sin. 0°, 1 and sin. 10°, 2 and sin. 20°, etc., on a sheet of squared paper.

A *curve of sines* expresses the fact that, if d represents the displacement of a vibrating body from its middle position after an interval of t seconds since it was at the middle of its course, then $d = a \sin. pt$ where a is the greatest displacement of the body from its middle position. This displacement is usually called the *amplitude of the vibration*. If T is the time of a complete vibration, it is easy to see that the equation is $d = a \sin. \frac{2\pi}{T} t$, or $a \sin. 2\pi f \,.\, t$ if f is the frequency or number of complete oscillations per second.

If we make the bob of a pendulum terminate below, in a tube which can act as a pencil-holder, and in which a well-fitting pencil can slide freely, and if we move a sheet of paper at a uniform rate underneath this pencil at right angles to the direction of motion of the pencil, a curve of sines will be traced out, if the pendulum swings without friction. But in practice we always find that, what with the friction at the point of support, friction with the atmosphere, etc., a pendulum's swings get smaller and smaller—that is, the **amplitude of the vibration gets less and less** as time goes on, until the pendulum at length comes to rest.

This motion is not a simple harmonic motion, but, within certain limits, each swing may be regarded as very nearly a simple harmonic motion. Practical men who deal with oscillating bodies, such as pendulums, ships, tuning forks, magnetic needles, and suspended coils of wire, usually assume that the motion during each swing is a simple harmonic motion. The frictional resistance to motion of any ordinary vibrating body in a fluid medium, or of a magnetic needle vibrating near any body capable of conducting electricity, is almost always such that the quicker the motion the greater the friction (*see* Art. 64)—that is, frictional resistance is proportional to speed; and in this case it is not difficult to show that, instead of the law $d = a \sin. \frac{2\pi}{T} t \ldots . (1)$, we have the law $d = a\epsilon^{-\alpha t} \sin. \frac{2\pi}{T_1} t \ldots . (2)$. That is, if the strength of the spring, or other governor of vibration, and the character of the vibrating body are such that without friction the law would be (1), then, when the vibration is damped by frictional resistance of the above character, the law of the motion becomes that given by equation (2). Here α is a constant which depends on the character of the friction. Thus α is greater when a pendulum swings in water than when it swings in air. Also, T_1, the periodic time of the vibration, is no longer the same T as it was for undamped vibrations, and the relation between T and T_1 is $\frac{1}{T^2} = \frac{1}{T_1^2} + \frac{\alpha^2}{4\pi^2} \ldots . (3)$; or, if f is the undamped frequency, and f_1 the damped frequency, then $f^2 = f_1^2 + \frac{\alpha^2}{4\pi^2} \ldots . (4)$. In order to get exact ideas on this subject of the **damping of vibrations**, the student ought to plot on squared paper a curve such as O A′ B′ C′ D′ E′ F′ G′ H′ I, Fig. 323, which corresponds with equation (2). Thus, let us suppose that a body undamped in its vibrations gets an impulse which sends it from its position of rest in such a way that its amplitude is 10 inches, and let the time of a complete oscillation be 1·6 second. Then the law of its motion would be $x = 10 \sin. \frac{6{\cdot}2832}{1{\cdot}6} t$, or $x = 10 \sin. 3{\cdot}927\, t \ldots . (5)$ where x is in inches, t in seconds, and the angle $3{\cdot}927t$ in radians.*

If, now, the friction is such that $\alpha = 0{\cdot}7$, we find from (3) that the time of an oscillation is practically unchanged. Find, therefore, the original curve of sines by calculating the second column of the following table. The numbers of the first two columns plotted on squared paper would represent the undamped vibrations. But for damped vibrations the numbers of the second column have all to be multiplied by $\epsilon^{-0{\cdot}7t}$; and if we denote this multiplier by the letter z, we see that z being $\epsilon^{-0{\cdot}7t}$, or $\log. z = -0{\cdot}304\, t$. We have calculated z for the various values of t, and placed the results in the third column. Multiplying, therefore, the respective numbers of the second and third columns together, we get the fourth column of numbers; and plotting the numbers of the first and fourth columns on squared paper, we find the curve which shows the nature of the damped vibrations.

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* We may write (5) in the form $x = 10 \sin. 225t$. In this case the angle $225t$ is expressed in degrees.

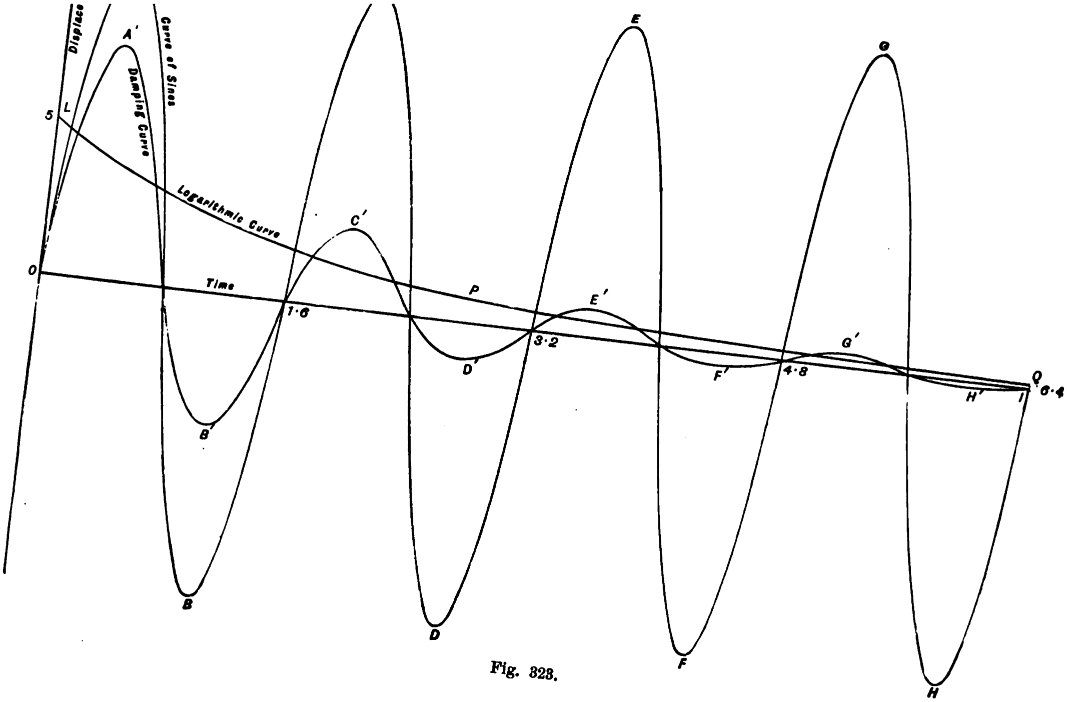
Displace
A'
Curve of Sines
Damping Curve
5
L
Logarithmic Curve
C'
0
Time
P
E'
G'
1·6
3·2
4·8
Q
6·4
D'
F'
H'
I
B'
B
D
F
H
E
G

Fig. 323.

t in seconds.	10 sin. 3·927 t, or 10 sin. 225 t, if angle is taken in degrees.	$\epsilon^{-0·7t}$	$\epsilon^{-0·7t}$ 10 sin. 3·927 t
0	0	1	0
0·2	7·07	·869	6·14
0·4	10	·756	7·56
0·6	7·07	·657	4·65
0·8	0	·571	0
1·0	− 7·07	·497	− 3·51
1·2	− 10	·432	− 4·32
1·4	− 7·07	·375	− 2·65
1·6	0	·326	0
2·0	10	·247	2·47
2·4	0	·186	0
2·8	− 10	·141	− 1·41
3·2	0	·106	0
3·6	10	·080	·8
4·0	0	·061	0
4·4	− 10	·046	− ·46
4·8	0	·035	0
5·2	10	·026	·26
5·6	0	·020	0
6·0	− 10	·015	− ·15
6·4	0	·011	0

457. Some students may find it as instructive to first draw a **curve of sines**, then draw the **logarithmic curve**, corresponding to column three, on the same sheet of squared paper, and multiply the ordinate of one curve by that of the other to get the ordinate of the real curve which exhibits the damped vibrational motion. This is what has been done in Fig. 323; O A B C D E F G H I is the curve of sines, L P Q is the logarithmic curve, showing how rapidly the amplitude of the vibration diminishes, and O A′ B′ C′ D′ E′ F′ G′ H′ I is the curve which represents the actual motion of the vibrating body. In this figure the logarithmic curve is drawn to such a scale as seemed convenient for showing its properties distinctly. It would be very easy to dilate on the nature of the resulting curve O A′ B′, etc., but this book is written to help students who are earnest enough to calculate the above numbers and plot the curve, and when they perform these operations they will have very clear notions about the motion we have been investigating.

Exercise.—A heavy disc, suspended by a wire, vibrates in each of a number of fluid media, its periodic time of vibration in all being sensibly the same, or 1·5 second. The ratio of the amplitudes of two successive swings in one direction being 0·9 in one fluid—that is, the second swing being only nine-tenths of the first, and the third being only nine-tenths of the second, and so on—and 0·8 in another fluid, and 0·7 in another, what numbers will express the **relative viscosities** of these fluids?

Here we have, taking *common* logarithms, $0{\cdot}9 = \epsilon^{-1{\cdot}5\alpha}$ for the first fluid, so that $-\log. 0{\cdot}9 = 1{\cdot}5\,\alpha \log. \epsilon$, or $\alpha = \dfrac{-\log. 0{\cdot}9}{1{\cdot}5 \log. \epsilon}$, that is, $\alpha = 0{\cdot}07$. In the same way $\alpha = 0{\cdot}15$ and $\alpha = 0{\cdot}24$ for the other fluids, and hence 7, 15, and 24 are the required numbers expressing the relative viscosities as measured by the vibrating disc method.

A very slowly swinging disc and pointer will enable us to lot the complete curve from actual observations. The nature f the motion when the friction is that of solids rubbing on lids is studied in my book on the Calculus.

CHAPTER XXVI.

MECHANISM.

458. In Art. 179 we pointed out how **skeleton drawings and models** may be made useful in cases where velocity ratios vary greatly. We shall now give a sketch or outline of a general theory from which students may find benefit if they fill it in for themselves. We cannot say that there is as yet any satisfactory treatise on this subject. The most interesting part of it (to us) is that which concerns the mechanism of steam and other engines, and this will be found in the author's book on the Steam Engine.

459. We have already referred to spur and bevil gearing used to drive one shaft from another at uniform velocity ratios. Consider at any instant two teeth in contact; each of them is a rigid piece rotating about a fixed centre and acting on the other by rubbing contact. We shall now briefly refer to this way of transmitting motion.

Let AW be a body moving in the plane of the paper about the axis A, let BV be another body moving about the axis B, and let these two bodies keep in contact, as we see them in contact at P. Now, this keeping in contact, what does it mean? It is that, whatever may be the relative tangential or sliding motion at the point of contact P, they have the same velocity in the direction of their common normal there at the instant when we study them. Consider their motions during an exceedingly short interval of time δt. There are two points P to be considered. One is on the body AW, and it moves to Q, where PQ is an arc of a circle about A as centre; the other P is on the body BV, and it moves to R, where PR is an arc of a circle about B as centre. The angular velocity of AW being a, the angle PAQ is $a \cdot \delta t$; the angular velocity of BV being b, the angle PBR is $b \cdot \delta t$. As δt is considered smaller and smaller, PQ and PR may be considered more and more nearly short straight lines. $PQ = AP \cdot a \cdot \delta t$ and $PR = BP \cdot b \cdot \delta t$. Now observe that if TPS is the direction of

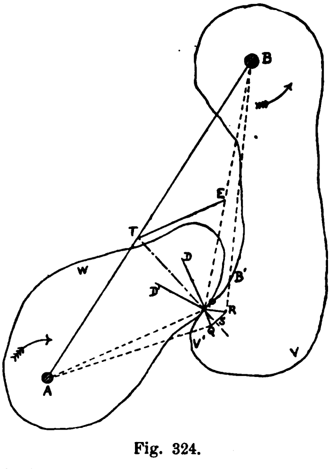

Fig. 324.

the common normal to the two bodies at P, the normal component PS of the motion PQ must be the same as that of the motion PR; in fact, the straight line QSR must be at right angles to the normal TPS. Now draw TE parallel to AP, and observe that the two triangles TEP and QPR have their sides QR and TP, RP and PE, PQ and ET respectively at right angles to one another, and hence they are similar; that is,

$$\mathrm{QR} : \mathrm{RP} : \mathrm{PQ} = \mathrm{TP} : \mathrm{PE} : \mathrm{ET} \;\ldots\ldots (1),$$

$$\text{or } \frac{\mathrm{RP}}{\mathrm{PQ}} = \frac{\mathrm{PE}}{\mathrm{ET}}.$$

$$\text{Hence } \frac{\mathrm{BP}\,.\,b}{\mathrm{AP}\,.\,a} = \frac{\mathrm{PE}}{\mathrm{ET}}, \text{ or } \frac{b}{a} = \frac{\mathrm{AP}\,.\,\mathrm{PE}}{\mathrm{BP}\,.\,\mathrm{ET}}.$$

But because TE is a line drawn parallel to AP, a side of the triangle APB, we have

$$\frac{\mathrm{AP}}{\mathrm{ET}} = \frac{\mathrm{AB}}{\mathrm{BT}} \text{ and } \frac{\mathrm{PE}}{\mathrm{BP}} = \frac{\mathrm{AT}}{\mathrm{AB}};$$

$$\text{therefore } \frac{\mathrm{AP}\,.\,\mathrm{PE}}{\mathrm{ET}\,.\,\mathrm{BP}} = \frac{\mathrm{AT}}{\mathrm{BT}}.$$

$$\text{Hence } \frac{b}{a} = \frac{\mathrm{AT}}{\mathrm{BT}} \;\ldots\ldots (2).$$

Hence the **ratio of the angular velocities** is inversely as the segments into which the common normal at the point of contact divides the line of centres. Hence, if the ratio of the angular velocities is constant, the common normal at the point of contact P passes always through the same point T in the line of centres AB. Notice also that $\frac{\mathrm{QR}}{\mathrm{PQ}} = \frac{\mathrm{PT}}{\mathrm{TE}}$. But $\mathrm{PQ} = a\,.\,\mathrm{AP}\,.\,\delta t$, $\mathrm{TE} = \mathrm{AP}\frac{\mathrm{BT}}{\mathrm{AB}}$, and $\frac{\mathrm{BT}}{\mathrm{AB}} = \frac{a}{a+b}$. Hence it will be found that

$$\mathrm{QR} = \mathrm{PT}\,(a + b)\,\delta t.$$

That is, the slipping speed at P is the speed at the end of a radius TP when the radius revolves at the angular velocity $a + b$. Hence there is always slipping at P, unless P is on the line of centres and the speed of slipping is proportional to the distance PT.

The student will find that when there is friction, if DPT = D′PT = angle of repose, then PD is the direction of the mutual force when AW rotates as shown, with the hands of a watch; whereas PD′ is the direction of the mutual force when AW rotates in the opposite direction, so that BV is the driver.

If we have given the shape of BVP, and we desire to find the shape of a piece AWP which will gear with it at a constant angular velocity ratio, make a template of BVP, and arrange that when this template is moved about the fixed centre B a sheet of paper shall move about A through the proper angular distances. If for each position of the two the curved shape of B′PV′ be drawn on the paper, the pencil marks will show the proper shape of AWP if it is to touch BPV. In fact, AWP will be the *envelope* of the shapes of B′PV′. When B′PV′ is circular with B as centre, AWP will also be circular. **True rolling** will be possible, for P will be at T, and as T will be constant in position the angular velocity ratio will be constant. When B′ P V′ is the arc of an ellipse whose focus is at B, and if the distance B A is equal to the major axis of the ellipse, A W P will be a similar ellipse; there will be true rolling, with changing angular velocity ratio. The other foci may be connected by a link. If B′ P V′ is shaped like an equiangular spiral whose pole is at B, A W P will be a similar spiral, and there will be true rolling between them, but with changing angular velocity ratio; in this way lobed wheels are formed to gear together.

460. Discs or cylinders touching each other, their axes parallel, are used for friction gearing. If the horse-power H is to be transmitted, and V is the common circumferential speed in feet per minute, P being the necessary tangential force, $P = 33{,}000\, H \div V$. Slipping is to be impossible, and therefore $P \div \mu$ is the force necessary to press the cylinders together. If this force acts through the bearings of the two shafts, it is usually found in practice that, unless at very high speeds and with small power, there is so much **practical difficulty** that the gear is never used. **Compressed paper** and **leather** have been used to work with iron. Sometimes nest-gears are used to produce the necessary pressure, but when the necessary pressure has been produced it has led to disintegration of the surface, or such local elastic changes of shape as produce annoying sound. In one case, where the driven pulley is very heavy and the pressure is produced by its own weight and that of the spindle and part of the weight of the rotating armature of a little dynamo machine, the gear has been used satisfactorily. Wedge-shaped grooves and projections have been cut in the rims of the pulleys, and sufficient grip has been produced in this way, but there is no longer true rolling.

461. When we attempt by **using teeth** to get the necessary driving forces we introduce **sliding contact**, using **spur, bevil,** and **skew bevil wheels**; the names pitch circles, pitch cones, and pitch hyperboloids being used for the friction gear, which would run with the same velocity ratios. When the axes are not in one plane, frusta of hyperboloids, generated by the rotation of the same straight line round both axes, will gear with one another, always touching along a straight line. But there will not be simple rolling; there is sliding along the line of contact.

Every student ought to study the **shapes of spur-wheel teeth**; it is easy to apply one's knowledge to other kinds of teeth and rubbing and rolling gear. Nothing illustrates the fact that we do not really *think*, so well as this, that all the principles for the proper construction of worm-wheel teeth and chain gearing were to be found in books many years before there existed any good **worm-wheel** teeth or **chain gearing**. The subject, like all other parts of machine design, is best studied when one draws things to scale, and there are now many books to assist the student in machine design. Perhaps it will be well to neglect almost all the mathematical parts of such books on strength.

462. Suppose we have two **curved rollers,** V T V moving about the axis B and W T W about the axis A, and suppose that these are capable of rolling on one another as they rotate; they touch at T, and the angular velocities about A and B are as B T to A T. Now suppose we wish wheels, with teeth centred at B and A, to have exactly the same angular velocity ratios as the rollers; if B′ P V′ and A′ P W′ are the shapes of the teeth in contact at P, it is necessary that during the motion the common normal at P should pass through the pitch point T. To effect this object it is necessary that B′ P V′ and A′ P W′ should be two **trochoidal curves**, generated by the rolling of a curve inside the curve T V and outside the curve T W. Thus in Fig. 325 imagine the curve V and the curve W to move about B and A, rolling on one another and keeping in contact at T in the straight line B A, and imagine the curve P P′ T to roll also, keeping in contact with both V and W at T. It is really rolling inside V and outside W. If any point of P P′ T, such as P, is always on the contours of the two teeth, the straight line P T is always normal to these teeth at P, their point of contact, and we have ensured that the common normal to both passes through the pitch point T. W and V may be ellipses, but they are generally circles, A and B being their centres. If P P′ T is a circle, rolling inside V and outside W, the trochoidal curves are called **hypo-** and **epi-cycloids**. When a number of wheels are to gear any one with any other, we usually choose one rolling circle for the insides and outsides of all the pitch circles; it is taken of a diameter equal to the radius of the smallest pitch circle. When it rolls inside this smallest pitch circle the hypocycloid is a radial line. A **rack** may be regarded as part of a wheel of infinite diameter. Sometimes the trochoidal curves are **involutes of circles**, the rolling curve being really a straight line or an infinite circle. Let W and V (Fig. 326) be the pitch circles. Draw any line C D through T, the pitch point, and describe circles with B and A as centres touching this line at D and C respectively. Draw through T′ G T H, the involute of the circle D G, and F T E, the involute of the circle C F. If G H and E F be the contours of teeth rotating about B and A, their common normal remains always C D. Thus at any point T′ in C D, if we draw

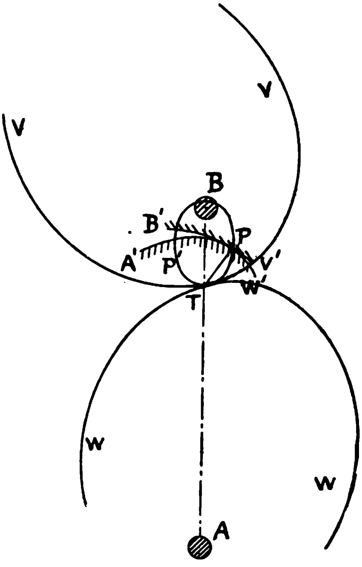

Fig. 325.

involutes to the two circles, they must touch there, because a tangent to a circle cuts all the involutes at right angles. It is evident that a pair of wheels with involute teeth may have less or more distance between their centres without any alteration of their velocity ratio, this being B T : A T or B D : A C. If there is no friction, C D is the direction of the driving force. It is usually more oblique to the line of centres in wheels with involute teeth than in cycloidal teeth. For the most accurate work the actual trochoidal curves are drawn by the rolling of templates, but there are well-known drawing-office rules by which we approximate to the true best curves by means of arcs of circles.

463. In Art. 462 we showed how, when given the **shape of a tooth**, to find the shape of another tooth to gear with it with constant velocity ratio. A good example of the application of this rule is found when designing the shape of the tooth of a worm wheel.

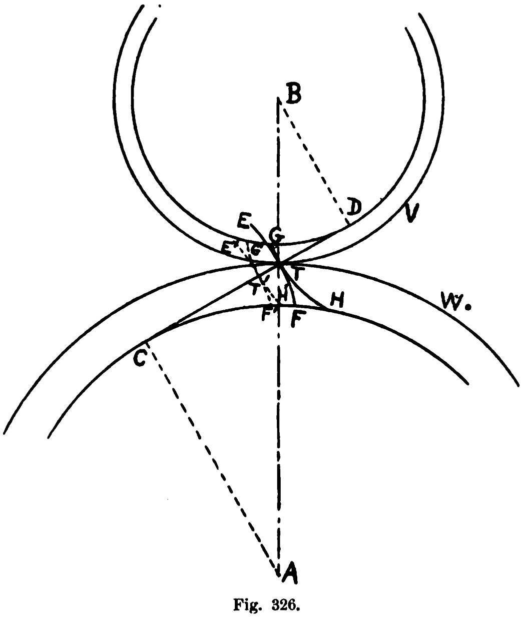

Fig. 326.

The shape of the **thread of the worm** being chosen, any section of the worm thread by a plane at right angles to the axis of the worm wheel may be drawn by elementary practical geometry. Regard it as the tooth of a rack. The section there of the tooth of the worm wheel must be the shape of a tooth to gear with the given rack tooth, and must be drawn by some such rule as we have given in Art. 462.

464. Sometimes pieces centred at A and B, acting upon one another, do not act directly as shown in Fig. 324, but through the agency of a third body. Thus, if pins on V and W are connected by a link W V, and if the pins are frictionless, we know that the direction of the force at any instant must be in the direction of the centres of the pins or in the line W V. It is easy to show as before that if a is the angular velocity of A W and b the angular velocity of B V, then $\frac{a}{b} = \frac{BT}{TA}$, if T is where W V cuts the line of centres A B. The complete study of the *relative* motion of four links such as A W, W V, V B, and B A is of course a very complicated business if we imagine them to be of all sorts of lengths. We may, if we please, imagine any one of them **fixed** and consider the motions of the others. Thus in Fig. 328 consider A B to be fixed. Think that A and B are merely pins in some fixed object of any shape whatsoever. Now consider the other pieces to be of any curious shapes and lengths. The motion is taken to be in the plane of the paper. W V is only a straight line joining the centres of the two pins W and V; but imagine W V to represent any curiously shaped body—we might wish to know the motion of any point in this body. The student's great aim is to get a **correct mental picture**, and if he recollects that W is moving about A as an **instantaneous centre** or at right angles to A W, and V is moving about B, the whole body W V must just at the present instant be moving about the point C as a centre. We have produced W A and V B to meet at C, the instantaneous centre of motion of the whole body W V. One has then a mental picture of the motion of any point whatsoever in the body W V.

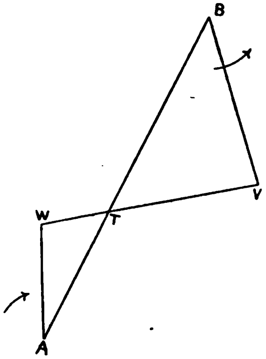

Fig. 327.

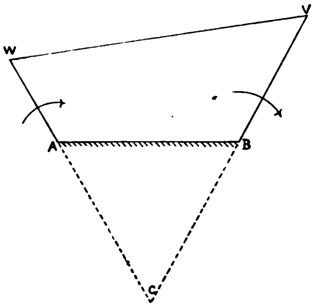

Fig. 328.

465. General motion of a body parallel to a plane.—Let us simply say a plane figure or body in its own plane. If we consider the motion of any two points, this settles the motion of every other point, so we shall speak only of two points. It is easy to prove that when we consider two positions of the body there is one point of the body whose position is the same. Thus, if w v is one position and w′ v′ is another position, bisect w w′ by a line at right angles to it, and let it meet the rectangular bisector of v v′ in c. Evidently c is a point in the body which has not altered its position; it is the instantaneous centre of the motion. Notice

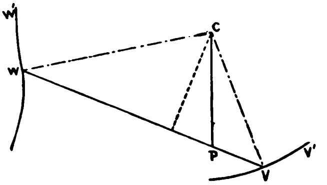

Fig. 329.

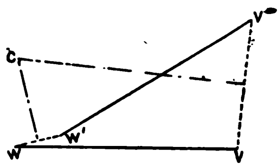

Fig. 330.

that if w w′ is parallel to v v′, c is at an infinite distance; we then say that the motion is one of translation merely. In Fig. 329 we have a piece, w v, whose ends move in the paths w′ w and v′ v. At any instant, if we draw w c and v c, normals to the paths, we find c the instantaneous centre.

Consider any point p in w v, and join p c. p traces out some path during the motion. Notice that p c is the normal to this path, and a line through p at right angles to p c is the tangent to it. If we want to consider the envelope of the straight line w v, notice that the foot of the perpendicular from c upon w v is a point in that envelope, because it is the only point of w v which moves in the direction of the length of w v. The student must not imagine that because at the instant every point of a body is moving about c, therefore c is the *centre of curvature* of the path of w, or of v, or of p. c is only for the instant the centre of motion, and such lines as w c, v c, and p c are the directions of the normals to the actual paths of these three points. As the figure moves in its own plane from one position to others successively, let $\mathrm{c}_1 \mathrm{c}_2 \mathrm{c}_3$, etc., be the successive points *of the figure* about which the rotations take place, and let $c_1 c_2 c_3$, etc., be the positions of these points *on the fixed plane* when each is the instantaneous centre of rotation. Then the figure rotates about c_1 (or c_1, which coincides with it) till c_2 coincides with c_2; then about c_2 till c_3 coincides with c_3, and so on. Hence, if we join $\mathrm{c}_1 \mathrm{c}_2 \mathrm{c}_3$, etc., in the plane of the figure, and $c_1 c_2 c_3$, etc., in the fixed plane, the motion will be the same as if the polygon $\mathrm{c}_1 \mathrm{c}_2 \mathrm{c}_3$, etc., rolled upon the fixed polygon $c_1 c_2 c_3$, etc. By supposing the successive displacements smaller and smaller, we have curves, and hence any motion whatever of a plane figure in its own plane may be imagined as produced by the rolling of a curve fixed to the figure upon a curve fixed to the plane. It immediately follows that any displacement of a rigid solid parallel to a plane may be produced by the rolling of a cylinder fixed in the solid on another cylinder fixed in space, the generating lines of the cylinders being at right angles to the plane.

466. What we have said about motion in a plane is true about motion of figures shaped to fit a spherical surface; straight lines in the one case corresponding to great circles of the sphere in the other case. It is easy to have a mental image of this. Now imagine that any point of the superficial spherical figure is joined to the centre of the sphere, and we see that (1) if a rigid body has one point fixed (imagine this the centre of an imaginary spherical surface), however it may move, in any two positions of the body there is one line of the body which is common to the two positions. (2) Any motion may be regarded as due to the **rolling of a cone** fixed in the body upon a cone fixed in space. What we have said, therefore, about the pieces A W and B V of Fig. 327 moving about axes at right angles to the plane of motion may be at once applied to pieces moving in a spherical surface about axes meeting in the centre of the sphere. Or the cylindric pieces A W and B V may be imagined to be conical pieces moving about axes A and B, meeting at the vertex of the cones. Hence all that we have said about spur wheels is at once applicable to bevil wheels, which are suitable for shafts whose centre lines or axes meet at a point.

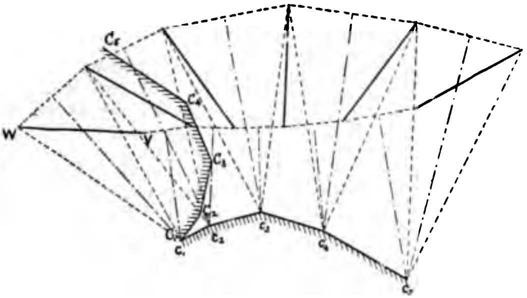

Fig. 331.

467. Fig. 329 is worth a very great deal of consideration. There is a link W V, and motion of a point W in it is known. The shape of the path of V is known, to find the motion of the whole link, and of any point in it, at every instant. When the paths of W and V are arcs of circles, we have the **four-bar kinematic chain** of Fig. 327. When W (a crank pin) moves in the arc of a circle, and V is a block moving in a straight slot, one particular case is called a **slider crank chain** (*see* Fig. 332). When both W and V are blocks moving in straight slots at right angles to one another, we have the ordinary **trammels** for describing an ellipse. We have the mathematical basis of the **elliptic chuck,** for we may imagine any part of a mechanism to be at rest, and then the relative motions of the others become absolute motions. The theory, then, of Art. 464 is the same for a very great many mechanisms.

468. In Fig. 327, if we imagine B A to be fixed, B in the position in which it is in the figure, W A and B A infinitely long, so that W's arc of a circle is really a straight line, we have the usual crank B V and connecting rod V W, which we can indicate in its simplest form by Fig. 332. Now take this as it stands. A piece B A X along which W slides, and the connecting-rod W V and the crank V B; think merely of the relative motion of each piece to the rest. You may imagine any one of the pieces fixed, and you may consider the motion of any point in any piece. If B X is fixed, V describes a circle, W a straight line, and any point in W V describes a curve which is somewhat like an ellipse, only blunter at one end than the other. Now imagine B V = V W: V will not make a rotation. Imagine W V extended beyond V as far again: that point will describe a straight line, and we have a parallel motion. A student must try these things with models. Now imagine B V fixed, and let V W turn uniformly. [Draw the fixed part of any curious shape, the frame of a machine, V and B here being two pins.] We get a **quick return** mechanism for shaping and other machines. Imagine V W fixed, and we have the mechanism of **oscillating cylinder** engines and of other machines as well. Now imagine W fixed, and guiding B X, so that it shall only move in the direction of its length, and we have a well-known **pump mechanism.** When we consider that, even from Fig. 332, with the motion of W in a line with B, we have obtained a number of different-looking mechanisms, and that these can be varied very curiously by taking various lengths of the parts, it will be seen that there is a nearly endless variety of forms derivable from the mechanism shown in Fig. 327. Now imagine a pin Y in the piece W V linked to Z, a pin in an arm Z B, and we have a much more complicated problem to study in the relative motion of six pieces. Any one of them may be imagined fixed, and they may be of all sorts of lengths.

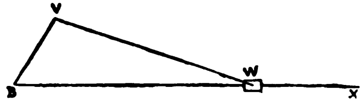

Fig. 332.

469. The following example is of geometrical interest. In the **Peaucellier cell** (Fig. 333), A B and A W of Fig. 327 are made equal in length, and so are B V and V W. Two pieces, W C and C B, equal to B V and V W, are added. It is easy to see that A C V is always a straight line. Also the positions of A, C, and V are such

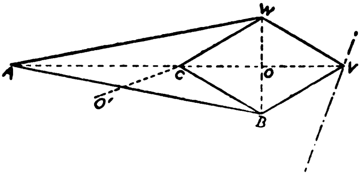

Fig. 333.

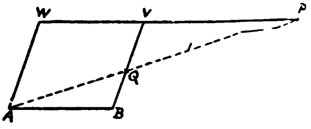

Fig. 334.

that A C . A V remains constant. For if A C V and W B be joined, and they meet in O, the centre of the rhombus, $AW^2 = AO^2 + OW^2$, and $VW^2 = VO^2 + OW^2$. Hence

$$AW^2 - VW^2 = AO^2 - VO^2 = (AO + VO)(AO - VO) = AV . AC.$$

Hence, if A is fixed, and C traces out any curve, V will trace out the reciprocal curve. If C is constrained by a link O′ C to move in a circle, and if O′ A = O′ C, V will describe the reciprocal curve, which in this case is a straight line. If O′ C is not equal to O′ A, V will describe the arc of a circle of much greater radius than C's.

470. When the four links of Fig. 327 form a parallelogram, as

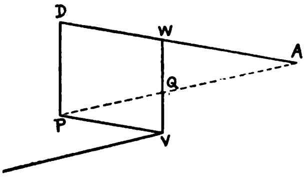

Fig. 335.

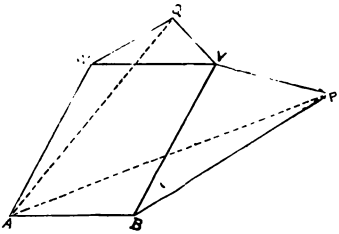

Fig. 336.

A W V B of Fig. 336, then, if P is a point anywhere rigidly part of the link W V, but in the line W V, and if at any time it is in the same straight line with A, and a point Q, which is rigidly a part of the link B V, and in the line B V, then P and Q will *always* be in a straight line with A. We can imagine now that either Q or P or A is a fixed point, and the other two will follow paths which are similar to one another.

Thus, for example, if A W V B (Fig. 335) is the four-bar mechanism of Fig. 327 (A B need not be drawn because we imagine it fixed), and if A W D is one piece, V P and P D being links equal to W D and W V respectively, then the pin P will travel in a path similar to that of Q if Q W : A W : : P D : A D. If Q travels very nearly in a straight line (if B V : A W : : Q W : Q V, we have a near approach to straight-line motion of Q, so long as B V and A W are nearly parallel to one another), then P will also travel very nearly in a straight line.

471. Again, if A W V B is a parallelogram, and if P is a point rigidly part of the link B V, and if Q is a point rigidly part of the link W V, and if P and Q are so placed that the ratios P V : V B : B P are the same as the ratios V Q : Q W : W V, then it can be shown that A Q and A P always make the same angles with one another, and that the ratio of the lengths A Q and A P keeps constant. In the triangles P V B and V Q W, the angles at P and V, at V and Q, and at B and W are respectively equal. Hence the triangles Q W A and A B P are similar, and it follows that the angle Q A P keeps constant in any motion of the mechanism, and the ratio of the lengths A Q and A P keeps constant. It follows, therefore, that if A is fixed and P is allowed to follow any curved path, Q will follow a similar path.

472. Note that Fig. 334, where P and Q are in the straight lines connecting W V and B V, is only a particular case of Fig. 336. Note also that if A W and W V are a pair of links guided at A and V along any paths, and if A B and B V are another pair of links whose A and V follow identical paths with the first pair's A and V, then for each point in the one mechanism there will be a corresponding point in the other which follows a similar path.

It is interesting to prove that if the links A W, W V, A B, B V, forming a parallelogram as in Fig. 336, be joined up as in Fig. 337, then any four points P Q R S in a line parallel to A V remain in a line parallel to A V, however the mechanism may alter in shape; and the distances P Q, Q R, R S are always such that P R . P Q is constant. Also P Q = R S.

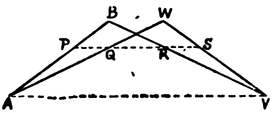

Fig. 337.

473. We study mechanisms, 1st, Because in designing new machinery we ought to have a general fairly exact knowledge of the sort of motion which each piece has in existing machines. For this we watch existing machinery, and work with rude models whose dimensions may be varied. 2nd, Because we wish to perfect an existing mechanism. For this we need a more exact study of the motion of every part, and we use skeleton drawings, algebraic and trigonometric analysis, or graphical methods such as I have described. 3rd, Because in these days of **increasing speed of machinery,** the forces necessary to produce the accelerations of all the parts have become important, both for the strength and stiffness of all the parts, and also for the effects of vibration.

474. In Fig. 329 we saw that, knowing the directions of motion of W and V, we can find the instantaneous centre. Now let W V get any motion whatsoever in the plane of the paper. Let the velocities of W and V be *w* and *v* in the directions marked. If we draw perpendiculars to *w* and *v*, meeting at C, the angular velocity of W about

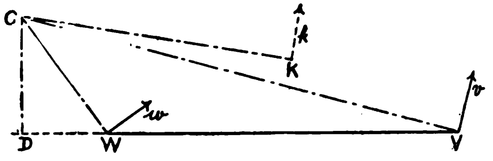

Fig. 338.

C must be the same as that of V, and this must be the same as that of any other point K, rigidly attached to W, about C. Hence, join any point K to C, the velocity of K is at right angles to C K, and is represented in amount by the length of C K. Hence, as soon as we find the instantaneous centre, we have a diagram of velocities of all points in the body. Notice that if we have the instantaneous centre C and the velocity of any one point, we have the whole diagram to scale. If C D is a perpendicular to V W, as the length of every line now in the diagram represents the amount of a velocity which is at right angles to its direction, C D is the component or common velocity of all points in the straight line W V in the direction W V. D W is the velocity at right angles to the direction W V of the point W, and D V is the same for V, hence W V represents V's velocity in excess of W's in this direction. Hence, if W V, V X, X Y are given links, and if we know the directions of the velocities of each joint at the instant, say the directions of the dotted lines, we can find the velocity of every point in any body which is rigidly a part of any of the links if we know the actual speed of any one point.

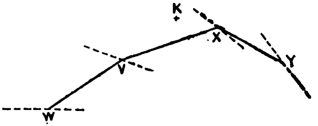

Fig. 339.

Choose a pole O. Draw lines from O at right angles to the velocities of all the joints—O *w*, O *v*, etc. Let the distance along any one of them represent the velocity of that joint to scale, and now draw the lines *wv*, *vx*, *xy* parallel to the real links. Then the lengths O *w*, O *v*, O *x*, O *y* represent the velocities of the joints. To prove this, drop a perpendicular O D from O upon any of the link directions in our diagram; we choose *vw*. If O *w* is the amount to scale and is at right angles to the velocity of W, then O D represents the velocity of W in the direction of the link W V; but this ought to be the same for V. We have, then, the amount O D of a component of V's velocity, and we have the direction of its whole velocity, and our construction is the very one which we should adopt to obtain V's velocity. If W and Y are fixed, the diagram *wvxy* is a closed polygon, in the present case a triangle.

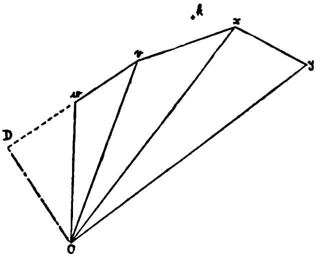

Fig. 340.

Notice that in any case, if from O we drop a perpendicular O D on a side of the diagram, say *wv*, then D *w* and D *v* represent the velocities of W and V at right angles to the length of D *w*. Hence the difference *wv*, divided by the actual length of W V, represents the angular velocity of W V. Thus, in Fig. 343, take A W V B as four links, A and B having no velocities. From the point O, which we may also call *a* or *b*, draw lines parallel to A W and B V, because we know that these are at right angles to the velocities of W and V, and draw *vw* parallel to V W. Then *bv* ÷ B V, *aw* ÷ A W, and *vw* ÷ V W represent the angular velocities of the three links.

475. In Fig. 339, if K is a point rigidly attached to the link V X (in the diagram find *k* so that *vkx* is a triangle similar to V K X), then o *k* is at right angles to and represents to scale the velocity of K. Professor R. H. Smith, who was, I believe, the inventor of this

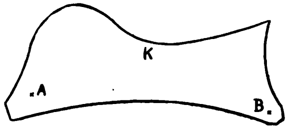

Fig. 341.

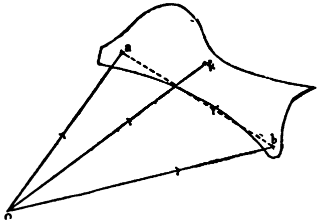

Fig. 342.

method, takes his radiating lines parallel to, instead of at right angles to, the actual velocities (Trans. R.S.E., Jan., 1885). He gives a similar construction for accelerations.

If a point B is fixed and a body rotates in the plane of the paper round it, the accelerations of any points A and K are in the directions A B and K B, and they are proportional to the distances A B and K B. Let A K B be a rigid body with a motion in the plane of the paper. Let the actual amounts of the accelerations of A and B be known; represent these in amount and direction by the lines o *a* and o *b*. Now make the figure *akb* exactly similar in shape to the real body A K B. The true acceleration of any point K is represented in amount and direction by o *k*.

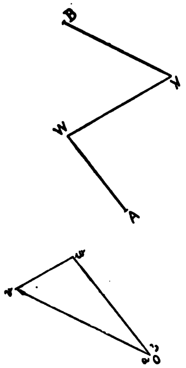

Fig. 343.

To prove this: The acceleration of A is the vector sum, acceleration of B + acceleration of A relatively to B. Hence *ba* represents this acceleration of A relatively to B. But the motion of A and the whole body relatively to B is a rotation, and hence *bk* represents the acceleration of K relatively to B to the same scale. The vector sum o *b* + *bk* = o *k* represents therefore K's acceleration.

Notice that the true acceleration of any point A is the vector sum of two accelerations—the first in the direction of motion, the second the centripetal acceleration if the path of A is curved.

476. Let a known force in any direction be applied to A, a point in the body A K B which has a known motion in the plane of the paper.

1st. Imagine the body divided into small equal masses. The construction of Fig. 342 enables us to find the forces with which these masses resist their accelerations. They may be combined with the weights of the parts to get what may be called the load diagram of the body.

Let it be known that there is another force of unknown amount, but known in direction, acting at A (for example, the guiding force, with or without friction, at the crosshead of a steam engine). It is easy to see that, by Art. 475, we can find the amount of this guiding force at A, and also the total force which must be acting at a given point B.

2nd. In Art. 475 we have a construction which enables us to find the stress at any point of any section of the structure AKB, if it may be imagined to be any quasi-prismatic structure like a beam or connecting-rod, or even if it be shaped like part of a metal arch. In such a case as that of **the connecting-rod,** the steam-engine maker ought to study the motion from many points of view, OV and VW being the crank and connecting-rod in any position. Produce WV to meet OB in B. OB is at right angles to the line of centres, OW. Then the lines OV and OB are at right angles to the velocities of V and W, and VB is in the direction of the link WV; consequently, OVB is (Art. 475) **a diagram of velocities.** OV representing to scale the constant velocity of V, OB represents to the same scale the velocity of W; also the distance VB represents the angular velocity of the connecting-rod. It follows from this at once that a force F at W in the direction WO would, if there were no friction and the connecting-rod were massless, produce a turning moment F × OB on the crank shaft.

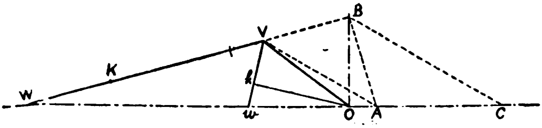

Fig. 344.

It can be shown that if we draw BA at right angles to WB, join VA, draw BC parallel to VA, let *w* be in WO as far from A as C is, but on the opposite side of A, and join *w*V, then O*w*V is an acceleration diagram such that if K is any point in the connecting-rod, and we let the points *w*, *k*, and V be relatively to each other as W, K, and V, then *k*O represents the acceleration of K in direction and magnitude to the same scale to which VO represents the centripetal acceleration of V. In fact, O*wk*V is a **diagram of accelerations.**

477. When one point of a body is fixed, and we may neglect centripetal accelerations, we may take up the problem as a particular case of the general problem already considered, or we may attack it analytically as follows:—If OP is the straight centre line of an arm, O being fixed, and if α is its angular acceleration; neglecting forces parallel to OP, let there be a force F acting at P; the acceleration at any point Q (neglecting radial acceleration) is $x\alpha$, if OQ is x. If m is the mass per unit length, the load due to acceleration per unit length is $m a x$; and if M is the bending moment at the cross-section (see Art. 357),

$$\frac{d^2 M}{dx^2} = m a x \; \ldots . (1).$$

Assume m to be some function of x, and let S be the shearing force at a section. Let $\int mx \, . \, dx = v$, $\int v \, . \, dx = u$, and let the values of v and u become v_1 and u_1, when $x = l$. Then

$$S = \frac{d M}{dx} = av + c \; \ldots . (2),$$

$$M = au + cx + e \; \ldots . (3),$$

$$F = av_1 + c \; .\because . \; c = F - a V_1,$$

$$D = au_1 + (F - av_1) \, l + e \; .\because . \; e = a \, (lv_1 - u_1) - F e.$$

If A is the area of cross-section and ρ is the mass per unit volume, $A \rho = m$. If we wish to have the **same maximum stress** f in every cross-section, and if Z is the strength modulus of the section, then $M = Z f$,

$$\frac{M}{m} = \frac{Z \, f}{A \, \rho} \; \ldots . (4).$$

Dividing, therefore, (3) by m, we know the value of Z/A everywhere, if the arm is to be of uniform strength. Thus, for example, let the section be rectangular, of breadth z and depth (in the plane of motion) y. Then $A = zy$, $Z = \frac{1}{6} zy^2$; so that Z/A is $\frac{1}{6} y$. Hence from (4) we have $\frac{fy}{6\rho} = \frac{M}{m}$ or $y = \frac{6 \rho}{f} \frac{M}{m}$, or

$$y = \frac{6 \rho}{fm} (au + cx + e) \; \ldots . (5).$$

Since $\rho zy = m$, $z = \frac{m}{\rho y}$, so that when we know y we know z;

$$z = \frac{f}{6 \rho^2} \frac{m^2}{M} \; \ldots . (6).$$

Example.—Let

$$\begin{aligned} m &= a - bx, \\ v &= \tfrac{1}{2} ax^2 - \tfrac{1}{3} bx^3, \quad v_1 = \tfrac{1}{2} al^2 - \tfrac{1}{3} bi^3, \\ u &= \tfrac{1}{6} ax^3 - \tfrac{1}{12} bx^4, \quad u_1 = \tfrac{1}{6} al^3 - \tfrac{1}{12} bl^4, \\ c &= F - al^2 \left(\tfrac{1}{2} a - \tfrac{1}{3} bl\right), \\ e &= al^3 \left(\tfrac{1}{3} a - \tfrac{1}{4} bl\right) - F \, l, \\ y &= \frac{6 \rho}{f \, (a - bx)} (au + cx + e). \end{aligned}$$

478. In discussing the **reciprocating motion** of a point, I have found in practice much simplification in my ideas when I have reduced the motion to some such shape as

$$x = a \sin. (qt + e_1) + b \sin. (2 qt + e_2) + \text{etc.}$$

where x is the displacement of the point from some fixed point in its path, and q is 2π multiplied by the frequency, or 2π divided by the periodic time. For many purposes we find the first term sufficient. In most **valve motions,** such as **link motions** and **radial**

gear, I find that only two terms are ever needed even for a very exact definition of the motion. One interest attaching to this method of working is in its showing how the *b* term becomes twice as important in the velocity, and four times as important in the acceleration as the fundamental term. Another important matter is this: in modern machinery vibration is becoming very important, and the above description of a motion is the one that lends itself most easily to a discussion of the vibrations which want of balance gives rise to. (See the author's books on the Calculus and on the Steam Engine.)

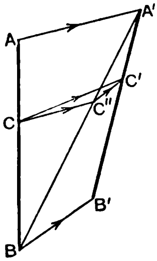

Fig. 345.

479. The following proposition is the foundation of most calculations on **motion communicated through links.** The points A C B are in a straight line. They move, keeping in a straight line, and at the same distances from one another. (In engineer's language A C B is a moving link, and C is a point in it.) Prove that the motion of C is the vector sum of the fraction $\frac{AC}{AB}$ of B's motion, and the fraction $\frac{BC}{AB}$ of A's motion. For let A A', B B', C C' be any displacements of A, B, and C. Join A' B. Draw C C'' parallel to A A', and join C'' C'. Now $\frac{AC}{AB} = \frac{A'C''}{A'B} = \frac{A'C'}{A'B'}$, so that C'' C' is parallel to B B'. Hence, as C C' is the vector sum of C C'' and C'' C', we have proved the proposition.

It is easy to extend our reasoning to motion which is not parallel to one plane. The motions of any three points A B C of the body, not in one line, define the whole motion; and when we are given the accelerations of A, B, and C, it is easy for anyone who knows descriptive geometry to make a diagram showing the acceleration of any point of the body.

480. Newton's great one law of motion for any, however complex, system of bodies is this: Look upon the rate of change of momentum of any small portion of matter of a system as a force in the opposite direction. **All such forces are in equilibrium with the forces which act on the system from the outside.** In most cases it is this most general way of stating the law that is most useful to engineers. Given their diagram of accelerations, they really have a force or load diagram, and the problems to be dealt with are now merely worked out by graphical statics.

481. Most men are led to their study of this subject through what is called **D'Alembert's principle.** This is a principle which served a very useful purpose. At a time when English mathematicians were stagnating, being academically learned as to Newton's methods, but being really ignorant of them, the French mathematicians were developing kinetics practically independently of Newton's methods. Newton's third law was quite misunderstood, and D'Alembert discovered a principle which gave the power of solving dynamical problems with certainty. It will not astonish anyone who knows academic methods to be informed that, although D'Alembert gave what is now seen to be merely a **rather cumbrous explanation** (easily misunderstood, because of certain technical terms which may be confounded with one another) of Newton's third law, it is through the clumsy explanation that the subject is nearly always approached. We believe that this is one of the greatest reasons why many engineers are so disgusted with higher studies in dynamics. There seems to be as much absence of **common sense now in academic persons** as there was in the time of Erasmus. He, the greatest scholar of the fifteenth century, wrote: "They are a proud, susceptible race. They will smother me under six hundred dogmas. They will call me heretic, and bring thunderbolts out of their arsenals, where they keep whole magazines of them for their enemies. Still, they are Folly's servants, though they disown their mistress. They live in the third heaven, adoring their own persons, and disdaining the poor crawlers upon earth. They are surrounded with a bodyguard of definitions, conclusions, corollaries, propositions explicit, and propositions implicit."

482. Let the engineer take Newton's law in its very simplest form, as above expressed, and he will have no difficulty in attacking the most complicated problems, for the dynamical becomes a static problem on the equilibrium of forces. It is usual to express part of the result analytically in the following way:—If in any direction, which we may call x, the small mass m has the acceleration $\ddot{x}$, then the resultant force acting from outside the system in that direction is equal to the sum of all such terms as $m\ddot{x}$. State this as being true in any three directions, and of course it is true in any direction whatsoever.

Now if the x of the centre of gravity of the whole system is $\bar{x}$, we know that $\bar{x}\Sigma m = \Sigma mx$. Differentiate with regard to time once, and again, and we see that the whole mass Σm, multiplied by the acceleration of the centre of gravity, is equal to the sum of all the masses multiplied by their accelerations in the direction x. It follows, therefore, that **the motion of the centre of gravity of a system is the same as if the mass of the system were collected there, and all the forces acting from outside on the system acted there.** The other part of Newton's law is, of course: **The resultant moment of all the outside forces about any axis is equal to the rate of change of moment of momentum of the whole system about the same axis.** We very often choose as our axis an axis through the centre of gravity, but it is well to notice that this is not necessary.

483. Angular Motion.—We know (Art. 92) that when a **rigid body** can only rotate about an axis, if the sum of the moments of the forces acting on the body is M, when the body moves through the angle $\delta\theta$ radians the work done is M . $\delta\theta$. If any little portion m of the mass of the body is at the distance r from the axis, and the body is rotating with the angular velocity a, so that ar is the velocity at the place, the energy stored up in the little mass is $\frac{1}{2}ma^2r^2$. If this is summed up for the whole body, we see that every little term contains $\frac{1}{2}a^2$; so that if we know the sum of all such terms as mr^2 (called I, the moment of inertia of the body about the axis), we can say that the kinetic energy is $\frac{1}{2}\mathrm{I}a^2$. When a force F acts on a body through the distance δs, the work done being $\mathrm{F} \,.\, \delta s$, and the kinetic energy of the body being $\frac{1}{2}mv^2$, by assuming that work done is equal to gain of kinetic energy, we are led to the law $\mathrm{F} = m \times$ acceleration. We have an exactly analogous set of terms for angular motion. The work $\mathrm{M} \,.\, \delta\theta$ corresponds with $\mathrm{F} \,.\, \delta s$; the energy $\frac{1}{2}\mathrm{I}a^2$ corresponds with $\frac{1}{2}mv^2$. The proof, therefore, is exactly as in Art. 497—namely, if a moment M acts, through the angle $\delta\theta$, on a body moving with the angular velocity a, and whose moment of inertia is I, so that its kinetic energy is $\frac{1}{2}\mathrm{I}a^2$, and if $a + \delta a$ is its new angular velocity, then

$$\mathrm{M} \,.\, \delta\theta = \tfrac{1}{2}\mathrm{I}\,(a + \delta a)^2 - \tfrac{1}{2}\mathrm{I}a^2 = \mathrm{I} \,.\, a\delta a + \tfrac{1}{2}\mathrm{I} \,.\, (\delta a)^2,$$

$$\mathrm{M} = \mathrm{I}\,a\frac{\delta a}{\delta\theta} + \tfrac{1}{2}\mathrm{I}\frac{\delta a}{\delta\theta} \,.\, \delta a.$$

As $\delta\theta$, and therefore δa, become smaller and smaller, we have

$$\mathrm{M} = \mathrm{I}\,a \,.\, \frac{\delta a}{\delta\theta} = \mathrm{I}\frac{\delta\theta}{\delta t} \,.\, \frac{\delta a}{\delta\theta},$$

or

$$\mathrm{M} = \mathrm{I}\frac{da}{dt} = \mathrm{I}\,\alpha$$

if α is used for $\frac{da}{dt}$, or the angular acceleration. This is exactly analogous with the force law in linear motion. $\mathrm{I}\,a$ is called the **moment of momentum** of the body, and $\mathrm{I}\,\alpha$ is the rate of change of the moment of momentum.

484. To arrive directly at the **moment of momentum of a rigid body about any axis**, consider a portion of mass m at the distance r from the axis; r making an angle θ with a fixed plane through the axis,

$$x = r \cos.\ \theta,\ y = r \sin.\ \theta,$$

$$\frac{dx}{dt} = -\,r \sin.\ \theta\frac{d\theta}{dt},\ \frac{dy}{dt} = r \cos.\ \theta \,.\, \frac{d\theta}{dt} \quad \ldots\ldots (1).$$

The moment of the momentum $m\frac{dx}{dt}$ about the axis is $-\,my \,.\, \frac{dx}{dt}$ in the direction of increasing θ, and the moment of momentum of $m\frac{dy}{dt}$ is $mx \,.\, \frac{dy}{dt}$, and the sum of these is

$$m\left(r^2 \sin.^2\theta\frac{d\theta}{dt} + r^2 \cos.^2\theta\frac{d\theta}{dt}\right), \text{ or } mr^2 \,.\, \frac{d\theta}{dt},$$

and the sum of all such terms is $\mathrm{I}\frac{d\theta}{dt}$, or $\mathrm{I}\,a$. The rate of change of this is $\mathrm{I}\frac{d^2\theta}{dt^2}$, or $\mathrm{I}\,\alpha$. This also may be obtained by differentiating

(1), and so finding $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$. Taking the moments of the virtual forces $-m\frac{d^2x}{dt^2}$ and $-m\frac{d^2y}{dt^2}$ about the axis, we arrive at $-\mathrm{I}\frac{d^2\theta}{dt^2}$, or $-\mathrm{I}\alpha$, which, together with the moments due to the outside forces, produce equilibrium; or, with signs changed, are equal to them, as above. Students ought to think of other ways of reaching these results concerning rigid bodies.

485. Kinetic Energy of any System.—Let m be the mass of a portion of the system in the position x, y, z. Let $\bar{x}, \bar{y}, \bar{z}$ at this instant be the position of the centre of gravity. The whole kinetic energy is the sum of such terms as

$$m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}.$$

Let $x = \bar{x} + x^1$, $y = \bar{y} + y^1$, $z = \bar{z} + z^1$; so that x^1, y^1, z^1 show the position of *m relatively to the centre of gravity*. Note that

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{d\bar{x}}{dt} + \frac{dx^1}{dt}\right)^2 = \left(\frac{d\bar{x}}{dt}\right)^2 + 2\frac{d\bar{x}}{dt}\cdot\frac{dx^1}{dt} + \left(\frac{dx^1}{dt}\right)^2.$$

But $\Sigma\, m \,.\, 2\frac{d\bar{x}}{dt} \,.\, \frac{dx^1}{dt} = 2\frac{d\bar{x}}{dt}\,\Sigma\, m \,.\, \frac{dx^1}{dt}$, and $\Sigma\, m\frac{dx^1}{dt}$ is 0 because it is the rate of change of $\Sigma\, m\, x^1$ and $\Sigma\, m\, x^1 = 0$. Hence

$$\Sigma\, m\left(\frac{dx}{dt}\right)^2 = \Sigma\, m\left(\frac{d\bar{x}}{dt}\right)^2 + \Sigma\, m\left(\frac{dx^1}{dt}\right)^2.$$

The first of these terms is the whole mass multiplied by the square of the x component of the velocity of the centre of gravity. It follows, therefore, that **the kinetic energy of any system** is equal to the kinetic energy which the system would have if it were all moving with the velocity of the centre of gravity + the kinetic energy due to the motions of all the parts relatively to the centre of gravity.

Now, the motion of a rigid body relatively to any point of it can only be a rotation. Hence for **rigid bodies we have the rule**: If the velocity of the centre of gravity is v, and if there is a rotation of angular velocity $\dot{\theta}$ about some axis, and if I_0 is the moment of inertia of the body about a parallel axis through the centre of gravity, and M is the whole mass, the kinetic energy E is $\frac{1}{2}\mathrm{M}v^2 + \frac{1}{2}\mathrm{I}_0(\dot{\theta})^2$. If $\mathrm{I}_0 = \mathrm{M}k^2$, then $\mathrm{E} = \frac{1}{2}\mathrm{M}\left\{v^2 + k^2(\dot{\theta})^2\right\}$. It is only in the case of a rigid body (in all this by rigid body I mean an infinitely rigid body) that we have this simple rule. In the case also of a rigid body we can find law (Art. 482) from the law of energy, as we do in Art. 495; but we cannot do this so easily for a **system having internal relative motion**, because there is internal potential energy, which may alter. The *virial* and other laws, which may easily be arrived at, do not concern engineering applications of mechanics.

486. In any ordinary elastic body the internal motions are vibrational. The momentum and moment of momentum of the system are therefore practically the same as if the body were quite rigid. This is not the same in regard to the energy. We say that it disappears or is changed to heat and other forms; this means merely that it has become molecular, and equations which regard the body as if it were rigid are quite inapplicable. Thus it is that the momentum equations are what we rely upon, because when we study the motions of bodies momentum cannot be hidden as energy may be.

487. When we work exercises on rigid dynamics we ought to apply Newton's law always in the shape given in Art. 480. When we work a new exercise let it be regarded as a new illustration, to be worked solely for the sake of making our knowledge of the fundamental principle clearer. Most details of one's theory will disappear from view in one's practical work. Let the fundamental notion be so well fixed that it cannot disappear.

488. *Exercise.*—In the compound pendulum of Fig. 346 find the force acting at S, the point of suspension. In Fig. 347 let G be the centre of gravity.

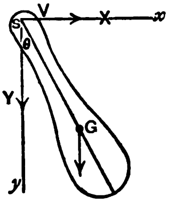

Fig. 346.

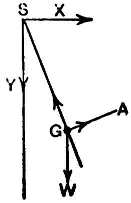

Fig. 347.

Let X and Y, as shown, be the horizontal and vertical components of the force at S. Let G S y be θ, and let the body be moving so that θ increases. If S G be called r, the velocity of G in its path is $r\dot{\theta}$. It has an acceleration $r\ddot{\theta}$ in its path, and $r(\dot{\theta})^2$, a centripetal acceleration in the direction G S. I. Regard now the whole system of forces, which are supposed to balance if they were to act at one point, G. Let M be the mass or W/g, we have $-\,Mr\ddot{\theta}$ in the direction G A, $-\,Mr(\dot{\theta})^2$ centripetal in the direction G S, W downwards (the weight of the body), X horizontally, and Y downwards. Resolve these in any direction whatsoever, they are to balance. Thus, resolve them horizontally and vertically,

$$X - Mr\ddot{\theta}\cos.\theta + Mr(\dot{\theta})^2\sin.\theta = 0 \;\ldots\; (1),$$

$$Y + W + Mr\ddot{\theta}\sin.\theta + Mr(\dot{\theta})^2\cos.\theta = 0 \;\ldots\; (2).$$

Observe that the forces in G A and G S are virtual forces, or forces qual and opposite to mass multiplied by acceleration. II. Now take noments about any axis. The most convenient seems to be S. W is the nly externally applied force that has any moment about S, and so

$$Wr\sin.\theta + I\ddot{\theta} = 0 \;\ldots\; (3),$$

if I is the moment of inertia about S. From these equations X and Y, in terms of θ and $\dot{\theta}$, may be calculated. We first express $\ddot{\theta}$ in terms of θ from (3). If $\text{I} = \frac{\text{W}}{g} k^2$, (3) gives

$$\ddot{\theta} = -\frac{gr}{k^2}\sin.\theta,$$

$$\text{X} = \frac{\text{W}}{g} r \cos.\theta\left(-\frac{gr}{k^2}\sin.\theta\right) - \frac{\text{W}}{g} r \sin.\theta \,.\, (\dot{\theta})^2,$$

$$\text{Y} = -\text{W} - \frac{\text{W}}{g} r \sin.\theta\left(-\frac{gr}{k^2}\sin.\theta\right) - \frac{\text{W}}{g} r (\dot{\theta})^2 \cos.\theta,$$

$$\frac{\text{X}}{\text{W}} = -\frac{r^2}{2k^2}\sin.\,2\theta - \frac{r}{g}\sin.\theta(\dot{\theta})^2,$$

$$\frac{\text{Y}}{\text{W}} = -1 + \frac{r^2\sin.\,2\theta}{k^2} - \frac{r}{g}\cos.\theta\,(\dot{\theta})^2.$$

$\ddot{\theta}$ will depend on the limit of the swing, and may have any value.

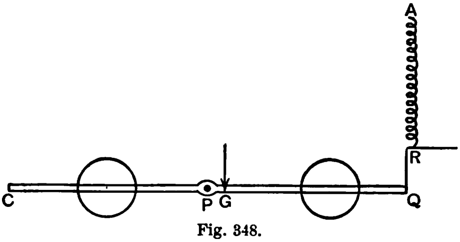

Fig. 348.

489. Vibration Indicator.—Fig. 348 shows an instrument which has been used for indicating quick vertical vibration of the ground. The mass CPQ is supported at P by a knife-edge, or by friction-wheels. The centre of gravity G is in a horizontal line with P and Q. Let $\text{PG} = a$, $\text{GQ} = b$, $\text{PQ} = a + b = l$. The vertical spring AR and thread RQ support the body at Q. As a matter of fact, AR is an Ayrton-Perry spring, which shows by the rotation of the pointer R the relative motion of A and Q. Let us neglect its inertia now, and consider that the pointer faithfully records relative motion of A and Q. It would shorten the work to only consider the forces at P and Q in *excess* of what they are when in equilibrium; but, for clearness, we shall take the total forces.

When a body gets motion in any direction parallel to the plane of the paper, we get one equation by stating that **the resultant force is equal (numerically) to the mass multiplied by the linear acceleration of the centre of gravity in the direction of the resultant force.** We get another equation by stating that **the resultant moment of force about an axis at right angles to the paper through the centre of gravity is equal to the angular**

acceleration, multiplied by moment of inertia about this axis through the centre of gravity. I shall use x, $\dot{x}$, and $\ddot{x}$ to mean displacement, velocity, and acceleration, or x, $\frac{dx}{dt}$, and $\frac{d^2x}{dt^2}$.

Let P and A have a displacement x_1 downward. Let Q be displaced x downward. Let the pull in the spring be $Q = Q_0 + c(x - x_1)$ where c is a known constant (c is the reciprocal of the h used in Art. 514). Let W be the weight of the body. Then if P_0 and Q_0 be the upward forces at the points marked P and Q, in the position of equilibrium,

$$Q_0 (a + b) = W a \text{ and } P_0 + Q_0 = W.$$

Hence

$$P_0 = \frac{b W}{a + b},\ Q_0 = \frac{a W}{a + b} \ldots . (1),$$

$$Q = Q_0 + c(x - x_1).$$

Now G is displaced downwards $\frac{b}{a + b} x_1 + \frac{a}{a + b} x$, so that

$$W - P - Q = \frac{W}{g} \left\{ b\ddot{x}_1 + a\ddot{x} \right\} \frac{1}{a + b} \ldots . (2).$$

The body has an angular displacement θ clockwise about its centre of mass, of the amount $\frac{x - x_1}{a + b}$. So that if I is its moment of inertia about G,

$$- Q b + P a = \frac{I}{a + b} (\ddot{x} - \ddot{x}_1) \ldots . (3).$$

Hence (2) and (3) give us, if M stands for $\frac{W}{g}$, and if $I = M k^2$ where k is the radius of gyration about G,

$$\frac{\ddot{x}}{a + b}\left(\frac{I}{a} + a M\right) + xc\left(\frac{b}{a} + 1\right) = \frac{\ddot{x}_1}{a + b}\left(\frac{I}{a} - b M\right) + x_1 c\left(\frac{b}{a} + 1\right) \ldots . (4).$$

If k_1 is the radius of gyration about P, we find that (4) simplifies to

$$\ddot{x} + n^2 x = e^2 \ddot{x}_1 + n^2 x_1 \ldots . (5)$$

if n stands for $\frac{l}{k_1}\sqrt{\frac{c}{M}} = 2\pi \times$ natural frequency and e^2 stands for $1 - \frac{al}{k_1^2}$. Call $x - x_1$ by the letter y, because it is really y that an observer will note, if the framework and room and observer have the motion x_1. Then, as $y = x - x_1$, or $x = y + x_1$,

$$\ddot{y} + \ddot{x}_1 + n^2(y + x_1) = e^2\ddot{x}_1 + n^2x_1.$$

So that

$$\ddot{y} + n^2y = (e^2 - 1)\,\ddot{x}_1 \;.\;.\;.\;.\; (6),$$

or

$$\ddot{y} + n^2y + \frac{al}{k_1^2}\ddot{x}_1 = 0 \;.\;.\;.\;.\; (7).$$

Thus let $x_1 = \text{A} \sin. qt$.

We are neglecting friction for ease in understanding our results, and yet we are assuming that there is enough friction to destroy the natural vibration of the body. We find that if we assume $y = a \sin. qt$, then

$$a = \frac{al}{k_1^2}\,\frac{q^2}{n^2 - q^2}\,\text{A}.$$

That is, the apparent motion y (and this is what the pointer of an Ayrton-Perry spring will show; or a light mirror may be used to throw a spot of light upon a screen) is $\frac{al}{k_1^2}\,\frac{q^2}{n^2 - q^2}$ times the actual motion of the framework and room and observer. If q is large compared with n—for example, if q is always more than five times n—we may take it that the apparent motion is $\frac{al}{k_1^2}$ times the real motion, and is independent of frequency. **Hence any periodic motion whatever** (whose periodic time is less than $\frac{1}{5}$th of the periodic time of the apparatus) **will be faithfully indicated.**

Note that if $al = k_1^2$, so that Q is what is called the point of percussion, Q is a motionless or "steady" point. But in practice the instrument is very much like what is shown in the figure, and Q is by no means a steady point. Apparatus of the same kind may be used for east and west, and also for north and south motions.

490. We did not think it necessary to interrupt our account of simple harmonic motion to speak about **the analogies between linear and angular motion**, although we began to use them in Art. 453. We see (Art. 482) that any motion whatsoever of a rigid body may be most simply studied as a **motion of translation** and a **motion of rotation**, and we shall find it best to take the translational motion of the whole body as the actual motion of its centre of gravity, and the rotation as one about an axis passing through the centre of gravity. When we do this, we shall find the following angular formulæ valuable, not merely for the motion of a body whose axis is fixed, but for any motion whatsoever. The only point in which the analogy between linear and angular motion fails is this: The mass or inertia of a body is independent of the direction of motion; the moment of inertia of a body is usually different about different axes of rotation.

COMPARISON OF LINEAR AND ANGULAR MOTIONS.

s = space, t = time.

v = velocity = $\frac{ds}{dt}$, or $\dot{s}$.

α = acceleration = $\frac{d^2s}{dt^2}$, or $\ddot{s}$, or $\dot{v}$.

m = mass or inertia = w/g.

F = force.

mv = momentum.

If a body has two displacements (or velocities, or accelerations, or momenta) given to it simultaneously, represented in magnitude and direction and sense by O P and O Q, they are equivalent to the displacement (or velocity, or acceleration, or momentum) O R if O P and O Q are the sides, and O R the diagonal of a parallelogram, the arrows being all difluent or confluent.

Forces represented by O P and O Q have a resultant O R.

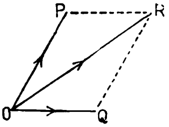

Fig. 349.

$\alpha = F \div m$.

Impulse = change of momentum $= \int F \,.\, dt$.

Average force = change of momentum ÷ time.

Work done = force × distance.

Space average of force = work done ÷ distance.

Kinetic energy, $\frac{1}{2} mv^2$.

Under uniform acceleration from rest at time 0, $v = \alpha t$, $s = \frac{1}{2} \alpha t^2$, $v^2 = 2 \alpha t$.

θ = angular space.

a = angular velocity = $\frac{d\theta}{dt}$, or $\dot{\theta}$.

α = angular acceleration = $\frac{d^2\theta}{dt^2}$, or $\ddot{\theta}$, or $\dot{a}$.

I = moment of inertia.

M = moment of forces or torque.

$I a$ = angular momentum, usually called moment of momentum.

If a body has two angular displacements (or velocities, or accelerations, or momenta [moments of momentum]) about *the axes* O P and O Q, and if the lengths of the lines O P and O Q represent these to scale, and if the arrows indicate positive direction, as that in which a right-handed screw moves, they are equivalent to the angular displacement (or velocity, or acceleration, or moment of momentum) represented in axis, amount and sense, by O R.

Moments of force or torque represented by O P and O Q (the direction of the line being the axis about which the moments of forces are taken, and the sense of the arrow-heads, as in the last case) have a resultant O R.

$\alpha = M \div I$.

Angular impulse = change of moment of momentum $= \int M \,.\, dt$.

Average torque = change of angular momentum ÷ time.

Work done = torque × angle.

Angular average of torque = work done ÷ angle.

Kinetic energy, $\frac{1}{2} I a^2$.

$a = \alpha t$, $\theta = \frac{1}{2} \alpha t^2$, $a^2 = 2 \alpha t$.

Simple harmonic motion if T is the periodic time, A is the amplitude, e is the lead

$$s \text{ or } \theta = \mathrm{A} \sin. \left(\frac{2\pi}{T} t + e\right),$$

$$v \text{ or } a = \mathrm{A} \frac{2\pi}{T} \cos. \left(\frac{2\pi}{T} t + e\right),$$

$$a = -\mathrm{A} \left(\frac{2\pi}{T}\right)^2 \sin. \left(\frac{2\pi}{T} t + e\right),$$

$$-\frac{s \text{ or } \theta}{a} = \frac{T^2}{4\pi^2}$$

A body moves backwards and forwards under the action of a variable force, which is always proportional to the distance of the body from its middle position, and which always acts towards this position; and if the force at a distance of 1 foot is 5 lbs., then the time of vibration is π times the square root of the quotient of the mass of the body divided by 5.

If a body vibrates about a fixed axis under the action of the torque (say from a spiral spring or twisted wire), so that it is always proportional to θ, the angular displacement of the body from its mean position, and if the torque is 5 pound-feet when the body is 1 radian from the mean position, then the time of a vibration is π times the square root of the quotient of the moment of inertia divided by 5.

If m is the mass; if the force of friction is b times the velocity; if the constraining force is N times distance from centre,

If I is moment of inertia, frictional torque is b times the angular velocity. If the constraining torque is N times the angular distance from the mean position,

$$s \text{ or } \theta = \mathrm{A}\,\epsilon^{-ft} \sin. (at + e)$$

where $f = \frac{1}{2} b/m$, or $\frac{1}{2} b/\mathrm{I}$, and

$$a = \frac{1}{2\mathrm{M}} \sqrt{4\mathrm{N}^2 - b^2} = 2\frac{1}{\mathrm{I}} \sqrt{4\mathrm{N}^2 - b^2}.$$

A body of mass m moves with constant speed v. At any instant its momentum is in the direction O E, and is represented to scale by the amount O E. If a centripetal force of constant amount is always acting at right angles to the direction E, the effect is to make the *point* E travel with *linear* velocity equal to F. That is, the direction of motion alters with an angular velocity F/mv.

A body whose moment of momentum is represented to scale, and the direction of its axis by O E, is acted on by a constant torque M, which is always about an axis at right angles to O E. The effect is to make the *point* E travel with linear velocity equal to M; that is, the axis O E has an angular velocity (towards the axis of the torque) whose amount is M ÷ moment of momentum Ia.

EXERCISES.

1. A top (Fig. 351) rotates at r radians per second about its axis O C. The axis rotates (or processes) at Ω radians per second in a conical path round the vertical O Z. Represent this motion by one cone rolling on another.

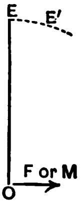

Fig. 350.

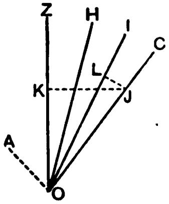

Fig. 351.

Answer.—If O C is the axis of the rolling cone of angular velocity r of which O I is the slant side, if O Z is the axis of a fixed cone of which O I is the slant side, at any instant O I is a line *in the rolling cone*, which is at rest, and about which it rotates with angular velocity ω. Let Z O C $= \alpha$, I O C $= \beta$. Draw J K and J L at right angles to O Z and O I. We see that J would go into the paper with the velocity Ω . K J, because J is moving in a horizontal circle whose fixed centre is K; also J goes into the paper with the velocity ω . L J, because L is fixed; and these are equal; so that Ω . K J $= \omega$. L J, or, since K J $=$ O J . sin. α, L J $=$ O J . sin. β,

$$\Omega \sin. \alpha = \omega \sin. \beta \;.\;.\;.\;.\; (1).$$

Again, ω about O I is the total spin of the top. It has components

$$\left.\begin{array}{l} \omega \cos. \beta = r \text{ about the axis O C} \\ \omega \sin. \beta \text{ or } r \tan. \beta \text{ about the axis O A} \end{array}\right\} \;.\;.\;.\;.\; (2),$$

if O A is at right angles to O C; and hence (1) may be written

$$\Omega \sin. \alpha = r \tan. \beta \;.\;.\;.\;.\; (1),$$

so that β is now known. All the lines drawn are in the plane of the paper at the instant.

2. If the moment of inertia of the top about O C is C and about O A is A, find the **resultant moment of momentum.**

Answer.—C . r is the moment of momentum about O C; A . r tan. β is the moment of momentum about O A. The resultant has an axis O H, its amount being C r . sec. γ. If γ is H O C,

$$\tan. \gamma = \text{A} r \tan. \beta \div \text{C} r = \text{A} \tan. \beta \div \text{C} \;.\;.\;.\;.\; (3).$$

3. If the distance O H represents the moment of momentum of a top about the axis O H, and a torque W h sin. α about an axis at right angles to the paper tends to send O H away from O Z, the effect is to make the horizontal velocity of the point H at right angles to the paper be W h sin. α. What is the angular velocity of O H about O Z?

Answer.—The point H moves in a circle of radius O H . sin. Z O H, and the linear velocity of H, or $\mathrm{W}\,h$ sin. α divided by radius gives angular velocity, or

$$\mathrm{W}\,h \sin.\,\alpha \div \mathrm{O\,H} \sin.\,\mathrm{Z\,O\,H},$$

$$\text{or}\quad \Omega = \frac{\mathrm{W}\,h \sin.\,\alpha}{\mathrm{O\,H}\,.\,\sin\,(\alpha - \gamma)} \quad\text{or}\quad \frac{\mathrm{W}\,h\,.\,\sin.\,\alpha}{\mathrm{C}\,r\,.\,\sec.\,\gamma\,.\,\sin.\,(\alpha - \gamma)}.$$

Hence $$\Omega\,\mathrm{C}\,r\,(\sin.\,\alpha - \cos.\,\alpha\,.\,\tan.\,\gamma) = \mathrm{W}\,h \sin.\,\alpha.$$

But tan. γ we found to be A tan. $\beta \div$ C, and hence

$$\Omega\,\mathrm{C}\,r\left(\sin.\,\alpha - \cos.\,\alpha\,.\,\frac{\mathrm{A}}{\mathrm{C}} \tan.\,\beta\right) = \mathrm{W}\,h \sin.\,\alpha.$$

But (1) gives us tan. $\beta = \Omega$ sin. α/r, and hence

$$\Omega\,\mathrm{C}\,r - \Omega^2\,.\,\cos.\,\alpha\,.\,\mathrm{A} = \mathrm{W}\,h,\quad \Omega^2 - \Omega\,.\,\frac{\mathrm{C}\,r}{\mathrm{A}} \sec.\,\alpha\,. + \frac{\mathrm{W}\,h}{\mathrm{A}} \sec.\,\alpha = 0.$$

From this we may calculate Ω, **the rate of precession of the top.** The student will find a value of r for which any precessional motion is impossible, and the top must fall. He may also consider the meaning of two different real values for Ω. We have given this exercise because it gives a good drilling in the use of the above rules. Note that a real top has not an infinitely sharp peg. It rolls and slides on the table, and we shall not consider this much more complicated problem.

CHAPTER XXVII.

CENTRIFUGAL FORCE.

491. Centrifugal Force.—If a body is compelled to move in a curved path, it exerts a force directed outwards from the centre, and its amount in pounds is found by multiplying the mass of the body by the square of the velocity in feet per second, and dividing by the radius of the curved path. We evidently get the same answer if we multiply the mass by the radius and by the square of the angular velocity in radians per second. Thus, a weight placed at the end of an arm like the arm of a wheel exerts a pull in the arm. If a body moves round an axis 20 times per minute in a circle whose radius is 3 feet, you can determine the centrifugal force by first finding the velocity of the body and using the above rule, or you may proceed as follows:—The weight of the body multiplied by 3 multiplied by the square of 20 divided by 2,937 is the centrifugal force.

Suppose a wheel, whose total weight is 20 tons, or 44,800 lbs., has its centre of gravity 0·4 foot away from the axis—that is, suppose the wheel to be eccentric—then if the wheel makes 50 revolutions per minute, the centrifugal force is 44,800 × 0·4 × 2,500 ÷ 2,937, or 15,253 lbs.—that is, 6·81 tons. This force acts on the bearings of the shaft, always in the direction of the centre of gravity of the wheel.

492. Anyone who wants to get clear ideas about centrifugal force ought to make experiments of his own. Unfortunately, although there are many toys made to illustrate the effects of centrifugal force, we know of only one piece of apparatus which enables the laws to be systematically experimented upon. Fig. 352 is a drawing of it. A represents a little, flat, cast-iron box, like a narrow drum; one drum-head, as it were, being replaced by a thin steel plate, so as to be strong and flexible. B is a glass tube which enters the box. Mercury fills the box and the tube to the level *b*, and when C, the centre of the steel plate, is pulled or pushed, although we cannot see much yielding in C, we can observe the mercury rise and fall in the tube. There is a screw, D, entering the box at the back; by means of this screw we can adjust the

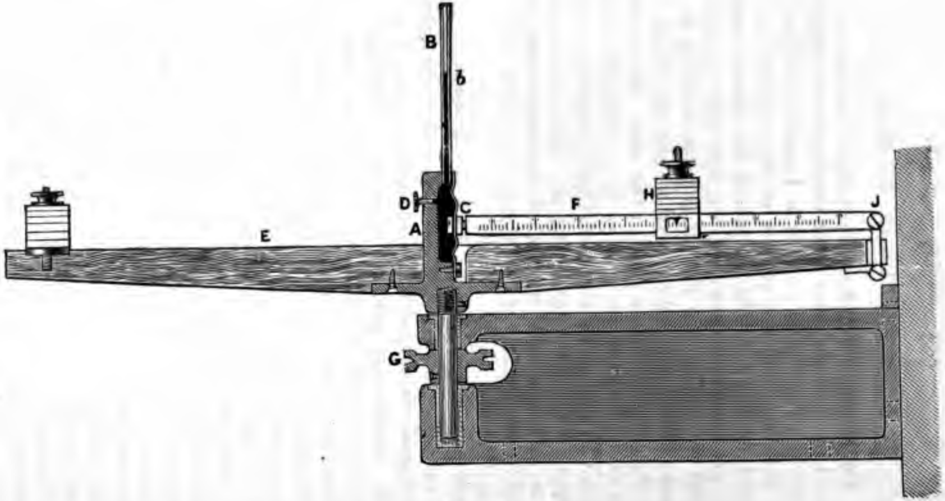

Fig. 352.

height of the mercury in the tube. The box is in the centre of a circular table, E, which can be whirled round at any speed we please, and the tube is exactly in the axis of rotation, so that the height of the mercury can be measured whatever be the speed. Fastened to the centre of the corrugated plate at C is a long brass rod, F, which is supported at J on the end of a little rocker, so that it can move backward and forward with less friction than if it were made to slide on a bearing. At any place along F we can clamp a weight, H, which we may alter as we please from 0·5 to 8 lbs. We can clamp it near the axis or one foot away, the radius of the circle described by its centre of gravity being measured by the scale in tenths and hundredths of feet marked on the rod. The experimenter, turning the handle which drives the pulley *g*, keeps his eye upon the mercury level, and is able to maintain a very great constancy of speed. He counts the revolutions of his hand, and one of his companions takes the time, so that no speed-counter or indicator is necessary. Now, the centrifugal force due to the rod and sliding weight causes C to be pulled out very slightly, and this causes the mercury to fall in the tube; and it is easy with a fixed vertical scale alongside the tube to measure this rise and fall. We usually get a spring balance, or a cord, pulley, and weights, and before our experiments begin we pull the end of the rod F, noting the height of the mercury for a pull of 1 lb., 2 lbs., etc., and in this way we can afterwards tell the value of our scale measurements. We also make a number of experiments when the sliding weight is removed from the rod F, to tell us the centrifugal force of everything else at different speeds, and this we subtract from our subsequent observations. We see, then, that we can measure the centrifugal force, in pounds, of various masses, from 0·5 to 8 lbs., moving at any speed in a circle whose radius may be as little as 1·5 inch and as much as 11·5 inches. With this instrument it is easy to test the law which is usually given, and without working with some such instrument we question if students are likely to have any but vague notions about centrifugal force.

493. There is another method of experimenting which suggests itself, with apparatus which anyone may fix up for himself, but it does not give such a thorough understanding of the law to the person who experiments. In Fig. 353, A is a leaden ball at the end of a silk thread, P A, fastened at P.

A is kept out from its natural position in the vertical by means of a horizontal thread in the direction A B. Now, if we pass the horizontal thread A B over a pulley and hang a weight at its end, we find that the force acting in A B is to the weight of A as the distance K A is to the distance K P. The body A is acted upon by three forces: its weight downwards in the direction A a, the horizontal force A B, and the pull in the string A P. The triangle of forces tells us that as A K P is a triangle whose sides A K and K P and P A are parallel to the three forces, then the horizontal force, acting in A B, is to the vertical force which is the weight of A as the distance K A is to the distance P K. Suppose, for example, the weight of A to be 4 lbs., the height P K to be 8 feet, and the distance A K 1 foot, then A K is one-eighth of K P, and we are sure that the horizontal force needed just to keep A in this position is 0·5 lb., for it must be one eighth of the weight of A.

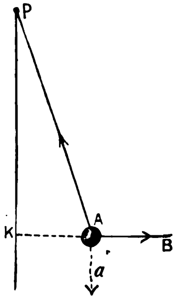

Fig. 353.

Now, let such a ball as A, hung by a thread P A, go round and round in a circle. Measure, as accurately as you can, K A, the radius of the circle, and K P, the vertical height from the ball to the point of suspension. Also count how many revolutions the ball makes per minute. The centrifugal force is now doing what the horizontal string did before, and we know how much it is. In fact, the centrifugal force is obtained by multiplying the weight of the ball by K A, the radius of its circle, and dividing by the vertical height K P. You can test if the centrifugal force law is true, therefore, by means of your measurements.

Conversely, if the weight of the ball is W, if A K is r, if P A is l, and P K is h, the centrifugal force is $\frac{W}{g}\frac{v^2}{r}$, and we see by the triangle of forces that this must be equal to $\frac{r}{h}$ W. Now, if the time of one revolution is T, then $v = 2\pi r \div T$, and hence $\frac{W v^2}{g\, r} = \frac{r\, W}{h}$ becomes $T = 2\pi\sqrt{\frac{h}{g}}$ . . . . (1)

A ball going round in this way is called a **conical**

pendulum, and we have in (1) found its periodic time. Its periodic time is that of a simple pendulum (Art. 446) whose length is the same as the h of the conical pendulum.

EXERCISES.

494. 1. A pendulum bob, weighing 20 lbs. moves with the velocity of 5 feet per second at the middle of its path. What force is tending to make the pendulum longer, in addition to the weight of the bob? The length of the pendulum is 15 feet. *Ans.*, $\frac{20}{32\cdot 2} \times \frac{25}{15} = 1\cdot 035$ lbs.

2. A locomotive weighing 35 tons travels at 50 miles an hour round a curve of 200 feet radius. What is the centrifugal force? *Ans.*, 29·23 tons.

3. The centre of a ball of 10 lbs. is at 4 feet from an axis, and revolves at 500 revolutions per minute. What is its centrifugal force? Its centrifugal force is to be balanced by those of two balls attached to the shaft, each of 20 lbs., revolving in circles whose planes are 4 feet and 1 foot from the plane of the first circle. At what distance is each of them from the axis? *Ans.*, 3,400; 0·4 foot; 2 feet nearly.

4. The rim of a pulley has a mean radius of 20 inches; its section is 6 inches broad, and average thickness $\frac{1}{2}$ inch. If it revolves at 200 revolutions per minute, what is the centrifugal force per inch length of rim? *Ans.*, 18·6 lbs.

5. A vehicle describes a horizontal circle of 600 feet radius with a velocity of 40 miles per hour; find the direction of the resultant due to gravity and centrifugal force. If it is a railway of gauge 1 metre, how much higher ought the outer rail to be than the inner one? *Ans.*, 10°·1 with vertical; 6·9 inches.

6. A skater describes a circle of 100 feet radius with a velocity of 18 feet per second; what is his inclination to the ice? *Ans.*, 84°.

495. 7. The law of energy states that a body of weight W and mass M falling from rest without friction along any path, at any instant being at the height h and having the velocity v, has the sum of its potential and kinetic energies constant, or

$$\mathrm{W}h + \tfrac{1}{2}\mathrm{M}v^2 = \text{constant} \quad \ldots . \ (1).$$

Show from this that force is equal to mass multiplied by acceleration. If s is distance along the path measured in the direction of motion, differentiating, we have $\mathrm{W}\frac{dh}{ds} + \mathrm{M}v \cdot \frac{dv}{ds} = 0$. But $-dh/ds$ is sin. a if a is the inclination of the path to the horizontal, and $v = ds/dt$, and hence

$$\mathrm{W}\sin. a = \mathrm{M}v\frac{dv}{ds} \quad \ldots . \ (2),$$

or, since

$$v\frac{dv}{ds} = v\frac{dv}{dt}\cdot\frac{dt}{ds} = \frac{dv}{dt},$$

$$\mathrm{W}\sin. a = \mathrm{M}\frac{dv}{dt} \quad \ldots . \ (3).$$

We see, therefore, that the resultant force (W sin. a) acting on the body is equal to the rate of change per second of the momentum. If α is 90°, so that the body falls vertically, then $\frac{dv}{dt}$ is called g, and (3) is merely

$$\text{M} = \text{W}/g \ldots . (4).$$

Students ought to familiarise themselves with this and other ways of showing that the law of energy leads to the law: force = mass × acceleration.

496. We think that **Newton's method, the Thomson and Tait method,** is very much to be preferred to any other method of starting in the study of mechanics. Every engineer ought to have his T and T′ (the elementary treatise), a well-thumbed book. Nevertheless, it is good to look at our fundamental notions from the energy point of view. The older proofs of statical principles were based on the idea of work, as in our Chapter XI. Assuming that mechanical energy is not lost by conversion into heat or other forms and that a total store remains constant, we get good working views of our subject. Underlying our notions are our speculations as to how matter performs attractions; but leaving out ideas of ethereal stress and strain energy and confining our attention to the idea that the sum of the potential and kinetic energies of a system remains constant, we have a working idea of great usefulness. Unfortunately many men forget that mechanical energy may be converted into heat, and so they make such mistakes as to calculate the force acting on a pile being driven as the space rate of conversion of energy.

497. Let us take **another example.** If a constant force F acts in the direction of motion of a body for a time δt through a space δs, increasing its velocity from v to $v + \delta v$, the work done by the force is F . δs, and this is spent in increasing the kinetic energy of the body from $\frac{1}{2} mv^2$ to $\frac{1}{2} m (v + \delta v)^2$. Hence

$$\text{F} \,.\, \delta s = mv \,.\, \delta v + \tfrac{1}{2} m \,.\, (\delta v)^2,$$

$$\text{or} \quad \text{F} = mv \frac{\delta v}{\delta s} + \tfrac{1}{2} m \frac{\delta v}{\delta s} \delta v \ldots . (1).$$

As we imagine δs, and therefore δv, to be smaller and smaller without limit, the last term in (1) gets to be nearer and nearer 0, and v gets to be better and better represented by $\delta s/\delta t$. Hence

$$\text{F} \doteq m \frac{\delta s}{\delta t} \cdot \frac{\delta v}{\delta s}, \text{ or } m \,.\, \frac{\delta v}{\delta t},$$

or force is equal to mass multiplied by acceleration, or force is the rate of change of momentum.

498. 8. So long as we deal with force in the direction of the motion of a body, there is no difficulty in showing that the law of work or the conservation of mechanical energy leads to the rule, "**Force is rate of change of momentum.**" When a point is moving in a curved path, we can say that the component of the force in the direction of motion is equal to the rate of change of momentum in the direction of motion. But what of the other component at right angles to this? Here it is necessary to observe that **mere change of direction of motion, and not merely of speed, indicates that force is acting on a body.** It is a matter of common observation that a centripetal force is necessary to keep a body moving in a circle, and that the body exercises a centrifugal force against those constraints which compel it to move circularly. When a curved surface gradually changes the direction of a moving body, these two forces act at the point of contact.* To get clear ideas, consider the conical pendulum of Fig. 354. The body P revolves in a circle at constant speed. To keep it at the constant distance $QP = r$ from the centre Q, we know that, if it were at rest, a force F must act. But it is in motion, and yet keeps out at the distance r. There is a centripetal force whose amount is known to us, W tan. a, and it is evidently balanced by what we can only call a centrifugal force of this amount which is due to the motion. Observe that we here have a case of centripetal force acting on a body, creating no increase of kinetic energy, and creating no change of any kind of energy, for when P comes round to the same position again everything is just as before, although the force has been acting for a whole round. Mathematically there is no great difficulty. We assume that we have proved that forces in directions at right angles to one another may be studied independently and may be combined as vectors. Consider circular motion in the plane of the paper. Constant speed v of P means a velocity v cos. θ in the direction QA and $-v$ sin. θ in the direction QC. The two accelerations are $-v$ sin. $\theta \,.\, a$ and $-v$ cos. $\theta \,.\, a$, a being $\frac{d\theta}{dt}$, the angular velocity, or $\frac{v}{r}$. If m is the mass of the body, we know that there are two forces acting upon the body: $-mv$ sin. $\theta \,.\, a$ in the direction QA, $-mv$ sin. $\theta \,.\, a$ in the direction QC. Forces always compound according to the vector law, as may be proved from the law of work (see Art. 495), and hence we see that the force acting upon P in the direction PQ is a centripetal force mvw or $m\frac{v^2}{r}$, and the acceleration of P is a centripetal acceleration v^2/r. We have seen (Art. 493) another way of arriving at the amount of the acceleration. In the very same way it is easy to show that when a small body moves, not merely in a circle, but in

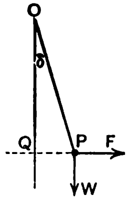

Fig. 354.

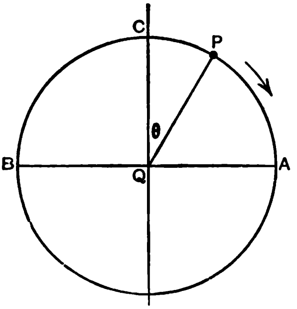

Fig. 355.

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* People may have notions of force that seem to be quite different from a metaphysical point of view, but which are really the same mathematically. Thus one man puts it that there is no such thing as force; we have only mass × acceleration. Another says—"Yes, we have a centripetal force; say, acting on a body which is whirled round at the end of a string, but it is not right to speak of the equal and opposite force, sometimes called centrifugal force." Now, if we only think of the body, it may be enough to speak only of centripetal force and centripetal acceleration; but a string in tension is really acted on by equal and opposite forces. If one of these is exerted towards the centre, the other is exerted outwards by the body, and we call it a centrifugal force.

any curved path, the force acting upon it has two components—one in the direction of the path, equal to its mass, multiplied by the rate of change of its mere speed; the other at right angles to the path, equal to its mass, multiplied by its linear velocity, multiplied by its angular velocity.

499. It is only when we have looked at the subject from the energy point of view, and in many others in which it will strike a thoughtful student to make experiments, that **the beauty of Newton's generalisation** comes home to us and we see how all the results of observation, experiment, and speculation are given to us in his statements. In fact, we gradually get to know that, whether acceleration of a small body is along the path or at right angles to it, force is equal to the vector rate of change of momentum. Now, this is Newton's definition of force, and we may begin our study of mechanics from this point of view. But speculation of the above nature is very far from being useless.

Taking up our subject from the easiest point of view, acceleration is the time rate of change of velocity, and **force is the time rate of change of momentum.** We first consider acceleration in the direction of motion, and then, if a body changes the direction of its velocity, the lateral rate of change of velocity or lateral acceleration must be considered. A velocity is a vector quantity, and velocities are added as all vectors are added. A velocity of 5 feet per second eastward, added to a velocity of 5 feet per second southward, are equal to a velocity of 7·070 feet per second southeastward.

500. When I speak of the motion of a body, I usually mean the motion of its centre of mass. If a body is at P moving along a curved path with the velocity of v feet per second, when it is at Q its velocity is in a direction making, let us say, an angle $\delta\theta$ with its old direction. In Fig. 357 let O B represent to scale the velocity at P; and let O C, equal in length, but making an angle $C O B = \delta\theta$, with O B, represent the velocity at Q. Now, in vector addition, $\mathbf{O B} + \mathbf{B C} = \mathbf{O C}$; so that if δt is the time taken by the body to move from P to Q, in that time there has been the lateral change of velocity **B C.** $B C = O B \cdot \delta\theta$ more and more nearly as $\delta\theta$ is made smaller and smaller, or $B C = v \cdot \delta\theta$. Hence the lateral acceleration, which is lateral change of velocity divided by the time of the

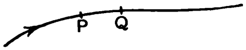

Fig. 356.

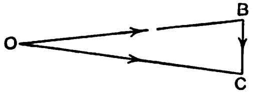

Fig. 357.

The academic person may be quite right to stick to his conventions as to force, because he never has to think of the medium (string or other) through which forces are exerted when he is working his problems. The engineer is compelled to deal with the larger question; he very wisely converts all his dynamical problems into statical problems, and all his forces are always *balanced.*

change, is $r \cdot \frac{\delta\theta}{\delta t}$ or $v \frac{\delta\theta}{\delta s} \cdot \frac{\delta s}{\delta t} = r^2 \frac{\delta\theta}{\delta s} = r^2 \times$ curvature, because our definition of curvature is $\frac{d\theta}{ds}$. In a circle the curvature $d\theta/ds$ happens to be the reciprocal of the radius. Hence we speak of this acceleration towards the concave side of a body's path as if it were moving in a circle (curiously enough, the fact seems always to be forgotten that the path is usually not a circle, nor even the very smallest arc of a circle) of radius r with a centripetal acceleration v^2/r.* The centripetal force causing the change of motion is the mass multiplied by this. The engineer is usually concerned with the equal and opposite force which the body exerts upon the constraints, and calls it centrifugal force. We see that it is mv^2/r or ma^2r or $wrn^2/2937$, where m is the mass of the body or w is its weight in pounds at London, r the radius of curvature of the path in feet, v the velocity in feet per second, a the angular velocity in radians per second, n the number of revolutions per *minute*. Observe that if masses m_1 and m_2 are attached to the same shaft at distances r_1 and r_2 from the axis, their centrifugal forces are in the proportion of $m_1 r_1$ to $m_2 r_2$; if we have half the mass at twice the distance, we have the same centrifugal force.

If a body changes in its speed v, and so has an acceleration in the direction of its path, its **total acceleration is the vector sum of the two accelerations**, and the resultant force acting on the body at any instant is the resultant of $m \frac{dv}{dt}$ along the path and $m \frac{v^2}{r}$ at right angles to and in the plane of the path.

Example.—A body of w lbs. is moving along a curve with a velocity v and an acceleration a in the direction of motion; the radius of curvature of the curve is r. What is the total acceleration and the force causing it? Here we have $\frac{v^2}{r}$ and a two accelerations at right angles to one another. The answer is, an acceleration $\sqrt{\frac{v^4}{r^2} + a^2}$ in the direction making an angle θ with the direction of motion, where $\tan.\,\theta = \frac{v^2}{ar}$. Multiply by $\frac{w}{g}$, and we have the total force.

501. Centrifugal Force in Belts or Ropes.—In Fig. 356 let P Q be a small portion of a flexible body of $w \,.\, \delta s$ lbs.; its centrifugal force is $\frac{w}{g}\,\delta s \,.\, \frac{v^2}{r}$, or $\frac{w}{g}\,\delta s \,.\, v^2 \,.\, \frac{\delta\theta}{\delta s}$, or $\frac{w}{g}\,v^2 \,.\, \delta\theta$. If the tension is T lbs., we know from Fig. 357 that $T \,.\, \delta\theta = \frac{w}{g}\,v^2 \,.\, \delta\theta$, so that $T = \frac{w}{g}\,v^2$.

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* To keep in our minds the fact that **all motion is relative**, we ought to remember that, relatively to the body, other bodies have a centrifugal acceleration. The words "centripetal" and "centrifugal" are technical terms now; their origins seem to have had something to do with the notion that a curve has millions of centres. It is all very well for the mathematicians to speak of a small part of a curve as being an arc of a circle, but the engineer knows that it is only in this matter of curvature or rate of change of θ with s that it is like an arc of a circle, for he knows that complete information about the very, very smallest portion of any curve implies a complete knowledge of the whole curve.

If a is the section in square feet, and f is the tensile stress per square foot, and w_0 is the weight of 1 cubic foot of the material, $w = aw_0$, $T = af$; so that $f = \frac{w_0}{g}\,v^2$.

Example.—What velocity will produce a tensile stress of 3,000 lbs. per square inch in the thin rim of a cast-iron pulley? Here $f = 3{,}000 \times 144$, and $w_0 = {\cdot}26 \times 1{,}728$. Hence

$$3{,}000 = \frac{{\cdot}26 \times 12}{32{\cdot}2}\,v^2 \text{ or } v = \sqrt{\frac{32{\cdot}2 \times 3{,}000}{{\cdot}26 \times 12}},$$

$$v = 175 \text{ feet per second};$$

or, in engineers' language, a velocity of 10,500 feet per minute will produce what is usually taken as the working tensile stress in cast iron.

502. If the plane of the path alters, if the plane rotates about the tangent to the path through the angle $\delta\phi$ in the distance δs, then $d\phi/ds$ is called the **tortuosity of the path.** When a body moves in a tortuous curve it has acceleration dv/dt along the path and v^2/r, a centripetal acceleration in the plane of the path, or the plane of curvature, as it is called. And if a model made of three very short pieces of wire, O S, O R, and O N, be made, the angles between O S, O R, and O N being right angles, and if we conceive O S to keep parallel to the path δs, if O R keeps pointing to the centre of curvature, then the angles turned through about the axes O N and O R per unit length of the path represent the curvature and the tortuosity. A student ought to make a model of a curve with wire and let a little frame like this slide along it, and study the matter for himself. A spiral path in which the curvature and tortuosity are constant is particularly interesting. If we refer the position of a particle to three axes of reference, its total acceleration at any instant is compounded of the three d^2x/dt^2, d^2y/dt^2, d^2z/dt^2. The three components of the resultant of all the forces which are acting are m times these, if m is the mass or inertia of the particle.

THE BALANCING OF MACHINES.

503. If a wheel is fixed eccentrically on its shaft, or if to a shaft there is attached any object whose centre of gravity is not exactly in the axis when the shaft rotates, **centrifugal force causes pressures on the bearings of the shaft** which are always in the direction of the centre of gravity of the rotating mass. In this case there is said to be a want of balance. If you wish to observe the effect produced by such *want* of balance, mount an axle to which a wheel is keyed on any support which is not very firm; fix a small weight on one of the arms of the wheel, and rotate it rapidly. You will find that, even if the weight is small, surprising effects are produced, and show themselves in a shaking of the supports; and the evil effects are four times as great at 200 revolutions per minute as at 100 revolutions per minute. Centrifugal force is proportional to the mass of a rotating part multiplied by the distance of its centre of gravity from the axis of rotation, multiplied by the square of the number of revolutions per minute.

504. If a number of bodies are attached to a shaft and are whirling round with it, each of them at any instant exerts a force on the shaft which can be calculated, and the resultant effect on the two bearings may easily be determined, just as easily as in the static problem of Art. 99. If the axis of rotation passes through the centre of gravity of all the rotating parts, the pressure on one bearing is equal and opposite to the pressure on the other; and by properly placing the masses, the pressure on either bearing may be reduced to nothing. Thus it is evident that when two masses are directly opposite to one another on a shaft, their centrifugal forces may be made to balance one another. When not opposite they cannot be made to balance, but two masses may balance one which is directly opposed to the resultant force of the two. When there is no pressure on either of the bearings, so that there is no tendency to change the direction of the axis, it is said to be the *permanent axis* of the rotating masses. **All axes of rotation in machines ought to be permanent axes.** When this is the case in a rotating machine, and it is suspended by ropes and made to work, there are no visible oscillations.

505. The balancing of a machine consists in adding masses in such positions, or re-arranging the positions of the existing masses so that the centrifugal forces due to their rotation are just able to balance the otherwise unbalanced forces which act on the various shafts. The student will find that the study of one problem in balancing will make him familiar enough with the method of calculation for its application to almost any other case which is likely to occur in practice. The most usual case for the student to take up is that of the locomotive engine, because want of balance in the locomotive is capable of producing very serious effects indeed.

358 shows an electromotor driving a shaft on which a mber of discs are keyed. Weights may be fastened on ese discs; the want of balance is evident when the shaft

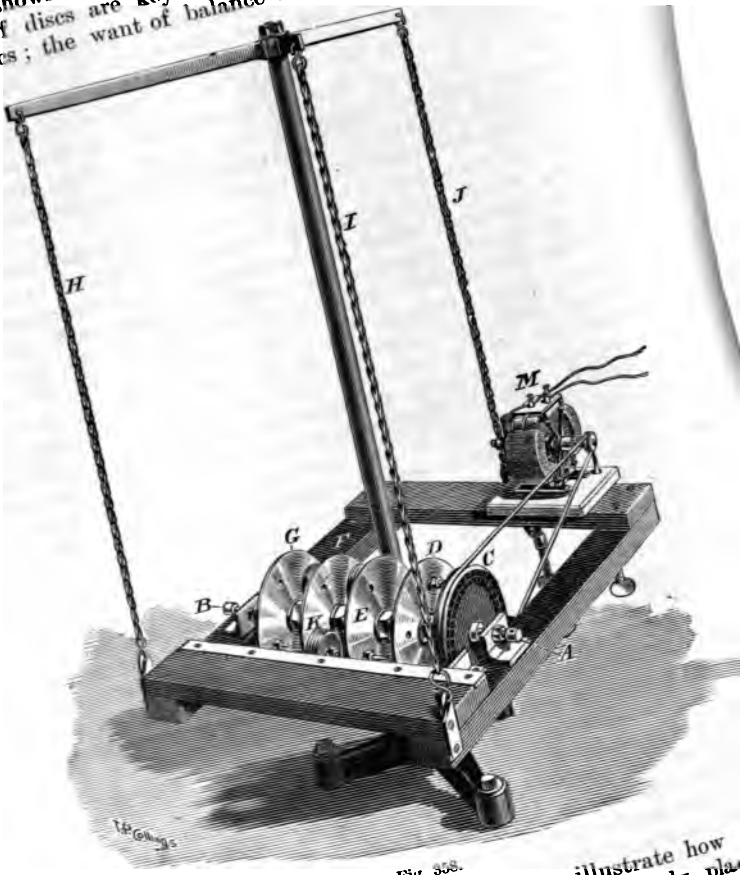

Fig. 358.

rotates, and students will find it easy to illustrate how shaft may be balanced by other weights properly pla They will see that when motions are merely rotatory we always have a perfect balancing of machinery.

506. *Example.*—It has been shown by experiment that the application of suitable balance weights is attended by a sensible reduction of resistance on railways at high speeds. **Locomotive engines** unbalanced cannot attain as high speeds as when balanced, with the same consumption of fuel. There are two separate sets of unbalanced forces acting on the crank shaft of a locomotive. (1) The centrifugal force of the crank, crank-pin, and as much of the connecting-rod as may be supposed, roughly, to follow the path of the crank-pin (say one-half of it). The mass or weight of each of these multiplied by the distance of its centre of gravity from the axis, divided by the length of the crank, gives the mass which, on the crank-pin, would produce the same centrifugal force. Let this weight be called w lbs. **In designing engines,** we consider half the connecting-rod to act as if collected at the crank-pin, the other half to be moving with the piston. At the end of the stroke, when the horizontal component of the centrifugal force is greatest and the vertical component vanishes, the horizontal pressure on the axle caused by the centrifugal force is

$$w \, \mathrm{R} \, n^2 \div 2{,}937,$$

R being the length of the crank in feet, and n the number of revolutions per minute. (2) We have the force due to the momentum of the reciprocating mass, including piston, piston-rod, slide, and the second half of the connecting-rod. The loss of momentum is most rapid just at the end of the stroke; and as loss of momentum per second is what we call force, the force acting on the axle at the end of the stroke due to this cause is easily found, and proves to be

$$\mathrm{W} \, \mathrm{R} \, n^2 \div 2{,}937$$

where W is the weight of the total reciprocating mass.

Now a weight w_1, or weights whose sum is w_1, may be placed on the driving-wheel or wheels at a distance r from the axis, such that the centrifugal force of w_1 may be equal to the sum of the above forces. This leads to

$$w_1 r = \mathrm{W} \, \mathrm{R} + w \, \mathrm{R};$$

and if we assume any distance, r, we can **calculate the balance weight** or weights, w_1.

507. Now, for the axis to be permanent in **inside-cylinder engines,** w_1 must be divided into two parts, one for each wheel, *inversely proportional to the distances of the wheels from the crank.* For outside-cylinder engines we get balance weights for the two wheels whose *difference* is w_1, and they are, as before, inversely proportional to the distances from the wheels to the crank in question. Hence, a consideration of each cylinder gives two balance weights, one usually much smaller than the other. As the cranks are at right angles, the balance weights ought to be 90° apart on each wheel. Instead of using these two, we can use one weight placed between their positions, so that its centrifugal force is the resultant of theirs. Thus, if we found 20 lbs. and 6 lbs. for the two placed at the same distance from the axis but 90° apart, make O A equal 20, and O B, at right angles to O A, equal to 6 according to any scale; complete the parallelogram, and O C represents on the same scale the weight which will replace them. It ought to be placed at just the same distance from the axis as they were supposed to be placed; and in position it makes the angle A O C with the larger weight. In this case it will be found that 20·88 lbs. placed 18·3° from the position which the weight of 20 lbs. might have occupied will be required to replace the two.

508. It often happens in **outside-cylinder engines** that the distance from one wheel, or rather from the centre of gravity of a balance weight, to the crank, is so little that the corresponding weight for the other wheel is very small, and may even be neglected. In inside-cylinder engines it will be found that, whereas the cranks are at right angles to one another, the balance weights on the two wheels on the opposite side of the axis to the cranks are often only 50° apart. In inside-cylinder engines with coupled wheels the *outside coupling rods and cranks are usually made to balance the inside moving parts*. These engines work very smoothly indeed. Outside-cylinder engines with coupled wheels are very unstable, from the use of small wheels requiring very rapid revolution of the crank axle: from the cylinders being farther apart than usual, so that the coupling-rods may have room, and from the number of reciprocating parts being increased. The conditions seem to admit of no remedy for these defects. The balance weight ought to be distributed over two or three of the spaces of the wheel, that the tire may not be unduly strained.

509. We have, then, the following easy, approximately correct **rules for locomotives**:—If R is length of crank, r the distance of centres of gravity of every balance weight from centres of wheels, e the distance apart of the centre lines of cylinders, d the distance apart of the wheels or centres of gravity of the balance weights, w the total weight of crank (referred to the pin), pin, connecting-rod, piston, slide, and piston-rod, A the angle which the position of centre of gravity of balance weight makes with near crank:

(1) **Inside-cylinder engines with uncoupled wheels.** Each balance weight $= \frac{w\,\mathrm{R}}{2\,dr}\sqrt{2\,d^2 + 2\,e^2}$, tan. A $= \frac{d - e}{d + e}$; so that A is easily obtained from a book of tables.

(2) **Outside-cylinder single engines with uncoupled wheels.** Each balance weight $= \frac{w\,\mathrm{R}}{r}$, A $= 180°$; so that in this case the balance weight is placed exactly opposite to the crank.

(3) **Inside-cylinder engines with wheels coupled.** Find by rule (1) if the weight of the coupling-rods, etc., is too great. If so, let counter weights equal to the difference be placed opposite the outside cranks. If too small, the difference must be made up with balance weights, as in rule (1). The positions of the outside cranks are found by rule (1).

(4) **Outside-cylinder coupled engines.** Find revolving weight of coupling-rods, etc., for each wheel. Also find sum of the weight

of the piston, rod, slide, and half connecting-rod. Divide this latter among the wheels, adding the given revolving weight already on them. Let this be used on each wheel according to rule (2).

510. We have dwelt upon these practical rules for balancing in locomotives, because they give good illustrations of centrifugal force. But the student ought clearly to see that it is only when a body rotates about an axis that we can exactly balance the forces. **A body with reciprocating motion can only be balanced by another body with a reciprocating motion**; and hence it is that, after much expense and quarrelling with persons complaining about the vibration of their houses, many electric lighting companies have discarded reciprocating steam-engines, replacing them with steam turbines. In Arts. 427 and 429 we give the principles on which the subject may be studied. I give practical examples of their use in my book on steam, gas, and oil engines. In that book I give the more exact constructions, which are so commonly taught now to advanced students, to find the forces at the ends of a connecting-rod for any position. It seems never to strike a student or his teacher that such elaborate calculations may possibly not give a very different result from what they may obtain by the simple assumption like what I have stated above. One of my students has made the comparison. His three weeks' constructions to find the forces acting on the frame of a **Willans engine,** the turning moment on the crank shaft, etc., nowhere differ appreciably in their results from those obtained by assuming that half the mass of the connecting-rod has the motion of the cross-head, and the other half that of the crank pin. The very much more important matter, the effect of the motion of the rod upon its strength as a **laterally loaded strut,** seems never to engage much attention.

EXERCISES.

1. A crank pin 4 inches in diameter and 6 inches long has to be balanced. If the length of crank be 9 inches, and the balance weights are placed directly opposite each crank arm, find the weights, the centre of gravity of each being 6 inches from the centre of the crank shaft.

Ans., 16·24 lbs.

2. A shaft is in balance under the action of three weights, one of 300 lbs., at a distance of 12 inches from the axis; another of 100 lbs., at a distance of 20 inches from the axis, on the opposite side, and 30 inches to the right of the first. How much must the third weight be, and where must be its position along the shaft, if its distance from the axis is 28 inches?

Ans., 57·1 lbs.; 37.5 inches to left of first weight.

3. In a locomotive the distance between the centre lines of the cylinders is 27½ inches. Balance weights are fixed at a horizontal distance apart of 59 inches, the centre of gravity of each describing a circle of 55 inches diameter. If the weight of the reciprocating masses for each cylinder be 400 lbs., and the stroke be 25 inches, find the position and magnitude of balance weights to counteract the horizontal and alternating force and couple.

Ans., At 160° with near crank; 142 lbs. each.

4. In a steam-engine the piston at the beginning of its stroke is exposed to a total effective steam pressure of 2,000 lbs., but the inertia of the piston is such that the thrust of the piston-rod is only 1,600 lbs. The speed of the engine is now raised until it becomes half as great again as before, while the steam-pressure is unchanged. What is the thrust of the piston-rod? *Ans.*, 1,100 lbs.

5. An engine is making 150 revolutions per minute. What is the acceleration of the piston at the commencement of each stroke, the connecting-rod being 4 feet long and the crank 9 inches?

Ans., 223 ; 153.

6. The weight of the reciprocating parts of a steam-engine is equivalent to 3 lbs. per square inch of the area of the piston. If the length of crank be 9 inches, find how much the initial effective pressure is reduced by the inertia of the reciprocating parts when the crank makes 70 revolutions per minute, the obliquity of the connecting-rod being neglected. *Ans.*, 3·8 lbs. per square inch.

CHAPTER XXVIII.

SPRINGS.

511. ANY contrivance which can store energy as strain energy, and give it out again readily, is a spring. Hence, any tie-bar or strut, any beam—in fact, any object whatsoever is a spring. The term is, however, generally applied only to such objects as can be changed in shape very much without fracture.

A tie-rod of indiarubber can be stretched to eight times its old length, again and again, without hurt; whereas a tie-rod of the best steel can only be stretched to $\frac{1}{80}$th of its old length, again and again, without hurt.

Hence the indiarubber tie-rod or a strut may be called a spring, just like the spiral spring; but it would not be right to speak of a tie-rod or strut of steel as a spring. The difference is, however, only one of degree, and indeed, a **mine cage** suspended by a **steel rope** half a mile long vibrates up and down just as if hung from any ordinary spring.

512. Springs are almost always used as **reservoirs of energy**—that is, as hydraulic accumulators are used, or fly-wheels of steam-engines, or cisterns of water, or electric accumulators.

The **mainspring of a clock** or watch takes a store of energy in winding-up, and gives it out gradually for about twenty-four hours in unwinding. A **bow** gets gradually a store of energy, which it gives out rapidly when the arrow is set free. A **buffer-stop spring** stores up all the kinetic energy of a train, and the stiffer it is the more quickly will it store the energy, and therefore the more suddenly will the train be brought to rest. Ordinary buffer springs are continually storing up and giving out energy, equalising only gradually the velocities of the two railway carriages, so that if one gets a sudden change of velocity, the other shall only be affected gradually. In the same way, when any two objects have a springy connection, if one of the objects alters its velocity suddenly, the other alters its velocity in consequence only gradually. These are cases in which a spring is used to **prevent shocks**, or, as we may put it, a spring is used to **lengthen the time of a blow**, and therefore to diminish the average force of a blow.

513. Now there are several distinct cases here to consider.

I. A body of mass B, moving with velocity v_b, overtakes a body A, moving with velocity v_a. The buffer between them is strained until they move with the same velocity. If the common velocity then is v,

$$B\left(v_b - v\right) = A\left(v - v_a\right)$$

$$\frac{B v_b + A v_a}{A + B} = v.$$

The energy now stored up in the spring is—

$$\tfrac{1}{2} B v_b^2 + \tfrac{1}{2} A v_a^2 - \tfrac{1}{2} (A + B) v^2.$$

In case the bodies B and A are themselves elastic, they themselves, to a greater or less extent, act as buffers, and less energy is stored in the buffer itself. Also as changes of shape are usually accompanied by friction, some of the energy is wasted. If no energy were wasted, the two objects would keep going faster and slower relatively to one another, as the spring was compressed and extended, and to some extent this does go on after the blow; but these vibrations are usually rapidly stilled as the energy is wasted in friction, partly, as we have already said, in the buffer spring itself, and partly by friction opposing generally the motion of the bodies.

II. If one of the bodies, A, is fixed to the earth, then the mass of A may be regarded as infinite; v_a and v in the above calculation become 0, but the same reasoning applies as before.

III. **Two carriages**, A and B, are at rest connected by a spring. A is made suddenly to move through a distance *a*. Now if the spring were infinitely stiff, B would just as suddenly move through the same distance *a*. As the spring is less and less stiff, B moves over the distance *a* with less and less suddenness, because the kinetic energy which must eventually be given to B is suddenly stored in the spring, but is only gradually given to B by the spring. If there is no friction, B will be left vibrating. If there is no friction opposing B's motion, but there is friction due to change of shape of the spring, B will vibrate and gradually come to rest.

If there is friction opposing B's motion, but there is no friction in the spring, the store of energy in the spring is greater than before. B must, of course, come to rest. With friction in the spring, B will more rapidly come to rest.

In all these cases, B comes to rest at the distance a from its old position.

IV. As in the last case—but A moves suddenly through a distance a, and after a short time T is moved suddenly back to its old position.

(1) If T is exceedingly small, B does not move.

(2) If T is great, B moves as described in III., and repeats its motion in the reversed direction.

(3) If T is neither too great nor too small, B has a motion intermediate between the (1) and (2) described motions.

(4) If A's motion is quickly vibrating through the amplitude a, B gets a **vibratory motion** of the same period; but superimposed on this is the natural vibratory motion of a period which depends on the stiffness of the spring and the mass of B; the natural vibrations will die away if there is friction.

514. It is well worth while for a student to illustrate this last case by means of a model. The crank Q, Fig. 359, is turned round regularly: if this is done by hand, a fly-wheel ought to be used to give steadiness of motion. By means of a connecting-rod, P gets an up and down motion which, diminished in the ratio OA/OP, is given to A; this creates a forced vibration in B. The **natural vibration** of B, when A is not moving, ought previously to have been studied—(1), when there is very little friction, and B goes on vibrating with nearly the same amplitude for a long time; (2) when there is fluid friction, B vibrating in a vessel of water. In this case there is really a change of inertia difficult to calculate; the water being set in motion. The amplitude of B's motion gradually diminishes because of friction. When A suddenly begins to vibrate, B may have a large natural vibration, but this gradually gets destroyed by friction, just as if there were no forced vibration. We shall now speak of the **forced vibration** only, and assume no friction. Let A vibrate with f times the **frequency** of the natural vibrations, and let A's amplitude be 1. Then the amplitude of B's motion will be as follows: when there is no friction B moves synchronously with A, so that B

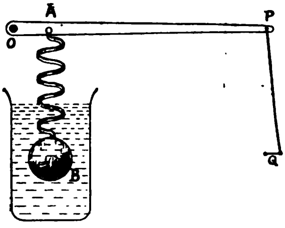

Fig. 359.

TABLE XII.

f	·1	·5	·8	·9	·95	·97	·98	·99	1	1·01	1·03	1·1	1·5	2·0	5·0	10·0
Amplitude of B.	1·01	1·333	2·778	5·263	10·26	16·92	25·25	50·25	∞	– 50	– 16·4	– 4·762	– 0·8	– 0·333	– 0·042	– 0·01

is at the top when A is at the top, or else (and this is indicated with a — sign) B is at the bottom when A is at the top.

To prove this, let W be the weight in lbs. of B, so that its mass is W ÷ 32·2; but we shall call it W/g. As before, a better approximation to accuracy is obtained by letting W include half the weight of the spring. Let the spring be such that a force of 1 lb. elongates it h feet. Then $\frac{W}{g}$ × the acceleration is the force in the spring; and if x is the distance in feet of the body below its position of equilibrium, $-h\frac{W}{g}\cdot\frac{d^2x}{dt^2}$ is the extra elongation of the spring when there is this acceleration. But if B is x feet below, and if A is y feet below their positions of equilibrium, then $x - y$ is this extra elongation, and hence

$$-h\frac{W}{g}\frac{d^2x}{dt^2} = x - y \ldots . (1),$$

or

$$\frac{d^2x}{dt^2} + \frac{g}{W\,h}x = \frac{g}{W\,h}y \ldots . (2).$$

Letting $n^2 = \frac{g}{W\,h}$, we see that n divided by 2π is the natural frequency of vibration. As in Art. 19, if we introduce our force of friction equal to b times the velocity, using $2\,F$ for bg/W, we have

$$\frac{d^2x}{dt^2} + 2\,F\frac{dx}{dt} + n^2x = n^2y \ldots . (3).$$

By putting $y = a$ sin. qt, it is easy to find the forced vibration. For simplicity, if we let F = O, then

$$x = \frac{a}{1 - q^2/n^2}\text{ sin. } qt.$$

The above table is obtained by letting q/n be called f.

Note in the model and from the tabled numbers that when the forced frequency is a small fraction of the natural, the forced vibration of B is a **faithful copy** of the motion of the point of support A; the spring and B move like a rigid body. When the forced frequency is increased, the motion of B is a **faithful magnification** of A's motion. As the forced gets nearly equal to the natural, the motion of B is an **enormous magnification** of A's motion. There is always some friction, and hence the vibration cannot become infinite. When the forced frequency is greater than the natural, B is always **a half-period behind** A, being at the top of its path when A is at the bottom. When the forced is many times the natural, the motion of B gets to be **very small**; it is nearly at rest.

515. Men who design **Earthquake** recorders try to find a steady point which does not move when everything else is moving. For up and down motion, observe that in the last case just mentioned B is like a steady point.

When the forced and natural frequencies are nearly equal we have the state of things which gives rise to **resonance** in acoustic instruments; which causes us to fear for suspension bridges or rolling ships.

It is obvious, then, that the simple statement of the problem, "How do springs prevent shocks?" presents, when we consider it very carefully, many quite different problems. It is worth while observing carefully the up and down motion of the body B of a waggon on the street when the wheels A go up and down over the stones. But we can study the subject better perhaps in the model. We see that although a spring connection between A and B does prevent shocks, the motion of B may be dangerously great. Thus, for example, when an earthquake occurs it does not always do to merely have an elastic connection between the ground and the house, as the earthquake leaves the house vibrating sometimes so much as to give quite a sea-sick feeling to the inhabitants.

We see that in all cases, unless there is friction opposing vibration of the body B—whether this friction exists in the parts of the spring itself, or more directly opposes the motion of B—there will be vibrations, sometimes dangerously large in amplitude. When by means of the spring connection we seek to diminish shocks, friction may be introduced by some **dash-pot arrangement,** which may consist of a porous piston moving in a cylinder filled with water or oil; or the piston may be solid, and there may be a pipe and cock connection between the opposite sides of it. Here the friction is mostly **fluid friction.** If it were all fluid friction, there would be no opposition to motion—there would, in fact, be no friction—if the motion were slow ; the friction is greater and greater as the motion is quicker and quicker. The friction between the **plates of a carriage spring,** which rub together every time the spring is changed in shape, is **solid friction,** which, if anything, is probably greater for slow motions than for quick motions. In all cases when vibrations are to be stilled it is better that the friction should be of the nature of fluid friction, but this is not always convenient. Its effect in stilling vibrations may very readily be studied in the laboratory. It will be found that by adjusting the cock of this dash-pot we can vary at will the rate of stilling of the vibrations of even very large masses, so that after a shock through the spring the body B may vibrate for a long time before it comes to rest, or it may come to rest after one slow lurching motion only. Now, it is necessary to understand that a dash-pot arrangement, or any other arrangement for introducing fluid friction, will not affect the static law of the spring in any way. It may be introduced on spring balances which are required to measure forces accurately, for example. But the sort of friction we find in carriage springs is very different. Here the plates of which the spring is made rub on one another, and there is solid friction, and such a spring as this **cannot be used for spring balances.** The law of the spring is altered ; we cannot depend upon the spring for measuring forces if there is any place where rubbing of solids takes place. Such springs are never used for purposes of measurement; they are only used for preventing shocks.

516. Springs are of many different forms; they are used as small as balance springs in watches, and they are sometimes so large as we see them in some buffer and locomotive springs. The following list is not given as by any means an exhaustive one :—

I. Cylindric spiral springs, subjected to axial loading.—In these it will be found that the wire, whether round or nearly square in section, is twisted like an ordinary revolving shaft transmitting power, and the strain is one of torsion in the material.

Examples.—Most forms of spring balance and dynamometers ; the springs of indicators ; many railway carriage and tram-car springs ; safety-valve springs for marine and locomotive boilers.

II. Cylindric spiral springs, subjected to a torque about the axis :—In these it will be found that the material is subjected everywhere to *bending.*

Examples.—The balance springs of the best chronometers; the springs used as elastic joints between two lengths of shafting.

III. Flat spiral springs, subjected to a torque about the axis.—In these it will be found that the material is subjected to *bending.*

Examples.—The main and balance springs of watches and many clocks; the springs used in nearly all contrivances which require to be wound up, or wherever a large reservoir of energy is required in a small space.

IV. Nearly straight strips of material subjected to bending.—These are used in a great variety of cases, sometimes in one piece, as in the limb of a tuning-fork.

V. More or less flat, or corrugated, circular, or of other outline, fixed or only supported at three or more points at the edges.

VI. Indiarubber springs, usually subjected to merely tension or compression, now being used largely for tram-cars. In all the above cases there is an absence of solid friction.

VII. The Ayrton-Perry spring, used in indicating the amount of a force by a large relative rotation of a pointer.—It will be found that the material in these springs is subjected to a combination of bending and twisting strains. The Ayrton-Perry twisted strip, in which a very small elongation is accompanied by great relative rotation.

VIII. More or less straight strips of material subjected to bending (like IV.), a number of pieces being used in one spring, these pieces rubbing on one another. When bending occurs, this introduces solid friction.

Examples.—Locomotive, waggon, and carriage springs, and nearly all the large springs used for minimising shocks, as buffer-stop springs.

IX. Gases in closed vessels, the volume of which may be altered, as in the air-chamber of some force-pumps; the cylinder and piston air-spring.

It is, however, obvious that the functions of springs cannot be specified in a few words. Springiness comes in usefully in the packing-rings of pistons, and in packing generally, to produce a good fit between pieces which rub on one another; in spring split rings, as washers, which prevent a nut from becoming loose; when a springy piece of material is used between two more rigid pieces which are bolted together, to give a uniform bearing without undue strains in the nuts. When, in fact, we discuss the elasticity of material generally, we see that everything in nature is a spring, and performs most of its functions in nature by means of this elastic property.

517. The main uses to which springs are put are these:—

1. Lengthening impacts, so as to diminish the forces of blows, and therefore absorbing, and in these cases usually also dissipating energy.
2. Regulating motion—that is, preventing large fluctuations in speed in driven pieces of machinery. This is not a common use of springs, because of the waste of energy.
3. As reservoirs of energy.
4. Regulating, as in watches and clocks.
5. As measurers of force.
6. As measurers of distance.
7. As measurers of angles.

518. The Best Materials to Use in Springs.—A spring's usefulness depends primarily on its being a reservoir of energy. In the first two cases of the preceding table this capacity for storage of energy, and of course cheapness and ease of manufacture, ought to settle for us the material of which a spring should be made. In the other cases we must also consider the question of perfect or imperfect elasticity and viscosity of the material. First, then, as to the energy which may be stored.

The energy which must be given to distort a spring before it takes a permanent set is called its **resilience**.

Now, it will be shown in Art. 535 that in all springs subjected to bending, as springs of classes 2, 3, 4, 5, 7, and 8, the resilience per unit volume of the material depends upon the resilience per cubic inch of the material when subjected to compressive or tensile force, and this is $f^2 \div 2E$ in inch-pounds, where f is the greatest tensile or compressive stress *which* the material will stand without taking a set, and E is

Young's modulus of elasticity. The following table shows the value of this constant for various materials :—

TABLE XIII.—SPRING MATERIALS SUBJECTED TO BENDING.

	f	E In Millions of Pounds per Square Inch.	$f^2 \div 2\text{E}$
Wrought Iron	24,000	29	10
Mild Steel	35,000	30	20
Mild Steel, Hardened ...	70,500	30	83
Cast Steel, Unhardened ...	80,000	30	107
Cast Steel, Hardened ...	190,000	36	501
Copper	4,300	15	0·62
Brass	6,950	9·2	2·62
Gun Metal	6,200	9·9	2·00
Phosphor Bronze	19,700	14	13·85
Glass	4,500	8	1·26

The numbers in Tables XIII. and XIV. are subjected to very considerable variations, especially in the cases of copper, brass, gun metal, phosphor bronze, and glass. Indeed, in our opinion, definite statements as to the values of $f^2 \div 2\text{E}$, or $f_1^2 \div 2\text{N}$, ought not to be made, until careful experiments have been made on such varieties of these materials as are actually used in spring-making, and this has not yet, we believe, been done. We have taken the values of f, E, and N from Table XXII. In the best spiral steel springs, for example, the value of f_1 (the proof shear stress) has been found to be rather 60,000 or 70,000 lbs. per square inch than the 145,000 given in the table.

It will be shown in Art. 535 that in all springs where the material is subjected to twisting merely, as springs of Class 1—the most important class, probably, for the use of mechanical engineers and instrument makers—the **resilience per unit volume** of the material depends upon the value of

$$\frac{f_1^2}{2\text{N}},$$

where f_1 is the greatest shear stress which the material will stand without taking a set, and N is the modulus of rigidity of the material. The following table shows the value of this constant for various materials :—

TABLE XIV.—MATERIALS FOR CYLINDRIC SPIRAL SPRINGS.

	f_1	N In Millions of Pounds per Square Inch.	$f_1^2 \div 2N$
Wrought Iron	20,000	10·5	19
Mild Steel	26,500	11	32
Mild Steel, Hardened ...	53,000	11	128
Cast Steel, Unhardened ...	64,000	11	186
Cast Steel, Hardened ...	145,000	13	809
Copper	2,900	5·6	0·75
Brass	5,200	3·4	4·00
Gun Metal	4,150	3·7	2·33
Phosphor Bronze	14,500	5·25	20

The numbers given in Tables XIII. and XIV. are supposed to express, then, the actual relative values of the various materials for spring-making.

It will be observed that **hardened cast steel** is very much better than any other material for spring-making; hardening makes it five times more valuable. It is about 35 to 40 times more valuable than phosphor bronze; more than 40 times more valuable than wrought iron (which is not so good as phosphor bronze). The fact that phosphor bronze makes probably the best non-magnetic material for springs has been known to me for fifteen years. I tested this result by a great deal of experimenting with various materials. But this is not the only virtue of phosphor bronze. In quite a remarkable degree it is free from many of the vices of other metals—sub-permanent set after small loads, effects due to fatigue, etc. It is worth while to mention that by the nature of the process of manufacture, the material may have initial strains in it; before applying it in an instrument it ought to receive a considerable set in the direction in which it will most usually be strained. **Phosphor bronze springs** in my electrical instruments receive a set from a load which is six times as great as the greatest load ever applied to the spring when it is in use.

The numbers in the tables give us guidance, but we must also consider special conditions. The **hardening** and **tempering of steel** require great care; so great is this that we may *almost* say that there is only one steel spring maker in the whole of England. Now, phosphor bronze and brass and copper receive their greatest hardness by drawing through dies or rolling. They can, in fact, be hardened very uniformly in the cold state quite readily, and springs of them are easy to make. Again, although tempered steel has usually an oxide of iron to protect it, and a soft iron spring of any kind can also be given such an oxide by Barff's process as a covering, yet, on the whole, steel and more especially iron springs are much more subject to rust when exposed to a damp atmosphere than copper, brass, or phosphor bronze. Again, in certain electrical instruments where springs are used, steel and iron must not be used because of their magnetic properties, and in other measuring instruments the properties catalogued in the tables may not always be all-important.

519. The following property is not of importance in springs used to prevent shocks, as in buffer and carriage springs :—

When a spring is loaded with even a small load it may continue to lengthen axially slowly if the load is kept on ; and afterwards, when the load is taken off, it may not immediately shorten to its original length, but needs time. This is usually called sub-permanent set, and is greater with greater loads. When such a spring is unloaded after it has experienced loads of various amounts and various periods of rest, it will not usually go back to its old length, but will slowly undergo slight shortenings and lengthenings of various amounts depending on its previous experiences. I sometimes call this property the "creeping" property of the material. Professor Ayrton and I have written a paper concerning this property. A material possessing much of it is quite unsuitable for the springs of measuring instruments.

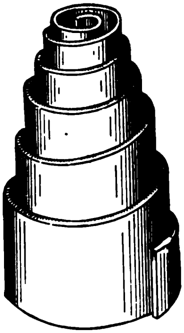

Fig. 360.

520. Spiral Springs.—A spiral spring is a wire, or rod, or strip of any constant or varying section (we shall always speak of it as a wire, of whatever size or shape its section may be), coiled so that the centre line of the wire lies everywhere on some surface of revolution. In most cases the wire is wound on a cylindric surface, the winding being perfectly regular—that is, the angle made by the centre line of the wire, with a plane at right angles to the axis of the spiral, is constant. Many cylindric spiral springs in use have wire of square or of elliptic section. In another form which is used as a buffer spring, the diameter of the coils varies as if the wire had been wound on the half of a very bulging barrel-shaped mandril; and we see that under pressure this spring can get very short axially without the coils coming into contact. In the form Fig. 360, the mandril was of a conical shape. Lastly, in Fig. 361, we have the flat spiral spring—what the conical form would become if it were squeezed until all its coils lay one plane.

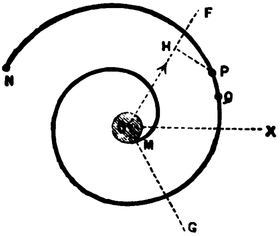

Fig. 361.

521. As an example of the **bending** of a strip of material, which might have been considered after Art. 327, let us take the case of a **flat spiral spring**, such as the main or balance spring in watches. Let N P M (Fig. 361) be such a spring, fastened to a case at N, and to an arbor or axle at M. When no forces are acting on the spring it has a spiral shape. Suppose that in this case, at a point P, the radius of curvature is r_0, and that when the spring is partly wound up there is at P a radius of curvature r, then $\frac{1}{r} - \frac{1}{r_0}$ is the **change of curvature** at P, and we know that the bending moment which produced this change of curvature is equal to $E I \left(\frac{1}{r} - \frac{1}{r_0}\right)$, where E is the modulus of elasticity of the material and I is the moment of inertia of the cross-section. (Thus, taking E at 36,000,000, if the breadth of the spring is 0·2 inch and its thickness 0·03, then E I is 16·2.) Now suppose the arbor to have turned through the angle X O G (which we shall call A) from the unstrained condition. What are the forces acting on P M and the arbor? Whatever these forces may be, they must be in equilibrium. If these forces were changed, there would be an alteration in the shape; but so long as these forces do not change, the shape and position of things do not alter. This is why we can apply to the spring, P M, and the arbor the **laws of forces acting on rigid bodies.** So long as P M O does not alter in shape, it obeys the laws of rigid bodies.

522. Now, the **forces acting on the arbor** may be very numerous —pressure of the pivots, pull of the fuzee chain, or pressure of teeth of wheels; but whatever they may be, we know that they can be represented by one force acting at O, the centre, together with a couple, C. If the spring is not in contact with the top or bottom of its case, and if the coils are not in contact with one another, no other forces act on the spring, M P, except at P. The particles of steel on one side of the section at P are acting on the particles on the other side; but whatever the forces at each of the particles may be, we know that the total effect at P is the same as that of one force and one couple. We cannot easily say what the force is, but if r is the radius of curvature at P, and if r_0 was the radius of curvature at P when the spring was unstrained, then the couple at P is what we have already called the bending moment,

$$\frac{\mathrm{E}\,bt^3}{12}\left(\frac{1}{r}-\frac{1}{r_0}\right).$$

Let us suppose, for simplicity, that the spring is everywhere of the same breadth and thickness, and let us use the letter e instead of $\frac{\mathrm{E}\,bt^3}{12}$, which is now, of course, the same everywhere. The couple at P is then $e\left(\frac{1}{r}-\frac{1}{r_0}\right)$. The only forces acting on P M O are: A force at O, of amount H, in the direction O H, say; a couple at O whose moment is $-$ L; a force at P; a couple at P whose moment is given above and is positive. Now, we know that the sum of the moments of all the forces about any point must be nothing. Take all the moments about the point P. The force at P has, then, no moment, and is to be neglected, and we have

$$-\mathrm{H}\times\mathrm{PH}-\mathrm{L}+e\left(\frac{1}{r}-\frac{1}{r_0}\right)=0.$$

In fact,
$$e\left(\frac{1}{r}-\frac{1}{r_0}\right)=\mathrm{L}+\mathrm{H}\,.\,\mathrm{PH}.$$

Let P Q be a short distance measured from P along the spring. Multiply every term of the above equation by P Q, and we find

$$e\left(\frac{\mathrm{PQ}}{r}-\frac{\mathrm{PQ}}{r_0}\right)=\mathrm{L}\,.\,\mathrm{PQ}+\mathrm{H}\,.\,\mathrm{PH}\,.\,\mathrm{PQ}.$$

Now, when P Q is very small it may be regarded as the arc of a circle whose radius is r; consequently $\frac{\mathrm{PQ}}{r}$ simply means the angle between the radius, or normal at P, and the normal at Q; in fact, it means the small angle which the tangent at P makes with the tangent at Q. Thus $\frac{\mathrm{PQ}}{r}-\frac{\mathrm{PQ}}{r_0}$ simply means the change which has occurred in the angle between the direction of the spring at P and the direction at Q. If now, instead of considering what occurs at the point P, we take the point Q, we shall get just a similar equation for another little length of the spring. Suppose we do this for every short length of the spring, and add up our results; we shall find that the sum of all terms such as $\frac{\mathrm{PQ}}{r}-\frac{\mathrm{PQ}}{r_0}$ means **the change** which has been produced **in the angle, between the tangents to the spring, at its two ends.** Thus, suppose the arbor has turned through the angle θ, and suppose that, whether or not the point of fastening at N has been moved, the direction of the spring at N has on the whole changed through an angle β: then we find that the sum of all the above-mentioned terms amounts to $\theta - \beta$. (θ may be called the amount of winding up of the spring; β may be called the amount of yielding in the fastening to the case.) Hence the sum of all the left-hand sides of all such equations as the above is $e(\theta - \beta)$.

Now let us consider the right-hand sides of the equations. Evidently the sum of all such terms as L × PQ will be L × length of spring; say $\text{L}\,l$. The sum of all such terms as H × PH × PQ is (Art. 109) equal to H multiplied by the length of the spring multiplied by the perpendicular distance of the centre of gravity of the spring from the line OH. This is, of course, the length of the spring multiplied by the moment of the force H about the centre of gravity of the spring. Summing up our results, we find that if the force on the arbor through the pivots, etc., has a moment about the centre of gravity of the spring of the amount G, if the length of the spring is l, if the angle turned through by the arbor from the unstrained position is θ, and if β is the angular yielding at N, and L is the couple with which the arbor tends to unwind itself, then $e(\theta - \beta) = \text{L}\,l - \text{G}\,l$, or $\text{L} = \frac{e}{l}(\theta - \beta) + \text{G}$. The term G depends on the position of the centre of gravity of the spring.

If the coils are numerous, each will be nearly circular, and the centre of gravity of the spring will nearly be at O, and G becomes insignificant: so that the equation becomes $\text{L} = \frac{e}{l}(\theta - \beta)$. If the spring is so rigidly fastened at its ends that there is no change of direction relatively to the barrel, $\text{L} = \frac{e}{l}\theta$, and the couple exerted by the spring in trying to unwind itself is simply proportional to the amount of turning of the arbor or the amount of winding up. If, then, the centre of gravity of the spring always remained in the centre of the arbor, and if the spring were rigidly fastened at N and M, we should have the **couple exerted simply proportional to the angle of winding**; and this is the condition for perfect isochronism in the balance spring of a watch. I need hardly say that this condition can never be perfectly satisfied. If we use a fuzee, the mainspring may be fastened as we please; but suppose we want the couple exerted by the spring to be nearly constant for various amounts of winding up, it is evident that the angle β ought to increase as fast as θ; that is, there ought to be a very considerable amount of yielding in the fastening of the spring to its case. The same effect will be produced by exerting considerable pressure on the arbor at its pivots, or in some way causing the arbor and its case to be not quite concentric with one another.

The watchmaker's usual plan to get moderately good isochronism is to make one of the above errors tend to correct another; that is, by allowing a greater yielding or greater stiffness of the outer attachment to counteract the results due to centre of gravity of the spring not remaining exactly in the axis of the balance.

523. Thus we see that by applying the law given in Art. 522 to a flat spiral spring fastened to a case at its outer end, N, and to an arbor or axle at its inner end, M, we find that if the spring is riveted firmly both at N and M, and if it is so long and its coils so nearly circular that its centre of gravity is always nearly in the centre of the axle, then, when partly wound up, the spring tends to unwind itself with a **turning moment which is proportional to the amount of winding up**. This is the case in the balance spring, and it is this condition that gives to the balance its character of taking almost exactly the same time to make a small swing as to make a great one. (See Art. 449.) When the end N is not riveted, but merely hinged or fastened in any way that will allow it to turn about N, the unwinding tendency is not proportional to the amount of winding up; it is proportional to the difference between the angle of winding and this angular yielding at N. If the strip is everywhere of the same breadth and thickness, the unwinding tendency is proportional to the moment of inertia of its own section—that is, to its breadth and to the cube of its thickness; it is also proportional to the modulus of elasticity of the material used, and is inversely proportional to the total length of the strip. Suppose we wind a cord round the barrel or case containing a mainspring of a watch whose arbor is fixed firmly, and, using a scale-pan with weights, we find the turning moment of the spring for various amounts of winding up. If we plot our results on squared paper, we shall find that the points lie in a curve like A O, B O, C O, or D O of Fig. 362, whereas for a balance spring we should get nearly a straight line through O.

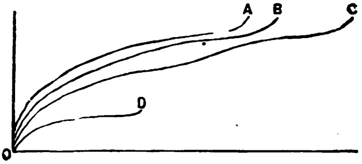

Fig. 362.

In Fig. 363 is represented an instrument which I have been in the habit of using in my laboratory,* to show the connection between the **turning moment** and the **angular winding in a flat spiral spring**. Different weights used at the end of the string give different readings of the pointer. By means of such an apparatus we are enabled to verify the laws described above. When we have performed one set of experiments with a spring, another set may be made on the same spring with its length diminished or increased by means of the arrangement for clamping, shown in Fig. 363. In this way we can experiment with springs of different breadths and thicknesses, as well as of different materials.

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* The woodcutter has represented too large a weight and too thin a spring.

524. The **flat spiral spring** just considered is a case of the bending of a strip of steel along its entire length. I will now take up a case in which there is no bending. Fig. 364 shows a **cylindric spiral spring** whose coils are very flat. Besides its own weight, it

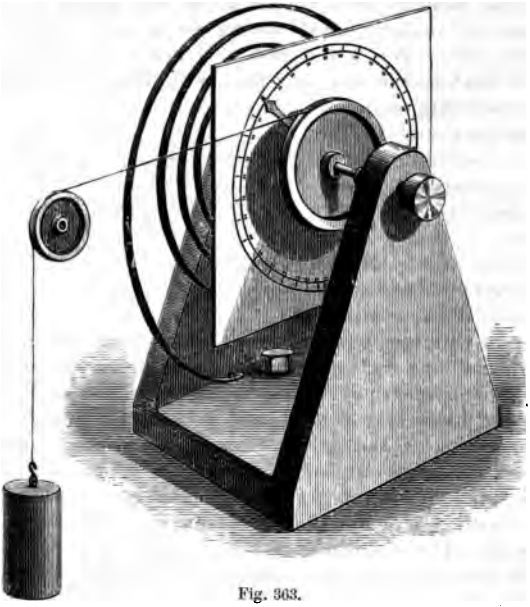


Fig. 363.

is acted upon by two equal and opposite forces in the direction of its axis, the supporting force at N and a weight at M. Now let us consider the equilibrium of the portion of the spring from any point P to M. Suppose the wire cut at P by a plane passing through the axis; this section will be more and more nearly a cross-section normal to the axis of the wire, as the spirals are more and more nearly horizontal. Let us regard it as a normal cross-section of the wire. Now, whatever may be the stresses at this cross-section, they must balance all the other forces acting on P M—namely, the force F at M, which is axial, and the weight of P M, which is very nearly axial. If we neglect the weight of P M, we have only to balance the force F acting at M. To do so we evidently need a **shearing force,** F, at P, distributed over the section, and a **twisting torque** which is equal to F . P H. It is easy to show that the shear is of much less importance than the torsion. Indeed, in many ways it is like the shearing force in beams (see Art. 369), and we shall neglect it. Again, since P H is the same for every part of the spring, every section of the wire is acted on by the same twisting couple, just as the shaft of Fig. 191 or the wire of Fig. 186 and its strength is calculated in the same way. Now, what is the amount of motion at M in consequence of this twist? As the wire is everywhere twisted, just as if it were a straight wire fastened at one end whilst at the other end there were a force, F, acting at the end of an arm whose length is equal to P H, the radius of the coils of the spring, the amount of the motion of M is just the same as the motion of the end of such an arm attached to the straight wire.

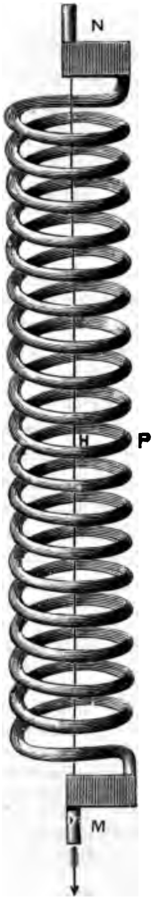


Fig. 364.

525. We have, then, the following pretty **illustration** (Fig. 365), which serves to keep the **rule for spiral springs** in our memory. Let two pieces of the same wire of the same length be taken; one of them kept straight, fixed firmly at A, and fastened at B to the axis of a pulley which can move in roller bearings. A cord, C, fastened to the rim of this pulley, carries the upper end of a spiral spring, D E, formed of the other piece of wire, the diameter of its coils being equal to the diameter of the pulley. Evidently, if a weight, W, is placed in the scale-pan, a point E gets just double the motion of a point C, for E gets C's motion as well as the lengthening of the spring. The scales F and G and the little pointers are for the purpose of making exact measurements. It is interesting to note how accurately the law is fulfilled, even in a roughly-constructed piece of apparatus such as anyone may easily put up for himself.

Example.—A spiral spring of charcoal iron spring wire, 0·1 inch diameter, 21·6 inches long, its coils having a radius of 1·3 inch, is extended by a weight of 10 lbs. Supposing that a piece of wire of the same material 1 inch long and 0·05 inch diameter, gets a twist of 24·2 degrees with a twisting moment of 2 inch-pounds, what is the extension of the spring? We see that if the trial wire were of twice the diameter the twist would be 24·2 ÷ 16, or 1·51 degrees, and with a twisting moment of 13 inch-pounds, which is 6·5 times as great, the angle would be 9·82 degrees, and on a wire 21·6 times as long would be 212· degrees, or 3·7 radians, and the arc of a circle whose radius is 1·3 inch, subtending this angle is 3·7 × 1·3 or 4·8 inches, the answer.

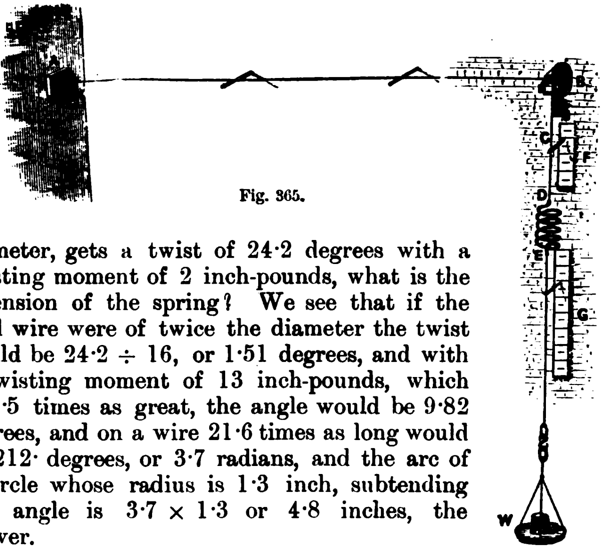


Fig. 365.

526. In designing a cylindric spiral spring it is very important to know the **greatest elongation** it will bear without taking a permanent set. If the material has internal strains given to it during its manufacture—and this it is very difficult to prevent in steel springs, unless great care is taken in tempering, and it is almost impossible to prevent in brass springs, because the elasticity added in manufacture is often regarded as a necessary quality which ought not to be destroyed by any annealing process—in this case the reader must keep in mind the considerations of Art. 294a. Otherwise, let f be the greatest shearing stress per square inch which the material can resist without getting a permanent set. Let M be the greatest twisting moment which a round wire of diameter d can bear without getting a permanent set. We see from Art. 296 that

$$M = \pi d^3 f/16 \; . \; . \; . \; . \; (1).$$

Now f will be approximately known from Table XIV., or M may be found by experiment for a given wire by any person who wishes to make a spring; and whether M or f is used in a formula, you now know how to calculate one when given the other and the size of the wire. If, then, we have a spring made of wire whose diameter is d, and if the radius of the coils as measured to the centre of the wire from the axis of the spring is a, we see that when W is the greatest weight with which the spring may be elongated without producing a permanent set,

$$\mathrm{W} = \frac{\mathrm{M}}{a} = \pi d^3 f/16\,a \;\ldots\; (2),$$

being independent of the length of wire employed.

From Art. 295 we see that if N is the modulus of rigidity of the material, T′ the greatest angular twist in radians which we can give to a wire of diameter d inches and length 1 inch, and m' the twisting moment which produces this twist, then

$$\mathrm{T}' = \frac{32\,m'}{\pi\,\mathrm{N}\,d^4} \;\ldots\; (3),$$

m' being what we have previously measured or calculated. N is approximately known for a material from Table XIV., or T′ may be found by experiment for a given wire; and whether T′ or N is used in a formula, you now know how to calculate one when given the other.

Putting the result of our reasoning in Art. 524 into an algebraic form, we see that a load, W, will elongate the spring by the amount

$$x = \frac{32\,\mathrm{W}\,la^2}{\pi\,\mathrm{N}\,d^4} \;\ldots\; (4),$$

and hence the greatest elongation which can be given to the spring without its getting a permanent set is

$$x' = l\,\mathrm{T}'\,a, \text{ or } \frac{32\,lam'}{\pi\,\mathrm{N}\,d^4}, \text{ or } \frac{2\,laf}{\mathrm{N}\,d} \;\ldots\; (5).$$

Combining (2) and (5), we see that when a spring is stretched to its elastic limit, the mechanical energy stored up in it, which is called its "*resilience*," being half the product of W′, the proof load, into the proof elongation, is

$$\tfrac{1}{2}\,m'\,l\,\mathrm{A}, \text{ or } \frac{\pi f^2 d^2 l}{16\,\mathrm{N}}, \text{ or } \frac{16\,lm'^2}{\pi\,\mathrm{N}\,d^4} \;\ldots\; (6).$$

527. Many interesting methods may be taken to express in words the meanings of these results. Thus the second expression in (6) shows that the **work which we can store up** in a spiral spring is simply proportional to the weight or quantity of material in it. It would be easy to show that we can **store more energy** in a spring formed of wire of circular section than in one of equal weight of the same material whose wire has any other than a circular section.

528. The following **readings of our formulæ** may prove to be useful:—1st. If d, the diameter of the wire, and a, the radius of the coils, be fixed, the elongation produced by any weight, W, will be proportional to l, the length coiled up to form the spring. 2nd. If a wire of a certain length and diameter be given to form a spring, the elongation produced by a certain weight, W, will be proportional to the square of the diameter which we may adopt for the coil. 3rd. If the diameter of the wire be fixed, and the axial length of the spring, when closed, so that the coils may touch one another, or, what is the same, the number of coils be also fixed, l must be proportional to a, and therefore the elongation due to a weight, W, will be proportional to the third power of the radius which we may adopt for the coil. 4th. If the length of the wire and the radius of the coil be fixed, the elongation due to a weight, w, will be inversely proportional to the fourth power of the diameter of the wire which we may adopt. 5th. With a given weight of metal and a given radius of the coil, the elongation due to a weight, w, will be proportional to l^3, or inversely to d^6, since l must be proportional to $\frac{1}{d^2}$.

We see that the **ultimate elongation** is—1st, proportional to the length of the wire, if the diameter of the wire and the radius of the coil be fixed; 2nd, proportional to the radius of the coil, if the length and the diameter of the wire be fixed; 3rd, inversely proportional to the diameter of the wire, if the length of the wire and the radius of the coil be fixed.

It will be found that a weight hung at M (Fig. 364) will tend to turn as the spring lengthens, unless the coils of the spring are very flat. This is due to the fact that the cross-sections of the wire are really subjected to a little bending as well as torsion.

529. We can cause the strain in such a spring to consist altogether of bending, if, without exerting any axial force such as I have shown in Fig. 364, **we exert a couple about the axis** such as we exerted on the wire in Fig. 186. The wire in Fig. 186 would be twisted, but the wire in Fig. 364 is subjected everywhere to bending without any twisting, or with only a very little twisting, due to the fact that the coils are not perfectly flat.

If a_0 is the radius of the coils to the centre line of the wire when unstrained, and the length of the coiled wire is l, then the number of coils multiplied by the circumference of each is the total length, so that the number of coils is $l \div 2\pi a_0$. If now the moment of inertia of the cross-section of the wire about the axis through its centre about which it bends is I, and if M is the moment which acts at the unfixed end of the spring to twist it, then the new radius, a, of every coil is obtained from our knowledge of the fact given in Art. 325.

$$\text{M} = \text{E I} \times \text{change of curvature},$$

or

$$\frac{\text{M}}{\text{E I}} = \frac{1}{a} - \frac{1}{a_0} \quad \ldots\ldots (1),$$

E being the modulus of elasticity of the material. I is $bd^3 \div 12$ for a wire of rectangular section, d being the dimension in inches of the section, measured radially out from the axis of such a spring; I is $\pi d^4 \div 64$ for a wire of circular section of diameter d.

Now $l \div 2\pi a$ would be the number of windings; so that, if n is the new number and n_0 the old number of windings, we have

$$\frac{\text{M}\,l}{2\pi\,\text{E I}} = n - n_0 \quad \ldots\ldots (2).$$

But one additional winding means 2π radians; so that if θ is the amount of winding up corresponding to $2\pi\,(n - n_0)$, we have

$$\theta = \frac{\text{M}\,l}{\text{E I}} \quad \ldots\ldots (3).$$

We see that it does not depend upon the radius of the coils, and is, therefore, the same formula as given in Art. 522 for a flat spiral spring whose radius varied continually. E I is called the flexural rigidity of the wire, being $E\,bt^3/12$ for a strip of thickness t; being $E\,\pi d^4/64$ for a circular section of diameter d; being $E\,s^4/12$ for a square of side s. The strength is as that of a beam subjected to the bending moment M.

530. From these considerations it is evident that a **spiral spring** like Fig. 364, **when it lengthens under the action of a weight,** has all its wire subjected to torsion. The spring itself is extended, but the wire of the spring is twisted. Again, if we subject the spring **to torsion as a whole,** the strain really going on in the wire is **a bending strain.** Usually, a spiral spring, as its coils are not perfectly flat, has its wire subjected to torsion principally, and a little bending as well, when the spring is extended; and when the spring is twisted as a whole its wire is mainly subjected to bending, but there is also a little twist in it. The extension of a spiral spring is proportional to the pulling force, and also to the length of the wire and to the square of the diameter of the coils; it is inversely proportional to the fourth power of the diameter of the wire if the wire is round. The twist given to a spiral spring as a whole is proportional to the moment of the twisting forces—it does not depend on the size of the coils; it is proportional to the length of wire, and inversely proportional to the fourth power of the diameter of the wire if the wire is round.

531. We have taken up at length the two cases of spiral springs in which the angle of spiral is 0. It will be found that when the angle is not 0, the stiffness of the cylindric spiral spring follows much the same law as for springs of small angle, but it is necessary to take up the general case.

532. The **theory of the cylindric spiral** spring is enough to study, because each small portion of any spiral spring may be regarded as part of a cylindric spiral spring. A rough model will help a student to understand the work better.

We shall imagine the upper end of a vertical cylindric spiral spring to be held fixed, and that at the other end, by means of a rigid arm coming in from the wire to the axis of the spring, we are able to apply an axial force F, by means of a weight, tending to elongate the spring, and also a couple, L, about the axis of the spiral tending to increase the number of coils; the directions of elongation axially and of greater winding up are our positive directions of motion. Let the axial elongation produced be called x, and the angular rotation produced be called ϕ. We shall neglect the weight of the spring itself as it will be quite easy afterwards to correct for this.

Let r be the radius of the coils—that is, the distance of the centre of the wire everywhere from the axis. Let the whole length of wire be l, the angle of the spiral α. Let B be the **flexural rigidity** of the wire in the osculating plane of the spiral B = E I where E is Young's modulus of elasticity, and I is the moment of inertia of the section of the wire about the line through its centre of gravity which touches the cylindric surface, and is at right angles to osculating plane. The bending moment, divided by the flexural rigidity, gives the change of curvature produced by bending. Let A be the **torsional rigidity** of the wire. The twisting moment applied to a wire divided by A gives the **angle of twist** produced per unit length of wire. In Table XV. a number of values of A and B are given for sections of wire which are in common use in springs. A is the torsional rigidity of the wire, being the twisting moment required to produce unit angle of twist per unit length. B is the flexural rigidity of the wire in the osculating plane of the spiral, being the bending moment required to produce unit change of curvature in that plane. The line P Q represents the axis of the spiral relatively to the wire.

N is the modulus of rigidity, and E the Young's modulus for the material. In the last two cases t is supposed to be small in comparison with b. Notice that A in the elliptic sections becomes $N\pi D d^3/16$ when d is small in comparison with D, and it becomes $N\,bt^3/3$ in the rectangular sections when t is small. Coulomb's wrong assumption was that $A = N I$ when I is the moment of inertia of a section about its centre. Now, for the elliptic section

$$I = \frac{\pi}{32}\left(D^3 d + D d^3\right), \text{ and hence}$$

$$\frac{\text{Coulomb's A}}{\text{correct A}} = \frac{1}{2}\,\frac{D^3 d + D d^3}{D^3 d^3}\left(D^2 + d^2\right) = \frac{1}{2}\left(x + \frac{1}{x}\right)^2$$

if x is D/d. Coulomb's value is correct when the section is circular, and we see that it is more and more wrong as the section gets flatter and flatter. The true value of A is always easy to calculate

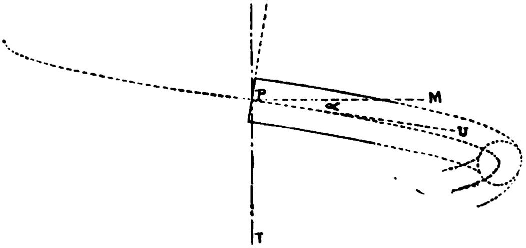
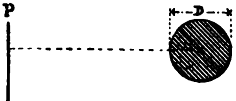
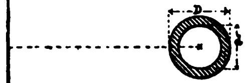
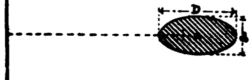
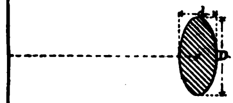
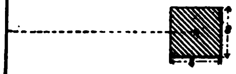
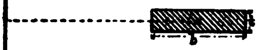
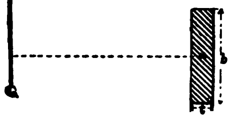


Fig. 366.

TABLE XV.

	A. Torsional rigidity.	B. Flexural rigidity.	W. Axial load when f is greatest shear stress.
	$\frac{N\pi d^4}{32}$	$\frac{E\pi D^4}{64}$	$\pi d^3f/16r$
	$\frac{N\pi}{32}(D^4 - d^4)$	$\frac{E\pi}{64}(D^4 - d^4)$	$\pi f(D^4 - d^4)/16D$
	$\frac{N\pi}{16}\frac{D^3d^3}{D^2 + d^2}$	$\frac{E\pi}{64}D^3d$	$\pi Dd^2f/16r$
	$\frac{N\pi}{16}\frac{D^3d^3}{D^2 + d^2}$	$\frac{E\pi}{64}Dd^3$	$\pi Dd^2f/16r$
	$0{\cdot}14058\ N s^4$	$\frac{E}{12}s^4$	$s^3f/4{\cdot}79r$
	$\frac{N}{3}\frac{b^3t^3}{b^2 + t^2}$	$E\frac{tb^3}{12}$	$b^3t^2f/3r(b^2 + t^2)$
	$\frac{N}{3}\frac{b^3t^3}{b^2 + t^2}$	$E\frac{bt^3}{12}$	$b^3t^2f/3r(b^2 + t^2)$

in the case of an elliptic section. Its value for a rectangular section is calculated from an infinite series, and it is therefore very important to know that Cauchy has proved that the torsional rigidity of a rectangle bears (approximately) to the torsional rigidity of an inscribed ellipse the proportion of their moments of inertia in the case when D is several times d (see Art. 313).

Consider the portion of spring below any cross-section P. This portion is in equilibrium. Hence the molecular forces exerted on it at the section P must balance F and L. The only conditions which we find it necessary to consider are these. Let Fig. 366 represent the elevation of a portion of the wire below the section P. Let T T be the elevation of the axis of the spring. Then about the axis P M we know that the moment due to the force F must be balanced by a *torque* of molecular forces whose amount is Fr; and about the axis P T there must be a torque of molecular force of amount L. Resolving these in the directions P S and P U in the plane of the paper, which is tangential to the cylindric surface at P, we have about P S the torque

$$Fr \sin. \alpha - L \cos. \alpha,$$

and about P U we have

$$Fr \cos. \alpha + L \sin. \alpha.$$

Now the moment about P S means a bending moment which produces an angular change per unit length whose amount is

$$\frac{Fr \sin. \alpha + L \cos. \alpha}{B} \quad \ldots. (1);$$

and the moment about P U means a twisting moment which produces an angle of twist per unit length of the amount

$$\frac{Fr \cos. \alpha + L \sin. \alpha}{A} \quad \ldots. (2).$$

Let us now consider how these two angular changes in unit length of the wire cause elongation and rotation at the bottom end of the spring. If the spring is sufficiently long, it is obvious that an axial elongation x and a rotation ϕ will occur at the free end of the spring, for we can then imagine that the lateral motions due to all the elements of the wire exactly counteract one another. It is therefore only necessary for us to obtain from the above expressions those elements which produce x and ϕ. Now any rotation of the body below P about any axis can be resolved into equivalent rotations about other axes, according to the laws for the resolution of vectors generally.

The rotation (1) about the axis P S is equivalent to (1) multiplied by cos. α about P T, and to (1) multiplied by sin. α about P M. The rotation (2) about the axis P U is equivalent to (2) multiplied by sin. α about the axis P T, and to (2) multiplied by cos. α about P M. Adding, we get, on the one hand, the rotation about P T, produced by the flexural and torsional strains in unit length of the wire; and this multiplied by l, the total length of the wire, gives ϕ. Adding, we get, on the other hand, the rotation about P M, produced by these same flexural and torsional strains; and it is obvious that this rotation causes a point on the axis of any portion of the spring below P to be lowered by a distance which is obtained by multiplying the rotation by r, the radius of the spiral. Multiplying by l, we obtain the whole **axial lengthening of the spring**—that is, putting our answer in its simplest shape,

$$\frac{\phi}{l} = \mathrm{F}\, r \sin.\, \alpha \cos.\, \alpha \left(\frac{1}{\mathrm{A}} - \frac{1}{\mathrm{B}}\right) + \mathrm{L}\left(\frac{\sin.^2\alpha}{\mathrm{A}} + \frac{\cos.^2\alpha}{\mathrm{B}}\right) \ldots . \ (3).$$

$$\frac{x}{lr} = \mathrm{F}\, r\left(\frac{\cos.^2\alpha}{\mathrm{A}} + \frac{\sin.^2\alpha}{\mathrm{B}}\right) + \mathrm{L} \sin.\, \alpha \cos.\, \alpha \left(\frac{1}{\mathrm{A}} - \frac{1}{\mathrm{B}}\right) \ldots . \ (4),$$

ϕ and x being the rotation and elongation produced in a spiral spring by an axial force F and a couple L acting together. In the theory ϕ and x are assumed to be very small. If we use $\delta\phi$ and δx for them, using ϕ for $2\pi n$ where n is the number of turns, and x for the axial length of the spring, so that x/n is what we sometimes call the pitch of the spiral; then $\frac{x}{n} \div 2\pi r = \tan.\, \phi \ldots . (5)$ and $\sin.\, \phi = x/l \ldots . (6)$, and by means of integration we can find accurately the effect of alteration in r and α as the spring changes in shape. Taking (3) and (4) as they stand, however, I find that young students can have no better easy mathematical exercises than are to be obtained by working problems on them.

For example, let l, r, α, A and B for a spiral spring be given, we can calculate x and ϕ when given F and L, or if given x and ϕ we can calculate F and L.

The very common case of α small, say $\alpha = 0$, leads to

$$\phi = \mathrm{L}\, l/\mathrm{B} \ldots . \ (7)$$

$$x = \mathrm{F}\, lr^2/\mathrm{A} \ldots . \ (8).$$

Thus **if the spirals are very flat**, whatever the nature of the section of the wire may be, the spring when subjected to axial force has no tendency to rotate at its end, and the spring when subjected to a couple merely has no tendency to alter in axial length.

Again, in any given spring, suppose we are informed that **the rotation of the end is prevented**, and that a force F acts, putting $\phi = 0$ we find L in terms of F from (3), and using this in (4) we have x in terms of F. Again, as in chronometer springs, **if the axial elongation is prevented** and only a torque L acts, put $x = 0$ in (4) and find F in terms of L; use this in (3) to find ϕ in terms of L.

Let $\alpha = \frac{\pi}{4}$ or 45°. (3) and (4) become

$$2\phi/l = \mathrm{F}\, r\left(\frac{1}{\mathrm{A}} - \frac{1}{\mathrm{B}}\right) + \mathrm{L}\left(\frac{1}{\mathrm{A}} + \frac{1}{\mathrm{B}}\right) \ldots . \ (9)$$

$$2x/lr = \mathrm{F}\, r\left(\frac{1}{\mathrm{A}} + \frac{1}{\mathrm{B}}\right) + \mathrm{L}\left(\frac{1}{\mathrm{A}} - \frac{1}{\mathrm{B}}\right) \ldots . \ (10).$$

If we apply to (9) and (10) the first condition given above, that is, $\phi = 0$, and an axial force F acting (9) gives $\mathrm{L} = -\,\mathrm{F}\, r\, \frac{\mathrm{B} - \mathrm{A}}{\mathrm{B} + \mathrm{A}}$ so that (10) gives

$$x = \frac{2\,\mathrm{F}\, lr^2}{\mathrm{B} + \mathrm{A}} \ldots . \ (11).$$

Again, if we apply to (9) and (10) the second condition given above, that is, $x = 0$ and a torque L acting, (10) gives

$$\text{F}r = -\text{L}\,\frac{\text{B} - \text{A}}{\text{B} + \text{A}},$$

so that (9) gives

$$\phi = \frac{2\,\text{L}l}{\text{B} + \text{A}} \quad \ldots . (12).$$

In estimating for general purposes the effect of altering the angle α; or the effect of constraint, such as preventing rotation of the end of a spring when an axial force is applied; or the effect of change of shape of section, etc., it is useful to remember that for most substances we may take it that approximately $\text{E} = 2{\cdot}5\,\text{N}$. It is very interesting to study the formulæ (8), (10), (12), using the Table XV. for the values of A and B for various forms of section. We shall leave this study to each student for himself—working out only the following interesting examples.

Students will please work carefully the following fifteen **exercises**:—Take $\text{N} = 10{\cdot}2 \times 10^6$, $\text{E} = 25{\cdot}5 \times 10^6$. Apply an axial load of 1 lb. and find the elongation in each case. There are 10 springs of wire, 100 inches long; diameter of coils 2 inches. The first has round wire of 0·1 inch diameter. All the others are of such sizes of section that there is the same total volume of material. Half the spirals have an angle of 45°. When the exercises are finished, divide all the elongations by that of the first spring, and enter the results on such a table as the following:—

TABLE XVI.—Relative Elongations for Same Axial Loads.

	$\alpha = 0$	$\alpha = 45°$	
	Whether the end is allowed freedom to rotate or not.	End free to rotate.	End restrained from rotating.
Solid round wire.	1	0·9	0·889
Hollow circular section, thickness of tube $\frac{1}{10}$th of outside diameter.	0·196	0·197	0·195
Square wire.	1·132	0·948	0·912
Rectangular wire, as in Fig. 6, Table XV. Breadth 4 times thickness.	2·029	1·108	0·349
Rectangular wire, as in Fig. 7. Breadth 4 times thickness.	2·029	2·544	2·440

In the theory of the spiral spring I considered the bend and twist given to each small portion of a spring, and assumed that the resultant action at the bottom of the spring was a rotation and an axial elongation—that is, that the lateral motions produced on the bottom by all angular motions of portions of the spring exactly balanced one another. As a matter of fact, however, it is only when the spring is very long that these lateral actions balance. To take the lateral actions into account is of no practical importance in such springs as we have been considering (cylindric spiral springs); but when the spires of a spring rapidly change their character, particularly in the case of flat spiral springs, the lateral actions are of importance (see Art. 522).

533. If any important object were to be served, I would calculate **the relative strengths of all the springs** under the various conditions here mentioned. It is only necessary to know the load F, which, when producing a bending moment $F r \sin.\alpha$ and a twisting moment $F r \cos.\alpha$ in a wire of the particular section at the same time, shall be capable of just producing permanent set. For a circular section this is well known—that is, it is well known that for a circular section the above two moments, acting together, are equivalent to a twisting moment

$$F r \sin.\alpha + \sqrt{F^2 r^2 \sin.^2\alpha + F^2 r^2 \cos.^2\alpha},$$

or

$$F r\ (1 + \sin.\ \alpha)$$

if the material has equal strength to resist shearing and tensile stresses. This is merely (3) of Art. 298. We see, then, that for a round wire the breaking-load for a spring of angle α is *less* than for one of the same size of wire and length of wire and diameter of coils, but quite flat in its spirals, in the ratio $1 : (1 + \sin.\alpha.)$ Thus a spring with $\alpha = 45°$ has a strength only a little over one-half of a spring with flat coils. This is the reason why, when a weight has been hung from a spring which produces permanent set, the spring so very rapidly **gets completely spoilt,** even although there is a counteracting influence due to the coils becoming smaller in diameter.

It is sufficient for most practical purposes, then, to say that if the load to produce permanent set in a round wire spring is 1 when the coils are flat, the load which will produce permanent set when the coils are not flat is

$$1 \div (1 + \sin.\alpha).$$

Again, in springs with flat coils, all of radius 1″, the loads to produce permanent set are simply equal to the twisting moments which, when applied to the wire, produce permanent set; and hence the following table is very useful.

W, in the last column of Table XV., gives the load which will produce the maximum shear stress f in each of the sections of wire in the case of cylindric spiral springs, the angle of the spiral α being 0. Hence, if f is the proof shear stress of the material, W is the proof load. The student will calculate the numbers of Table XV. as an exercise.

In the following table the load to produce permanent set and the resilience are given for cylindric spiral springs, so that students who are in the habit of only calculating the strength and stiffness of springs with round wire may be able to arrive easily at the strength and stiffness of springs made with other kinds of wire.

Comparison of the loads which will produce permanent set in springs of the same diameter of coils and same area of cross-section of wire: in all of them $a = 0$—that is, the coils are supposed to be as flat as possible. This is really a comparison of the torsional strength moduli of sections of the same area. Let students calculate the various numbers here given.

TABLE XVII.

	Load to Produce Permanent Set.	Resilience per Cubic Inch.
Circular	1	1
Hollow circular, thickness of tube $\frac{1}{10}$ of outside diameter	2·733	More and more nearly 2, as tube is thinner and thinner.
Square	·828	·938
Rectangular wire, as in Fig. 6. Breadth, 4 times thickness ...	·110	·224
Rectangular, as in Fig. 7. Breadth, 4 times thickness ...	·110	·224

CHAPTER XXIX.

RESILIENCE OF SPRINGS.

534. An examination of our formulæ will show that for nearly round wire, even when a is considerable, the rotation ϕ depends almost altogether on the couple L, and the elongation x depends almost altogether on the axial force F. But when the section of the wire is very different from circular, the angle a enters very materially into the calculation. Many examples of great interest may be taken up. Nearly flat coils are presumed.

1. If r alters in a spring, and if we want the resilience of the material to be the same everywhere; that is, the wire being round, if we want the material to be equally ready to break everywhere, we must make $\frac{d^3}{r}$ constant. That is, if the coils get twice as large, the diameter of the wire becomes $\sqrt[3]{2}$, or 1·26 times as large.

2. Round wire all of same diameter d, but the coils capable of lying just within one another if the spring is compressed by axial force.

Let n be the number of coils to any point starting from the end of the wire at the small coil side, where the radius is r_0. We may take $r = r_0 + nd$, $l = r_0\, 2\pi n + 4\pi^2 n^2 d$; and adding together the elongations produced on each elementary length into which we may imagine the wire divided, we find

$$x = \frac{16\,\text{F}}{\text{N}\,d^5}\left\{(r_0 + nd)^4 - r_0^4\right\}.$$

The strength of this spring is to be calculated as if all the coils were equal in size to the largest.

Some students may perhaps be interested in working the problem:—Find the cylindric spiral spring which, made of the same wire with the same number of turns, will receive the same axial elongation or compression from the same axial force.

The answer is, if n is the total number of turns, R is the radius of coils in the new spring,

$$\text{R}^3 = r_0^3 + \tfrac{3}{2} r_0^2 nd + r_0 (nd)^2 + \tfrac{1}{4} (nd)^3.$$

Thus, as a numerical example, if the first and last coils were 4 and 9 inches in radius, $d = 1$ inch, $n = 5$ turns, R is 6·8 inches.

535. Resilience.—The average resilience per cubic inch of a spring is the whole resilience divided by the volume. It is only in the case of uniform stress and strain in the material everywhere that we have maximum resilience for the whole spring. In actual cases this only occurs in tie-rods and struts, and in spiral springs made of a wire which is a thin tube.

$\frac{1}{2}$ M θ is the resilience per unit length of a wire, if M is the proof twisting moment and θ is the proof angle of twist, or if M is the proof bending moment and θ is the proof change of curvature produced. It is easy, therefore, to work out the following table of values, and every student ought to do this as an exercise :—

TABLE XVIII.—RESILIENCE PER UNIT VOLUME IN INCH-POUNDS.

Simple compression or extension. Only convenient in springs made of indiarubber	$\frac{1}{2}\frac{f^2}{E} =$	400
Bending if the maximum stress is reached in every section, and section is rectangular. As in beams of uniform strength, in well-made coach springs, and in spiral springs subjected to a torque only; also C-springs...	$\frac{1}{6}\frac{f^2}{E} =$	133
Bending. Uniform strip fastened at one end and loaded at the other, or supported at the ends and loaded in middle; also C-springs	$\frac{1}{18}\frac{f^2}{E} =$	44
Simple shearing. Possible in springs of indiarubber; also possible in spiral springs made of thin tubes circular in section	$\frac{1}{2}\frac{f^2}{N} =$	1,000
Torsion. As in spiral springs of round wire subjected to axial loads	$\frac{1}{4}\frac{f^2}{N} =$	500
Torsion. As in spiral springs of square wire subjected to axial forces.	$\cdot 235\frac{f^2}{N} =$	469
Torsion. As in spiral springs of oval or rectangular wire subjected to axial forces, anything less than for circular depending on ratio of diameters or of breadth to thickness.		

536. The student may not yet have been sufficiently impressed with the importance of knowing the **resilience per unit volume** of the material in all the springs. The resilience per unit volume tells us the value of a particular shape of spring if the spring is—as is almost always the case—to store and unstore energy.

Suppose that a spiral spring of any given material is to be used for any purpose. The greatest load is stated, and the elongation or compression due to that load. Then whatever other conditions may be given as to the coils being of the different sizes, etc., we know that the spring may be made of any shaped section of wire, but that the **thin tubular circular section is the very best** and the solid circular section is half as good, and any other sections are not half as good, and that the **spiral spring form is better than any other.**

In any spring, if a force F produces a known motion x in its own direction, then the resilience per unit volume, multiplied by the volume of material, gives the proof load F$'$ multiplied by half the motion x' produced by it. This principle enables easy calculations to be made.

Again, if in any spring a torque L produces a known angular motion ϕ in its own direction, then the resilience per unit volume, multiplied by the volume of material, gives the proof load L$'$ multiplied by half the angular motion ϕ' produced by it.

In nearly any case we consider it will be found that the important thing looked for when we choose a particular shape of spring is total resilience—total energy stored up.

Thus for example: We want a spring to exert a force F$'$ and only to alter, say, w lb., for a change of shape indicated by a motion in the direction of F$'$ by, say, b inches. Evidently here the law of stiffness of the spring is $b \propto w$, say $b = k\,w$, where k is a known number.

And, therefore, if x' is the greatest motion, $x' = \frac{b}{w}\mathrm{F}'$; and the total resilience is $\frac{1}{2}\mathrm{F}'\,x'$, or $\frac{1}{2}\frac{b}{w}\mathrm{F}'^2$.

That is, we are given F$'$, w, and b, and therefore the total resilience of the spring, and if we know the type of the spring we can find the volume of the substance required and therefore its weight. What people generally mean by the "springiness" of a spring means that it shall not change much in the force with which it acts, for a considerable amount of alteration in shape. Now b is the change of shape and $\frac{w}{\mathrm{F}'}$ is the fractional change of shape, so that $\frac{b}{w}\mathrm{F}'$ represents the *springiness* of a spring. But the value of a spring also depends upon the greatest force it can exert. That is, its value depends on $\frac{b}{w}\mathrm{F}'^2$; that is, on its resilience.

Looked at from almost any point of view, we see that the value of a particular form of spring is represented mainly by its resilience. Of course, its shape and cost of manufacture are also of importance. We may know that a tubular spiral spring may be the best for weight, and yet a **C-spring** shape may be best suited to the use to which it is to be put, and we put up with the disadvantage of using from twenty-two to twenty-eight times the weight of metal because of some other convenience.

537. If the angle of the spiral is α, the length of each turn, instead of being $2\pi r$, as we take it in approximate calculations, is really $2\pi r/\sin. \alpha$. If the axial length of a spring is x, the angle α is such that $\cos. \alpha = \mathrm{x}/l$. If the axial length changes to $\mathrm{x} + x$, the angle changes to α^1, such that $\cos. \alpha^1 = (\mathrm{x} + x)/l$. We have not taken these matters into account, assuming that our elongations were small or that our calculations were to be only approximate. The student who knows a little calculus may use δx and $\delta \phi$ instead of x and ϕ in Art. 532, and by integration obtain general and accurate expressions.

A conical spiral spring of round wire of diameter d, whose spires vary gradually from greatest radius r_1 to smallest r_0; the proof load is evidently to be calculated from r_1 or $\mathrm{w}^1 = \pi d^3 f/16 r_1$. The elongation for a given load w ought to be calculated for short lengths and added up. Mathematically it is evident that

$$x = \frac{32\,\mathrm{w}}{\pi \mathrm{N} d^4} \int_0^l r^2 . \, dx \; . \; . \; . \; . \; (2).$$

Taking it that if s is distance from the end of the wire where $r = r_0$, $r = r_0 + s \frac{r_1 - r_0}{l}$, and it is easy to show that

$$x = \frac{32\mathrm{w}}{\pi \mathrm{N} d^4} \frac{l}{3} (r_1^2 + r_1 r_0 + r_0^2) \; . \; . \; . \; . \; (3).$$

In fact, we see that in a conical spring, instead of r^2 for a cylindric spring we take

$$\frac{1}{3} (r_1^2 + r_1 r_0 + r_0^2).$$

EXERCISES.

1. If a conical spring is formed of round wire 0·2 inch diameter, 40 inches long, the coils varying from $r_1 = 2$ inches to $r_0 = 1$ inch, what is the proof load, and the axial shortening with this load?

Ans., 47 lbs. if $f = 60{,}000$; 0·4 inch.

2. An Ayrton-Perry spring is made of strip section, as in Table XV., which nearly covers a cylindric surface, and $\alpha = 45°$, so that $lb = 2\pi r \mathrm{x}$ if x is the axial length, and $\mathrm{x} = l \sin. \alpha$, or $l/\sqrt{2}$. Hence $lb = 2\pi r l/\sqrt{2}$, $b = \pi\sqrt{2} r$. If, then, $r = 0{\cdot}1$ inch, $b = {\cdot}44$ inch. If $t = {\cdot}001$ inch, and we have been able to obtain very broad strips of steel of this thinness, $\mathrm{A} = {\cdot}0019$ $\mathrm{B} = {\cdot}00132$ according to Table XV. Hence a load F will produce an elongation $x = 12{\cdot}8\,\mathrm{F}l$ and a rotation $\phi = -11{\cdot}5\,\mathrm{F}l$. It is to be noticed that the strip section of wire in a spiral spring (as, for example, in what are sometimes called volute springs for buffers) is very wasteful of metal for ordinary purposes.

3. Spring of strip 4 inches parallel to the axis 0·25 inch thick. If α is taken to be 0, and the coils lie inside one another, touching; if there are n turns, the smallest of radius r_0 and the largest of radius r_1, the

thickness of the strip being t, then the average radius is $\frac{1}{2}(r_1 + r_0)$, the length is $l = \pi(r_1 + r_0)n$ and $nt = r_1 - r_0$, so that $l = \frac{\pi}{t}(r_1^2 - r_0^2)$. Hence, as when t is small compared with b, W of column 3, Table XV., becomes $bt^2f/3r$, we have in this case

$$\text{W}^1 = bt^2f/3r_1 \;\ldots\ldots (1).$$

(8) of Art. 532 gives for the shortening under a load F, $x = \text{F}\,lr^2/\frac{\text{N}}{3}bt^3$, and as in the case of our conical spring we must take instead of the constant r^2 the value $\frac{1}{3}(r_1^2 + r_1r_0 + r_0^2)$, we have

$$x = \text{F}\frac{\pi}{t}(r_1^2 - r_0^2)\,\frac{1}{3}(r_1^2 + r_1r_0 + r_0^2)/\frac{\text{N}}{3}bt^3$$

$$x = \frac{\text{F}\,\pi}{\text{N}\,bt^4}(r_1 + r_0)(r_1^3 - r_0^3).$$

EXERCISES ON SPRINGS.

Take proof $f = 140{,}000$ lbs. per square inch, $\text{N} = 13 \times 10^6$ lbs. per square inch for the best spring steel.

1. A spring of coils 4 inches in radius is of round steel, 1 inch in diameter and 12 feet long. What are its proof axial load, its proof shortening or lengthening, and its resilience?

2. A spring of round steel wire the radius of whose coils is 2 inches is to be 20 inches long when its coils lie close together; it is to elongate 2 inches for a load of 400 lbs., and this is to be its proof load. Give its dimensions.

Here, if n is the number of coils, $n\,2\pi r = l$ nearly, $nd = 20$, so that

$$l/2\pi r = 20/d \;\ldots\ldots (1),$$
$$400 = \pi d^3 \times 14 \times 10^4/16r \;\ldots\ldots (2),$$
$$2 = 32 \times 400\,lr^2/\pi\,\text{N}\,d^4 \;\ldots\ldots (3).$$

We have only to find l, r, d from equations (1), (2), and (3).

3. A safety-valve spring of square steel wire is to shorten 0·4 inch when the pressure of steam is 150 lbs. per square inch, the effective area of the valve then being 12 square inches. The radius of the coils is 2 inches. A load of 400 lbs. per square inch on the valve produces the proof load on the spring. Find l and d.

CHAPTER XXX.

CARRIAGE SPRINGS.

538. **Carriage springs** usually consist of strips of steel in contact, as shown in Fig. 367. They are fastened at the ends and middle to the objects through which the loads are applied, in various ways, which must be examined by the student in actual examples. The ends of the strips are usually shaped, as in Fig. 368 or Fig. 369. These plates, before being tempered, are curved very accurately to a template; whilst dark red at the end of this shaping they are dipped into cold water, the two ends entering the water first and then the whole plate being lowered beneath the surface, so that if there is a variation in the hardness at different places it shall be a symmetrical variation. Each plate has a little pin or feather which enters a slot in its neighbour, so that the plates shall not be laterally displaced relatively to one another. Sometimes this is effected by making a short wrinkle or corrugation lengthwise at the end of each plate. A bolt passes through holes in the middles of all the plates, and when its nut is tightened up the plates are made to lie closely against one another. When springs of the general shape of Fig. 367, or of half of it, are made of one solid piece instead of a number of plates, the calculation of strength and stiffness is made by the method of Art. 339.

539. Carriage springs are usually **tested** by means of a hydraulic press or a steam press which forces them to become quite straight three or more times, and a spring is thought to be satisfactorily made if after this it is found not to have taken any permanent set. When the student works the following exercises, he will see that if the greatest load is that which produces straightness in all the strips, **their initial curvatures ought to be the same.** This will be found also to give sufficient tightness when the plates are bolted up.

I think that carriage spring makers and buyers are too particular in their wishing to have all the plates lying very tightly together when bolted up. There is too much of such tightness. Usually, too, in actual use a carriage spring never becomes unloaded, and its plates are therefore always very much more tightly pressing together than when the spring is examined unloaded. If, however, such tightness is really thought to be necessary, and if to obtain it we must have such great differences in curvature as I have found in springs by the best makers, it would be advisable to make the shorter plates thinner than the rest.

But if all the plates are of the same thickness, they ought certainly to have the same curvature before bolting up. If the student draws a set of such plates, he will see that when they are bolted up they will be sufficiently tight, and if such a spring is straightened the stresses in all the plates will be exactly the same.

540. The following exercises on bending are to be worked by the student to lead him to the theory of these springs. These exercises deal almost altogether with the best construction of carriage springs.

EXERCISES.

1. A beam of constant strength; what is its curvature everywhere?

Ans., the strength modulus Z is $\text{I} \div \frac{1}{2} d$ if d is the depth at any place and I is the moment of inertia of the cross-section there. Hence $\frac{\text{M}}{2\,\text{I}} d = f$, a constant for all sections, being the greatest stress in all, or $\frac{\text{M}}{\text{E}\,\text{I}} = \frac{2f}{\text{E}\,d}$; and this is the curvature.

2. If a beam has had the same change of curvature everywhere, show that if it is of constant strength it must be of constant depth.

3. If a beam of rectangular section of constant breadth and variable depth d is loaded at one end and fixed at the other, what must its depth everywhere be for the curvature to be constant?

Ans., if x is distance from the load W, $\frac{12\,\text{W}\,x}{\text{E}\,bd^3}$ must be constant, and therefore $d \propto \sqrt[3]{x}$.

4. In the last example, what is the greatest stress in each section?

$\frac{6\,\text{W}\,x}{\text{E}\,bd^2} = f$, so that $f \propto \frac{x}{d^2}$, or $f \propto \frac{x}{x^{\frac{2}{3}}}$, or $\propto x^{\frac{1}{3}}$. In fact, f is simply proportional to the depth, if the depth is proportional to the cube root of x. If we compare the answers to exercises (2) and (4), we see that the overlapping parts in carriage springs are most economically made of constant depth but varying breadth.

5. n strips, each of thickness t, make up a carriage spring if the length of the top strip (or the length of the curve ACB) is $2\,l$ inches; the overlap being λ, $l = n\lambda$. If OC is small compared with r_1, the radius of curvature of the top strip, $\text{OC} = l^2/2\,r_1$ nearly. Hence change in $\text{OC} = \frac{1}{2} l^2 \times$ change of curvature of top strip.

6. If when loaded the radius of the top strip is R_1, instead of being infinite, what is the radius of every strip when the spring is taken apart if the resilience is to be uniform? Also what is r_1, the radius of the top strip in the unloaded spring? This is a more tedious exercise, but is easy enough.

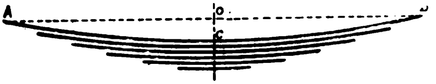


Fig. 367.

Ans. First, if the radius of the top strip when separate is ρ_1, ρ_1 is known, because $\frac{1}{\rho_1} - \frac{1}{\mathrm{R}_1} = 2f/\mathrm{E}\,t$, and f is the maximum stress.

$$\frac{1}{\mathrm{R}_1} - \frac{1}{\rho_1} = \frac{1}{\mathrm{R}_1 + st} - \frac{1}{\rho_s} \dots . (1)$$

where ρ_s is the radius of the s^{th} strip when separate. We now have the interesting problem: When a number of strips whose radii when free are known, are fastened together to form a spring, what will r_1 now be? The radius of the s^{th} strip was ρ_s when free, and is now $r_1 + st$, and so its change of curvature is $\frac{1}{\rho_s} - \frac{1}{r_1 + st}$. Hence the force W_s at its end, producing the bending moment $\mathrm{W}_s \lambda$, must be such that $\frac{\mathrm{W}_s \lambda}{\mathrm{E\,I}}$ produces this change of curvature. Hence

$$\mathrm{W}_s = \frac{\mathrm{E\,I}}{\lambda}\left(\frac{1}{\rho_s} - \frac{1}{r_1 + st}\right) \dots . (2).$$

We must, therefore, write out equation (1) for every strip to find the values of ρ_s for each, in terms of ρ_1, which is known, and also equation (2), thus calculating every value of W_s, the unknown r_1 being in every term. Now we make the statement that the sum of all the values of W_s is 0, and this enables r_1 to be found. Thus (2) and (1) give

$$\mathrm{W}_s = \frac{\mathrm{E\,I}}{\lambda}\left(\frac{1}{\mathrm{R}_1 + st} - \frac{1}{\mathrm{R}_1} + \frac{1}{\rho_1} - \frac{1}{r_1 + st}\right) \dots . (3),$$

and we must find r_1 by inserting 1, 2, 3, etc., for s in the equation

$$\Sigma \left\{ f + \frac{\mathrm{E}\,t}{2}\left(\frac{1}{\mathrm{R}_1 + st} - \frac{1}{r_1 + st}\right)\right\} = 0 \dots . (4).$$

If st is small compared with r_1 and R_1, we find that, if there are n strips altogether,

$$\left\{ \frac{2f}{\mathrm{E}\,t} + \frac{1}{\mathrm{R}_1} - \frac{1}{\mathrm{R}_1^2}\frac{t}{2}(n-1) \right\} r_1^2 - r_1 + \tfrac{1}{2}\,t\,(n-1) = 0 \dots . (5),$$

a quadratic to find r_1.

It is not necessary to pursue the subject farther. Students have the means of working all sorts of practical problems.

541. Tempering Carriage Springs.—In the last part of the work on the strips, a plate being red hot, by an interesting manipulation between several pairs of smiths' tongs held by two workmen it is fitted to lie everywhere close to a curved piece of metal, so that it receives a definite shape. It is now a dark red, and the workman dips the plate into water, letting the two ends go in simultaneously and the middle of the plate last. The plate is now placed in an air furnace, where it gradually heats; it is taken out once or twice and rubbed with a piece of partly charred wood. The experienced workman can tell by the nature of the smoke coming from the wood whether the temperature of the plate is high enough; when it has been heated to a sufficiently high temperature it is withdrawn from the air furnace and allowed to cool, lying on a metal table with others which have preceded it.

542. Tempering Spiral Springs.—The processes employed by Messrs. Salter are, unfortunately, unknown to me. In all cases, however, they probably consist in winding wire or rod in the red hot state on a smooth iron mandrel, shaping the end parts of the spring, if the wire is small, by pliers; if large, by special tools which can readily be designed for special cases. The formed spring is now heated to a dull red heat in an air furnace and plunged into **hot oil**, just of such a temperature as will produce the **blue colour** peculiar to spring steel. It is in this dipping of the hot steel into the oil bath that any trade secret can exist. If all the steel could from the same instant and at the same rate lose its heat, so that the hardened steel would be uniformly hard everywhere, the process would be perfect. Large spiral springs for safety-valves are dipped so that their axes remain vertical. To lay them sideways into the bath might cause the lower half of each coil to cool more quickly than the upper half. It is quite possible that good spring makers may not only dip their springs axially but also give to them a rotatory motion round their axes as they enter the bath.

543. Very small spiral springs for the **balances of chronometers and watches** require special care, whether they are cylindric or flat spiral springs. This is on account of the thinness of the material and the rapidity with which it may cool. They are usually enclosed in a box—very small springs are enclosed in a platinum box—either on their mandrels if cylindric, or coiled up together with others if flat, surrounded by powdered charcoal. The box is heated to redness for a sufficient time to ensure the springs inside being all red hot, and the box is then immersed in water. The little box is manipulated by means of a long handle. It is now placed upon an iron pan placed over a flame, and lying beside the box on the pan is a piece of brightened steel. As the pan gets heated the brightened piece of steel takes different colours, and it is presumed that the steel springs have about the same temperature. When the dark blue colour is reached, which is so characteristic of spring steel, the box is removed from the pan and allowed to cool.

544. I have never seen this process carried out by electrical heating as a substitute for the air furnace, but it is obviously quite easy to make the substitution. The strip or wire to be tempered receives an electrical current from a certain roller A, over which it moves, and another roller B at a short distance from the first; and at B, or even before it reaches B, the steel passes into an **oil bath.** In this case we have no sudden great hardening and then a softening process; the tempering is all done in one operation, a cooling from red heat to the temperature of hot oil. As the wire or strip is still kept heated when in the oil, the cooling is gradual, and obviously the **softness of the steel** will depend on how much of it is kept heated in the oil and on the average temperature of the oil.

545. Tempering Flat Spiral Springs.—In many cases these springs are not shaped before tempering. Great lengths of steel strip are passed through an air furnace from one set of rollers to another, at a rate which depends upon the particular purpose for which the strip is designed, upon its breadth and thickness, upon whether it is Bessemer or crucible steel. Just when in the red hot state and leaving the air furnace, the strip is passed through a vessel through which **cold water is kept circulating,** and is consequently made very hard. It then passes between pieces of cotton waste kept well soaked in oil, and then over a flame which keeps the oil ignited; and on leaving this region the strip is gradually allowed to cool before being wound upon a roller. If allowed to move too slowly, the strip and the oil covering its surface are for too long a time subjected to the heating effects of the second furnace, and the steel becomes softer. I have seen strips which were said to be continuous and a mile long which had been tempered in this way.

The very fine strip steel which is used in the Waterbury watch is tempered in this way. Great quantities of cheap Bessemer steel strip for use in ladies' corsets and numerous other purposes are also tempered in this way.

In well-made carriage springs the resilience is

$$\frac{1}{6}\frac{f_1^2}{E} \text{ per cubic inch,}$$

and in designing a spring a knowledge of this fact enables us somewhat to shorten the work.

Thus, suppose a spring is wanted to take a proof load of 3 tons, with a deflection of 3 inches just making it straight. The total resilience is

$$\tfrac{1}{2} \times 3 \times 2{,}240 \times 3, \text{ or } 10{,}080 \text{ inch pounds.}$$

If $f' = 30{,}000$ and $E = 30{,}000{,}000$, the resilience per cubic inch may be 5 inch pounds, so that the volume of steel required for the spring is 2,016 cubic inches. Now if there are n plates, this volume may be taken as

$$n\,b\,t\,l \text{ (l being half the length of the longest strip),}$$
$$\text{or } n\,b\,t\,l = 2{,}016 \quad . \; . \; . \; . \quad (1),$$

which is one equation towards the solution of some problem.

EXERCISES.

1. What ought the initial curvature of a $\frac{1}{2}$-inch plate to be if the proof stress of the material is 30,000 lbs. per square inch (exists when the plate is straight) and E is 30,000,000?

Here, by formula of Art. 540,

$$30{,}000 = 15{,}000{,}000 \times \tfrac{1}{2}\,w,$$
$$w = \cdot 004.$$

The radius of curvature is $\frac{1}{\cdot 004} = 250$ inches.

2. If the above plate is 36 inches long, what is its initial dip, *oc*, in Fig. 367? Using the approximate formula $oc = l^2 \div 2\,r$, we have

$$oc = \frac{(\frac{1}{2} \,.\, 36)^2}{2 \times 250} = 0{\cdot}64 \text{ inch.}$$

3. If the spring is formed of six plates, the longest being 36 inches, the overlap on one side of plate on its neighbour is

$$36 \div 12, \text{ or } 3 \text{ inches.}$$

4. If the breadth of each plate is 3 inches,

$$D = \cdot 000259\ W.$$

5. The load W' which will produce the deflection 0·64 inch—that is, which will straighten the spring—is

$$W = \frac{0{\cdot}64}{\cdot 000259}, \text{ or } 2{,}490 \text{ lbs.}$$

6. How many plates of $\frac{3}{8}$ inch thick and 3 inches broad, the longest being 30 inches in length, of the above-mentioned steel will be required for a spring whose proof load is to be 1 ton?

$$2{,}240 = \frac{2n \times 3 \times \frac{9}{64} \times 30{,}000}{3 \times 30},$$

$$\text{or} \quad n = 8.$$

Hence the spring ought to be made of eight plates.

546. The answers to the above exercises guide us as to the best way of constructing carriage springs so that the strips throughout shall have the strength of the overlapping parts. The following rules are based upon the dimensions of these overlapping parts, which are merely little cantilevers.

If λ is the overlap, Fig. 367, and l is the half-length of the longest strip, so that n being the number of strips $n\lambda = l$; then t being thickness of each strip, b its breadth, d the deflection of the spring, f the maximum stress, E Young's modulus, W the load at A and at B, there being a load 2 W at the middle, it is evident that

$$f = 6\,\text{W}\,l \div n\,b\,t^2,\ \text{D} = 6\,\text{W}\,l^3/n\,\text{E}\,b\,t^3.$$

If f' is the proof stress, the resilience in a well-made spring is $f'^2 \div 6\,\text{E}$ inch pounds. Carriage springs are usually made of Bessemer steel, and we may take $f' = 30{,}000$ lbs. per square inch and $\text{E} = 30{,}000{,}000$.

Examples.—1. If the overlap λ is 2 inches, and there are 10 strips each $\frac{3}{8}$ inch thick, so that the half-length of the whole spring is $l = 20$ inches, $t = \frac{3}{8}$, and if $b = 2\frac{1}{2}$ inches; find the load which will deflect the spring 2 inches. Here $\text{D} = 2$

$$2 = 6\,\text{W}\,20^3 \div 10 \times 3 \times 10^7 \times 2\tfrac{1}{2} \times (\tfrac{3}{8})^3, \text{ and hence}$$
$$\text{W} = 5{\cdot}926 \times 10^5 \text{ lbs.}$$

If the plates were not of the same initial curvature—that is, if the spring was, in this respect only, badly made, then the law for deflection is still true for the badly made spring; but the rules for strength are untrue. We only calculate the increase in f due to a load W.

547. In the above theory I have assumed no friction between the plates. This friction makes the bending to be less for a given load if the load has been increasing, but if the load has been diminishing the bending will be greater than the formula (1) gives, on account of friction. When we test a carriage spring, the effect of friction in causing the deflections to be greater with diminishing than with increasing loads is very noticeable. I have already pointed out the unsuitableness (on this account) of these springs for all **measuring purposes,** and how suitable they are for carriages and under other conditions where vibrations must be rapidly stilled by frictional forces.

548. In a spring such as Fig. 367, if 2 W is the upward load

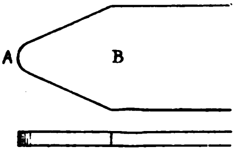


Fig. 368.

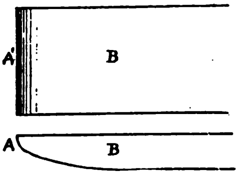


Fig. 369.

applied at the middle, the downward supporting forces at A and B are each W, and the end of each plate may be regarded as applying to the plate above it this force W. Thus in Fig. 368, if A B is the overlap of one plate on another, every part of the plate except the overlap part A B at each end is subjected to a bending moment W . A B everywhere, so that the change of curvature everywhere is the same.

Also the part A B ought to be fashioned like a beam of uniform strength and curvature fixed at B and loaded at A. We find from Exercise 2, Art. 540, that we can make it of

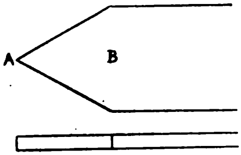


Fig. 370.

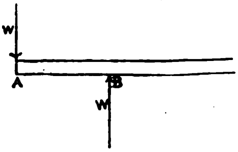


Fig. 371.

uniform strength and uniform curvature at the same time by varying the breadth of the plate but not its thickness, the proper shape being shown in Fig. 370. This is roughly approximated to in many springs where the ends are shaped as in Fig. 368.

If we keep the breadth constant, as in Fig. 369, but alter the thickness so as to have constant curvature everywhere (so that the end part of one plate may fit properly against another plate), we see from Exercises 3 and 4 that the thickness ought to vary as the cube root of the distance from A; but it is not possible to have also uniform strength in this part of the plate.

APPENDIX.

TABLE XIX.—USEFUL CONSTANTS.

Time. One sidereal day = 86,164 seconds.
Mean solar day = 86,400 seconds.
One year = 365·24224 mean solar days.

Length. British standard, the yard = 3 feet = 36 inches.
1 chain = 66 feet = 100 links.
1 mile = 1,760 yards = 5,280 feet = 80 chains = 8 furlongs.
1 nautical mile = 6080 feet (average).
= 10 cables = 1,000 fathoms (nautical).
1 fathom = 6 feet.

French standard, the metre = 39·37 inches = 3·2809 feet.
1 centimetre = ·3937 inches.
1 kilometre = ·6214 miles.
1 inch = ·0254 metre = 2·54 centimetres.
1 foot = 30·48 centimetres.
1 yard = ·9144 metres.
1 mile = 1,609 metres = 1·609 kilometres.

Surface. 1 square inch = 6·451 square centimetres.
1 square foot = ·0929 square metre.
1 square yard = ·8361 square metre.
1 acre = 43,560 square feet = 4,047 square metres.
1 square mile = 640 acres = 2·59 square kilometres.

Volume. 1 cubic inch = 16·387 cubic centimetres.
1 cubic foot = ·02832 cubic metre = 28·31 litres.
1 litre = ·22 gallon.

For many **practical calculations** it is **sufficiently correct** to take:—

Length. 1 inch = 25 millimetres, but 25·4 is more correct.
1 metre = 3 feet 3 inches and ⅜ths of an inch.
10 metres = 11 yards.
20 metres = 1 chain.
8 kilometres = 5 miles.

Surface. 1 square inch = 6½ square centimetres.
6 square yards = 5 square metres.
1 acre = 4,000 square metres.

Volume. 4 cubic yards = 3 cubic metres.
1 gallon = 4½ litres.
4 litres = 7 pints.
100 litres = 22 gallons.
1 cubic foot = 28·3 litres.

Weight. 1 gramme = 15½ grains.
10 kilogrammes = 22 pounds.
50 kilogrammes = 1 hundredweight.
1,000 kilogrammes = 1 English ton.

Speed. 1 knot = 1 nautical mile per hour.
= 1·15 miles per hour.
= 1·7 feet per second.
= 101 feet per minute.
= 51·5 centimetres per second.
60 miles per hour = 88 feet per second.

Weight and Force. British engineer's unit of force = the weight of the standard pound in London.
Weight of 1 lb. = 16 ounces = 7,000 grains = 453·6 grammes.
= 445,000 dynes.
= ·4536 kilogrammes.
1 ounce = 28·35 grammes.
Weight of 1 grain = 63·57 dynes.
1 kilogramme = 2·2046 lbs.
1 ton = 2,240 lbs. = 1,016 kilogrammes.
1,000 kilogrammes = 1 ton (metric) = 2,205 lbs. = ·9842 ton (British).

g = 981 centimetres per second per second.
= 32·2 feet ,, ,,

Value of g at London = 32·182 feet per second per second.
,, ,, Equator = 32·088 ,, ,, ,,
,, ,, the Poles = 32·253 ,, ,, ,,
Inertia or mass of a body = weight in lbs. at London ÷ 32·2.

1 gallon of water at 62° F. weighs 10 lbs. by English law.
1 cubic foot of water weighs 62·3 lbs.
1 cubic foot of air at 0° C. and 1 atmosphere weighs ·0807 lb.
1 cubic foot of hydrogen ,, ,, ,, ,, ·00557 lb.

For academic calculations we usually take the weights in pounds of 1 cubic foot of each of the following to be :—
Brickwork, 112 ; concrete, 150 ; grindstone and Portland stone, 131 ; granite and marble, 164 ; dry oak, 58 ; dry fir, 47 ; cork, 15 ; coal, 80 ; clay, 120.
For weights per cubic inch, see Art. 6.

Work, Energy. 1 erg = 1 dyne × 1 centimetre.
1 gramme-centimetre = 981 ergs.
1 foot pound = $1·356 \times 10^7$ ergs.
1 kilogramme metre = 7·233 foot pounds.
1 horse-power = 33,000 foot pounds per minute.
= 550 foot pounds per second.
= $7·46 \times 10^9$ ergs per second.
= 746 watts.
1 force-de-cheval = ·9863 horse-power.

Energy obtainable from 1 lb. of coal
= 8,500 Centigrade heat units.
= 15,000 Fahrenheit heat units.
= 12×10^6 foot pounds.

Joule's equivalent—
1 pound Fahrenheit unit of heat = 780 foot pounds.
1 pound Centigrade unit of heat = 1,400 foot pounds.

TABLE XX.—MODULI OF RIGIDITY.

Substance.	N = Modulus of Rigidity in Millions of Pounds per Square Inch.
Steel Plates, ¼ per cent. Carbon	13
” ” ½ ” ”	
” ” 1 ” ”	
Steel Boiler Plates	13·5
Cast Steel (tempered)	14·0
” ” (untempered)	12·0
Soft ” (unhardened)	11·0
” ” (hardened)	11·0
Cast Iron	5·0 to 7·6
Iron Boiler Plates	14·0
Wrought Iron Bars	10·5
” ” Plates	9·5
Soft Iron	10·8 to 11·3
Brass	5 to 5·5
Copper	5·6 to 6·7
Lead	·27
Zinc	5·1 to 5·4
Tin	2·2
Gold	4·0 to 5·6
Silver	3·8
Platinum	8·9 to 9·4
Aluminium	3·4 to 4·8
Delta Metal (rolled)	5·25
Gun ”	3·7
Phosphor Bronze	5·2
Glass	3·3 to 3·9
Wood	·1 to ·17
Granite	1·8
Marble	1·7
Slate	3·2

Authorities: Kelvin, Unwin, Wertheim, Cagniard Latour, Kowalski, Kiewiet, Katzenelsohn, Posati, Gray, Milne.

TABLE XXI.—MODULI OF COMPRESSIBILITY.

Substance.	Modulus of Compressibility in Pounds per Square Inch.	Temperature.	Authority.
Steel	$21\cdot6 \times 10^6$		Amagat.
Steel	$26\cdot7 \times 10^6$		Thomson's "Elasticity."
Iron	$21\cdot1 \times 10^6$		" "
Brass	$15\cdot3 \times 10^6$		" "
Copper	$17\cdot1 \times 10^6$		Buchanan.
Copper	$24\cdot4 \times 10^6$		Thomson's "Elasticity."
Delta Metal ...	$14\cdot4 \times 10^6$		Amagat.
Lead	$5\cdot3 \times 10^6$		"
Glass	$5\cdot8 \times 10^6$		"
Distilled Water ...	$3\cdot2 \times 10^5$	15° C.	Paglianni.
Alcohol	$1\cdot76 \times 10^5$	0° C.	Amaury and Deschamps.
Alcohol	$1\cdot62 \times 10^5$	15° C.	" "
Ether	$1\cdot35 \times 10^5$	0° C.	" "
Ether	$1\cdot15 \times 10^5$	15° C.	" "
Carbon Bisulphide.	$2\cdot32 \times 10^5$	14° C.	" "
Glycerine... ...	$5\cdot85 \times 10^5$	20·5° C.	Quincke.
Petroleum ...	$2\cdot11 \times 10^5$	16·5° C.	Martini.
Mercury	$7\cdot85 \times 10^6$	15° C.	Amaury and Deschamps.
Mercury	$4\cdot35 \times 10^6$	0° C.	Colladon and Sturm.
Mercury	$3\cdot74 \times 10^6$	0° C.	Amagat.

TABLE

Material.	Melting Point (Fahr.).	Specific Gravity.	Weight of One Cubic Foot in Pounds.	Breaking Stress, in lbs. per square inch.
				Tensile.
Soft Steel (unhardened)	2,400° to 3,000°	7·85	490	60,000 to 100,000
Soft Steel (hardened)				120,000
Cast Steel (untempered)				90,000 to 150,000
Cast Steel (tempered)				—
Steel Plates				60,000 to 80,000
Steel Bars				100,000 to 130,000
Cast Steel (drawn)		7·7	480	120,000
Steel Wire (English), drawn...				120,000 to 140,000
Steel Wire (Common), tempered blue ...		7·4	460	330,000
Steel Pianoforte Wire, English		7·73	480	
Manganese Steel, Cast ...				85,000
Nickel Steel, unhardened ...				75,000
Nickel Steel, hardened ...				190,000
Wrought Iron Bars and Bolts	3,000° to 3,300°	7·7	480	55,000 to 70,000
Wrought Iron Plates, with fibre...				51,000
Wrought Iron Plates (across fibre)				46,000
Wrought Iron Plates (mean)				48,500
Cast Iron	2,000° to 2,500°	7·2	450	14,000 to 30,000
Aluminium	1,300°	2·6	162	33,000
Aluminium (annealed) ...				13,500
Brass, Yellow	1,700° to 1,850°	7·8 to 8·4	490 to 525	17,500
Brass, Sheet				30,000
Brass, Tube				80,000 to 100,000
Brass, Cast		8·0	500	20,000
Brass, Wire				50,000
Copper, Wrought	2,000°	8·8	550	33,000
Copper, Cast				20,000
Copper Wire, hard-drawn ...		8·9	555	58,000
Copper Wire, annealed ...				47,000
Zinc, Cast	750°	7·0	436	7,500
Zinc, Sheet		7·2	450	30,000
Lead	615°	11·4	712	1,900
Lead, Cast		11·2	700	3,000
Tin	450°	7·4	462	4,600
Platinum Wire...	3,300°	21·5	1,340	50,000
Gold (drawn)	2,200°	19·2	1,200	38,000 to 41,000
Silver (drawn)	1,850°	10·4	650	42,000
Phosphor Bronze (cast) ...				55,000
Phosphor Bronze Wire (hard)...				100,000 to 150,000
Phosphor Bronze Wire (annealed)				50,000 to 60,000

XXII.

Breaking Stress, in lbs. per square inch.		Stress which produces Permanent Set, in lbs. per square inch.			Safe Limit of Stress, in lbs. per square inch.			Young's Modulus of Elasticity, in millions of lbs. per square inch.
Compressive.	Shear.	Tensile.	Compressive.	Shear.	Tensile.	Compressive.	Shear.	
		35,000		26,500	17,700	17,700	13,000	30
								30
								30
								36
								29 to 42
								27
								28
								26
								29
		45,000						
		35,000						
		56,000						
50,000	50,000	24,000	24,000	20,000	10,000	10,000	7,800	29
								25
								27
		20,000	20,000	15,000	10,000	10,000	7,800	26
50,000 to 120,000	28,500	10,500	21,000	8,000	3,500	10,500	2,700	14 to 23
10,500		7,000		5,000	3,600		2,700	9
								9·2
								14·2
58,000		4,300	4,000	3,000	3,600	3,200	2,300	15
		3,200						
7,300		1,500						·72
								6
								24
								12
								11
		20,000		14,500	10,000		7,400	14

TABLE

MATERIAL.	Melting Point (Fahr.).	Specific Gravity.	Weight of One Cubic Foot in Pounds.	Breaking Stress, in lbs. per square inch.
				Tensile.
Aluminium Bronze, Cu 95 Al 5		8·25	515	60,000
Aluminium Bronze, Cu 90 Al 10		7·7	480	100,000
Manganese Bronze				65,000 to 85,000
Delta Metal (cast)				47,000
Delta Metal (rolled)				74,000
Muntz Metal				45,000
Sterro Metal				60,000
Gun Metal	1,900°	8·6	536	25,000 to 50,000
Ebony		1·7	73	
Oak, European...		·93	58	14,500
Mahogany, Spanish		·85	53	15,000
Ash		·8	50	17,500
Pitch Pine		·7	44	12,000
Red Pine				10,500
Birch		·54	34	14,500
Beech		·7	44	11,500
Fir, Larch		·53	33	11,000
Fir, Spruce		·54	34	12,500
Hornbeam		·76	47	15,000
Teak, Indian		·78	49	15,000
Lancewood		·95	59	20,000
Elm, British		·55	34	14,000
Lignum Vitæ		·65 to 1·33	41 to 83	16,000
Sycamore				13,000
Cedar of Lebanon		·59	37	11,400
Granite		2·7	170	
Marble		2·8	175	700
Limestone		2·8	175	
Sandstone		2·3	144	800
Slate		2·8	175	11,000
Brick, Red		2 to 2·2	125 to 137	300
Brick, Fire				
Brickwork		1·8	112	
Concrete...		1·9 to 2·2	119 to 137	
Leather				4,000
Hemp Rope (in ordinary state)		1·3	—	10,000
Glass, Plate		2·7	170	2,700
Ice		·917	57	
Quartz Fibre (Professor Boys')				140,000

XXII. (*continued*).

Breaking Stress, in lbs. per square inch.		Stress which produces Permanent Set, in lbs. per square inch.			Safe Limit of Stress, in lbs. per square inch.			Young's Modulus of Elasticity, in millions of lbs. per square inch.
Compressive.	Shear.	Tensile.	Compressive.	Shear.	Tensile.	Compressive.	Shear.	
		17,000						12
		50,000						13
								10
18,000								
10,000	2,300							1·5
								1·4
								1·6
6,000	650							1·4
								1·65
								1·35
								1·1
								1·6
12,000								2·4
7,000								
10,000								
								1·04
								·48
17,000								7
6,000								6
5,000								
5,000								2
15,000								13
800								
2,000								
500								
2,000								
								·025
26,000								8
								8·5

LOGARITHMS. TABLE

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

XXIII.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7908	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9809	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS. TABLE

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

XXIV.

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2		4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE XXV.

Angle.	Radians.	Sine.	Tangent.	Co-tangent.	Cosine.		
0°	0	0	0	∞	1·0000	1·5708	90°
1	·0175	·0175	·0175	57·2900	·9998	1·5533	89
2	·0349	·0349	·0349	28·6363	·9994	1·5359	88
3	·0524	·0523	·0524	19·0811	·9986	1·5184	87
4	·0698	·0698	·0699	14·3006	·9976	1·5010	86
5	·0873	·0872	·0875	11·4301	·9962	1·4835	85
6	·1047	·1045	·1051	9·5144	·9945	1·4661	84
7	·1222	·1219	·1228	8·1443	·9925	1·4486	83
8	·1396	·1392	·1405	7·1154	·9903	1·4312	82
9	·1571	·1564	·1584	6·3138	·9877	1·4137	81
10	·1745	·1736	·1763	5·6713	·9848	1·3963	80
11	·1920	·1908	·1944	5·1446	·9816	1·3788	79
12	·2094	·2079	·2126	4·7046	·9781	1·3614	78
13	·2269	·2250	·2309	4·3315	·9744	1·3439	77
14	·2443	·2419	·2493	4·0108	·9703	1·3265	76
15	·2618	·2588	·2679	3·7321	·9659	1·3090	75
16	·2793	·2756	·2867	3·4874	·9613	1·2915	74
17	·2967	·2924	·3057	3·2709	·9563	1·2741	73
18	·3142	·3090	·3249	3·0777	·9511	1·2566	72
19	·3316	·3256	·3443	2·9042	·9455	1·2392	71
20	·3491	·3420	·3640	2·7475	·9397	1·2217	70
21	·3665	·3584	·3839	2·6051	·9336	1·2043	69
22	·3840	·3746	·4040	2·4751	·9272	1·1868	68
23	·4014	·3907	·4245	2·3559	·9205	1·1694	67
24	·4189	·4067	·4452	2·2460	·9135	1·1519	66
25	·4363	·4226	·4663	2·1445	·9063	1·1345	65
26	·4538	·4384	·4877	2·0503	·8988	1·1170	64
27	·4712	·4540	·5095	1·9626	·8910	1·0996	63
28	·4887	·4695	·5317	1·8807	·8830	1·0821	62
29	·5061	·4848	·5543	1·8040	·8746	1·0647	61
30	·5236	·5000	·5774	1·7321	·8660	1·0472	60
31	·5411	·5150	·6009	1·6643	·8572	1·0297	59
32	·5585	·5299	·6249	1·6003	·8480	1·0123	58
33	·5760	·5446	·6494	1·5399	·8387	·9948	57
34	·5934	·5592	·6745	1·4826	·8290	·9774	56
35	·6109	·5736	·7002	1·4281	·8192	·9599	55
36	·6283	·5878	·7265	1·3764	·8090	·9425	54
37	·6458	·6018	·7536	1·3270	·7986	·9250	53
38	·6632	·6157	·7813	1·2799	·7880	·9076	52
39	·6807	·6293	·8098	1·2349	·7771	·8901	51
40	·6981	·6428	·8391	1·1918	·7660	·8727	50
41	·7156	·6561	·8693	1·1504	·7547	·8552	49
42	·7330	·6691	·9004	1·1106	·7431	·8378	48
43	·7505	·6820	·9325	1·0724	·7314	·8203	47
44	·7679	·6947	·9657	1·035	·7193	·8029	46
45	·7854	·7071	1·0000	1·0000	·7071	·7854	45
		Cosine	Co-tangent	Tangent	Sine	Radians	Angle

INDEX.

The References are to Pages.

A

Absorption Dynamometers, 233
Acceleration, 13, 262
—— Angular, 25
—— Linear, 13
—— in Mechanism, 583
—— in S.H.M., 548
Accumulator, 187
Advantage, Mechanical, 59, 88
Air-vessel, 506
Algebra, 1
Alloys of Copper, 288
—— of Iron and Manganese, 287
—— of Iron and Nickel, 287
Aluminium, 285, 289
—— Bronze, 289
Ampères, 92
Amplitude, 20
—— of Vibration, 565
Analogies, Linear and Angular Motion, 593
Angle, Measurement in Degrees, Radians, 23
—— of Repose, 110, 112, 125
—— Sine, Cosine, Tangent of, 23
—— of Twist, 225, 349
Angular Acceleration, 25
—— Motion, 586
—— Ratio, 571
—— Velocity, 25
Aqueduct, 208
Arch, 113, 391, 457
—— Least Thrust at Keystone of, 165
—— Line of Resistance of, 163
—— Load Carried by, 163
—— Masonry, 166
—— Metal, 478
—— Middle Third of Section, 163
—— Rib of Iron, 166
—— Thrust at Crown of, 163
Arched Rib, 162
Area, Centre of, 136
—— of Curve by Integration, 17
—— of Curve by Planimeter, 7
—— of Irregular Figure, 10
—— Moment of Inertia of, 136
—— Projection of, 27
Areas of Figures, 6
Armstrong's Hydraulic Crane, 194
Ash, 278
Attwood's Machine, 64, 245
Average Curvature, 22
—— Space, Time, 272, 274
—— Velocity, 12
Axial Flow Turbine, 531
Axis, Neutral, 381
—— of Suspension, 560
Axles, Railway, 83
Ayrton-Perry Coupling, 239
Ayrton and Perry's Dynamometer, 225

B

Babbit's Metal, 289
Bailey's Testing Machine, 294
Balance, Chemical, 118
—— Compensation of, 559
—— of Watch, 558
—— Spring, 40
Balancing of Hoists, 201, 203, 204, 205
—— of Locomotive, 609
—— of Machines, 606
—— of Reciprocating Body, 611
Ball-Bearings, 69, 86
Ballistic Pendulum, 499
Barge, 208
Barker's Mill, 486
Basin, Water in, 539
Bauschinger's Experiments, 310
Beam, 127
—— Bending Moment Diagrams, Six Cases, 415, 428
—— Cast Iron, Weight of, 9
—— Curvature of a, 386
—— Fixed at the Ends, 413, 443
—— Not Rectangular in Section, 417
—— Shear Stress in, 459
—— Shearing Force in, 414, 427
—— Stiffness of, 431
—— of Uniform Depth, 446
—— of Uniform Strength, 419
—— Well-known Rules about, 410
—— Without Compression, 471
Bear, Punching, 179
Bearing, Diameter of, 68
—— Friction at, 68
Bearings, Ball, 69, 86
—— of Fans, Centrifugal Pumps, Dynamos, 70
Beech, 278
Bell-mouthed Pipe, 522
Belt, Centrifugal Force on, 327
—— Length of Crossed, 26
—— Length of Open, 26
—— Slip of, 86
Belting, Creep in, 86
—— Strength of, 281

Belts, Difference of Pulls in, 226
Bending, 380
—— and Crushing, 457
—— Moment, 128, 383
—— Moment in Beams (more difficult portion), 441
—— Moment Diagrams, Six Cases, 414, 427
—— of Prism 369
—— of Struts, 464
—— Theory of, 318
—— Unsymmetrical, 387
Bends in Pipes, 519
Bevel Gearing, 36, 95, 102
Bicycle Problems, 49
—— Resistance to Motion of, 50
Bicyclist, 33
Bifilar Suspension, 564
Blue Prints, 2
Block, Pulley, 98
—— Snatch, 102
Blocks and Tackle, 98
Board of Trade Rules for Railway Girders, 305, 402
Body Turning about an Axis, 114
Boiler, Bursting of, 181
—— Cylindrical, 320
—— Spherical, 320
—— Strength of, 319
—— Testing of Water-tube, 93
Bolts, 100
—— Tightening of, 311
Boundary Condition, 372
Boussinesq, M., 379
Boy's, Prof., Quartz Fibres, 299
Braced Triangular Pier, 148
Bracket, 459
Brake Block, 240
—— Horse-power, 91, 95
—— Knot, 236
Brass, 288
—— Weight of, 9
Bricks, 276
Bridge, Forth, 299
—— Suspension, 161
Bronze, 288
Brotherhood's Water-pressure Engine, 191
Bulk, Diminution in, 316
—— Modulus of Elasticity of, 316
Bull Engine, **important example**, 257, 269
Bullet, 14, 79, 500
Bursting of Boiler, 181
Buttress Thread, 101
Buttresses, 166, 458

C

Calculus, Differential and Integral, 15
Calorific Power of Coal-Gas, 91
Cam, 37
Canal, 170, 206
Candle, 496

Cannon, Momentum of, 486
Carbon, 280, 281, 285
Case-Hardening of Wrought Iron, 285
Castings, Chilled, 183, 284
—— Cooling of, 282
—— Internal Strains in, 282
—— Malleable, 284
—— Softening of Hard, 281
—— Strain in, 183
Cast Iron, 280
—— Density of, 281
—— Shell, 173
—— Toughening of, 284
—— Weight of, 9
Cast Steel, 183
Catenary, 169
Cauchy's Theorem, 363
Cedar, 278
Cement, 276
Centre of Area, 136
—— of Fluid Pressure, 215
—— of Gravity, 6, 136
—— of Inertia, 136
—— Instantaneous, 575
—— of Mass, 135
—— of Oscillation, 560
—— of Percussion, 499, 560
—— of Pyramid or Cone, 9
Centrifugal Force, 597, 606
—— Force in Belts or Ropes, 327, 605
—— Pump, 33, 264, 540
Centripetal Force, 603
—— Acceleration, 603
C.G.S. System of Units, 267
Chain Gearing, 232, 572
—— Hanging, 167
—— Horizontal Pull of, 162
—— Loaded, 161
Chains, Admiralty Rule for, 315
Change Wheels, 101
Charcoal Iron, 284
Chemical Balance, 118
—— Energy, 42
Chemistry, 290
Chilled Castings, 183
Chipping, 488
—— and Filing, 56
—— Hammer, 264
Chisel, Tempering of, 291
Chree, Dr., 371, 380
Chromium, 287
Chronometer, 649
Chuck, Elliptic, 577
Circle, Circumference of, 6
Circuital, 30
Circular Stream Lines, 541
Clinure, 29, 122
Clock, 38
Closed Polygon, 120
Coal, Energy of 1 lb. of, 42, 43
Co-efficient of Friction, 65
—— of Friction, Alteration of, 75, 236
—— of Viscosity, 76
Collars, Leather, 170, 176
Colours in Tempering, 291, 292

The References are to Pages.

Columns, 473
Combination of S.H.M.'s, 555
Combined Bending and Twisting, 353
Commercial Tests, 306
Compound Interest Law, 20, 99, 230
—— Pendulum, 559
—— Wheel and Axle, 102
Concrete, 183, 277
Conduit, 516
Cone, Area of Surface of, 8
—— Rolling of a, 577
—— Surface of, 8
Coned Pulley, 37
Cones, Stepped, 26
Conical Pendulum, 546
Confluent, 30
Connecting Rod and Crank, 222
Construction of Gun, 330
—— Materials Used in, 275
Contracted Vein, 533
Contraction of Section, 306
Cooling of Castings, 282
Copper, 287
—— Wire, 289
Cores, 10, 282
Cork, Floating, 547
Corrugated Flue, 326
Cosine of an Angle, 23
Cottars, 101
Cotton Press, 184
Couple, 127
Coupling, 225
—— Ayrton-Perry, 239
—— Dynamometer, 96
—— Flange, 227
—— Rod, 471
Crab, 102
Crane, 88
—— Double-Powered, 195
—— Hook, 458
—— Hydraulic, 193
Crank and Connecting-Rod, 222
—— Overhung, 355
Creep in Belting, 86
Creeping, 299
Cricket Bat, 500
Critical Velocity, 83
Crosshead of Engine, 113, 547
Crushing and Bending, 457
Cupola, 282
Curvature of a Beam, 386, 405
—— Definition of, 22
—— Radius of, 22
—— of Steel Spring, 388
Curve of Cosines, 555
—— Elastic, 390
—— Integral of, 18
—— Logarithmic, 568
—— of Sines, 78, 555, 565, 568
Curved Rollers, 522
Curves, Damping, 78
—— Trochoidal, 573
Cutting of Metals, 45
Cylinder, Rotating, 369
—— Thick, 183, 323

The References are to Pages.

D

Damping Curves, 78
—— of Vibration, 566
D'Arcy's Experiments, 85
Dash-pots, 239
Deadloads, 305
Definition of a Radian, 23
Definitions of Trigonometrical Ratios, 23
Deflection due to Shearing, 334, 460
—— in Beams, 407, 429, 431, 460
Delta Metal, 288
Descriptive Geometry, 124
Diagonal Pieces, 162
—— of Velocities, 583
Diagram of Accelerations, 583
—— of Bending Moment, six cases, 413–427
Diagrams of Shearing Force, 427
Differential Co-efficient, 16
—— Pulley Block, Efficiency of, 104
—— Pulley Block, 103
Diminution of Swing, 563
Displacement of a Ship, 55
—— Relative and Absolute, 35
Distance of Rubbing, 68
Draining of Fields, 516
Drawing Office, 1
Drill, Watchmaker's, 291
Dunkerley, Mr., Stability of Shafts, 477
$\frac{dy}{dx}$, 15
$\frac{d^2y}{dx^2}$, 18
Dynamo Machine, 43, 92
Dynamometer Coupling, 96
—— Froude's, Thorneycroft, 234
—— Hefner-Alteneck, 235, 240
—— Raffard's, 86
—— Transmission, 92
—— Transmission and Absorption, 233

E

Earth, Pressure of, 167
—— Rankine's Rule for, 349
Earthquake, 497–617
Eccentric, 222
Economical Efficiency, 89
Economy, General, 339
Eddies, 518
Effect of Friction, 60
Efficiency, 95
—— Economical, 89
—— Hydraulic, 529
—— of Screw Jack, 110
—— of Turbine, 522
Elastic Curve, 390
Elasticity, More Difficult Theory of, 361
—— of Bulk, Modulus of, 317
—— Shearing, 333
—— Young's Modulus of, 299

The References are to Pages.

Electric Energy, 92
—— Lamps, 43
—— Motor, 42
—— Supply Station, 92, 94
Electrical Horse-power, 92, 95
Electricity, 290
Ellipse, Area of, 8
Elliptic Trammel, 577
—— Chuck, 577
Elm, 278
Energy, Chemical, 42
—— Indestructibility of, 59
—— Kinetic, Potential, 40, 242, 505, 511
—— Loss of, due to Friction, 67
—— Loss of, in Change of Strain, 303
—— Lost, Fluid in Pipe, 53, 84
—— of 1 lb. of Coal, 42–43
—— Pressure, 505–511
—— in a Rotating Body, 247, 254
—— Storage of, in Fluids, 326
—— Store of, 188
—— Stored up in any Machine, 253
—— Strain, 85
—— of Strained Spring, 41
—— Transmission of, Hydraulically, 170
—— Waste of, in Conductors, 43
—— of Waterfall, 59
Engine, Bull, 257
—— Hero's, 486
—— Steam, Gas, Oil, 43
—— Water Pressure, 189–191
—— Willans', 611
English, Colonel, 55
Epicyclic Train, 35
Epicycloids, 573
Equilibrant, 30
—— of Forces in One Plane, 122
Equilibrium, Conditions of, 121
—— in One Position, 104
Equi-pollent Loads, 366
Equi-potential Surfaces, 218
Errors of Observation, 62
Escapements, 554
Ewing's, Prof., Extensometer, 294
Expansion, Cubical, Co-efficients of, 5
—— Linear, Co-efficients of, 4
—— of Girder, 133
Experiments by Bauschinger, 310
—— by D'Arcy, 85
—— by Fairbairn, 309
—— by M. Tresca, 301
—— by Reynolds, Prof. O., 81
—— by Tower, M. B., 73, 81
—— by Wöhler, 305, 309
—— on Attwood's Machine, 245
—— on Balancing, 608
—— on Crane, 88
—— on Deflection of Beams, 431
—— on Fluid Friction, 77, 79
—— on Forces at a Pin Joint, 151, 152
—— on Friction, 60
—— on Friction of Cords, Belts, 228
—— on Friction of a Machine, 63
—— on Friction and Speed, 71, 72
—— on Guns, 14

The References are to Pages.

Experiments on Hydraulic Jack, 175
—— on Inclined Plane, 104
—— on Jack, 63
—— on Loaded Links, 160
—— on Moments, 115
—— on Shearing, 332
—— on Strain, 293
—— on Triangle of Forces, 56
—— on Twisting of Wires, 349
—— to Find Co-efficient of Friction, 67
—— to Find Energy in Rotating Body, 247
—— to Find Modulus of Rigidity, 563
—— with Discs in Fluids, 78
—— **with Gas-engine,** 91
—— **with Oil-engine,** 91
—— with Pulley Block, 103
—— with Small Carriage, 262
Extensometer, Prof. Ewing's, 294

F

Factor, Load, 94
—— of Safety, 304
Fairbairn's Experiments, 309
Falling Weight, Work Done by, 39
Fatigue, 559
Ferguson's Paradox, 36
Fir, White, 277
Firwoods, 277
Flat Plate, 330
Flexible Pipe, 197
Flexural Rigidity, 634
Flexure, Non-uniform, 378
Flow of Gas from Orifice, 537
—— of Metals, 335
—— of Water, Sudden Stoppage of, 192
—— of Water through Notch, 514, 536
Flue, Corrugated, 331
Fluid Friction, 76
—— Friction, Cause of, 80
Fluids in Motion, 505
—— Steady Motion in, 532
—— Storage of Energy in, 326
—— Whirling, 216
Force Acting on a Body, **important example,** 265
—— Centrifugal, 597, 606
—— Constant, 267
—— Due to Pressure of Fluids, 214
—— Lines of, 217
—— Moment of, 115
—— of a Blow, 490
—— **of a Blow, Time Average of,** 273
—— of Friction, 64
—— Polygon, 124
—— Shearing, 383, 385
—— Space Average of a, 45
—— Time Average of a, 45
Forces Acting at a Point, 123
—— Graphical Representation of, 30
—— in One Plane, Analytical Rule, 121
—— in Space, Examples on, 143, 144, 145
—— in Structures, Determination of, 156

The References are to Pages.

Forces Not All in One Plane, 124
—— Parallelogram of, 30
—— Resultant of, 29
—— Triangle of, 30, 56
Forging, Hydraulic, 170
Forth Bridge, 299
Four-Bar Kinematic Chain, 577
Fracture Surface, 306
Frame of a Machine, 35
Framed Structures, 143
Francis, Mr., 536
Frequency, 20
Friction and Speed, 71
—— at Bearing, 68
—— at Different Speeds, 73
—— at Joints, 112
—— at Leather Collar, 171, 176
—— Between Cord and Pulley, 228
—— Circles, 113
—— Co-efficient of, 65
—— Cumulative Effect of, 99
—— Effect of, 60
—— Experiments on, 60
—— Fluid, 76
—— Fluid, Experiments of, 77, 78
—— Force of, 64
—— Internal, 495
—— Laws of Fluid and Solid, 80
—— Never Negligible, 248
—— of a Pivot, 68, 71
—— of Pulley, 58, 248
—— of Railway Brake, 75
—— of Ships, 56
—— on Inclined Plane, 107, 108
—— Quasi-Solid, 177
—— Skin, 54
—— Statical, 75
—— Wheels, 69
Frictional Loss, Reduction of, 68
Frictionless Hinges, 151
Froude, 54
Froude's Dynamometer, 234, 238
Fuel Consumption, 93
Fuller, Prof., 164
Functions, Graphing of, 21

G

Galileo, 244
Gas-Engine, 48
—— **Experiment with a**, 91
Gas Flowing from Orifice, 537
Gauche Polygon, 121, 143
Gauge Pressure, 505
Gearing Chain, 232, 572
—— Spur, Bevel, 95, 102
General Plane Motion, 575
Geology, 275
Geometry, Practical, 1
German Silver, 289
Girder, 398
—— Bending Moment in, 393, 396
—— Board of Trade Rules for, 305
—— Calculation of Forces in, 151

The References are to Pages.

Girder, Diagonal Braces of, 393, 396
—— Flanges of, 393, 396
—— Railway, 401
—— Rolled, 419, 421
—— Shearing Force in, 393, 396
—— Web of, 393, 397
Glass, 279
—— Sudden Cooling of, 279
—— Toughened, 280
Graphical Statics, 1, 149, 151
Graphing of Functions, 21
Greenhill, Prof., on Stability of Shafts, 475
Grid, Hydraulic, 212
Guide Blades, 529
Gun Experiments, 14
—— Lifting of, 199
—— Making of, 325
—— Metal, 288
—— Tensions in, 183
—— Wire, 325
Gyration, Radius of, 138, 560
Gyrostat, 498

H

Hammer, Chipping, 264
—— Tilt, 500
Hammering of Iron, 285
Hanging Chain, 167
Hardening, Case, 285
—— of Steel, 290
Harmonic Motion, 20
—— Law, 20
Heat, 290
—— Energy, Unit of, 42
Heavy Disc, Vibration of, 568
Hemp-Packing, 177
Hero's Engine, 486
Hinge, Quasi, 166
Hinged Arch, 114
Hoists, Balanced, 178
—— Balancing of, 201, 203, 204, 205
—— Hotel, Mill, Warehouse, 200
Horse-power, 39, 89
—— Brake, 91
—— Electrical, 92
—— Indicated, 92, 95
Hydraulic Crane, 193
—— Efficiency of Turbine, 529
—— Forging, 170
—— Grid, 212
—— Intensifier, 186
—— Jack, 68, 178
—— Jack, Efficiency of, 175
—— Limestones, 276
—— Mean Depth, 85
—— Power Company, 94, 187
—— Power, Loss of, 53
—— Press, 170
—— Ram, 501
—— Transmission, 52
Hydraulics, 170
Hypocycloids, 573

The References are to Pages.

I

I, for Various Sections, 251
Idea of Law of Dependence of two Variable Things, 63
—— of a Rate, 14
—— of Slope, 15
—— of Velocity, 12
Impact, 487, 505
Impulse, 264
Inclined Plane, 99, 104
—— Friction on, 107
Indiarubber, 305
—— Cup, 177
—— Shaft, 318
—— Spring, 41
Indicated Horse-power, 95
Indicator Diagram, 40, 274
—— Centre of, 149
—— Inertia, 40
—— Moment of, 135
—— Principal Moments of, 137
—— Rule for, 7
Instantaneous Centre, 575
Integral of a Curve, 18
Integrals, Table of, 16
Intensifier, 186, 188
Involute of Circle, 573
Inward Radial Flow Turbine, 531
Iron, Alloys of, 287
—— Cast, 280
—— Charcoal, 284
—— Galvanised, 287
—— Pig, 284
—— Rolling and Hammering, 285
—— Wrought, 284
Irregular Figure, Approximate Area of, 10

J

Jack, Hydraulic or Screw, 63, 178
Jet Pump, 516
Joint, Masonry, 458
Joints, Riveted, 336
—— with Friction, 112
Joule, 42
Journal, Friction at, 68
Journals and Footsteps, 73

K

Kater's Pendulum, 560
Kelvin, Lord, 302
Kelvin's Analogy, 358
—— Tide-Predicting Machine, 555
Kerosene, Energy of, 42
Keys, 101
Keystone of Arch, 165
Kinematics, 1
Kinetic Energy, 242, 505–511
—— of Any System, 588
Kinetics of Mechanism, 220

The References are to Pages.

Kirchhoff, 366
Knife Edges, 118
Knot Brake, 236
Kohlrausch, 303

L

Laboratory, Mechanical, 1
Ladder, Example on, 134
Lake of Water, 512
Landing Stage, 206
Larch, 278
Lathe, Screw-Cutting, 101
Law Connecting Variable Things, 63
—— of Crane, 89
—— of Friction for a Machine, 63
—— of Moments, 115
—— of Worth, 58
Laws of Fluid and Solid Friction, 80
Leather Collars, 170, 176
—— Friction at, 171
Level Surface, 22
Lever, 116
Limestones, 275
—— Natural Hydraulic, 276
Line of Resistance, 113, 126, 161, 163
—— Projection of, 27
—— Vertical, 22
Lines of Force, 217
Link Motions, 584
Link Polygon, 129
Links, Loaded, 160
Live Load, 305
Load Factor, 94
—— Rolling, 402
Loaded Chain, 161
—— Links, 160
Loads, Equi-pollent, 366
—— Suddenly Applied, 306
Loam, 282
Local Strengthening, 306
Lock Nuts, 311
Locomotive, Pull of, 76
Logarithmic Curve, 568
Logarithms, 1
Loss of Energy, by Impact, 488
—— of Energy due to Friction, 67
—— of Energy due to Fluid in Pipe, 84
—— of Energy in Change of Strain, 303
—— of Energy per lb. of Water, 518
—— of Power at Different Parts of Engine, 90
Love's Treatise on Elasticity, 361
Lowell Formula for Rectangular Notch, 515, 536
Lubricant, 66
—— Body of, 75

M

M, for Various Sections, 251
—— of a Wheel, 247, 248, 254
Machine, Bailey's Testing, 294

The References are to Pages.

Machine, Cotton Baling, 186
—— Design, 70
—— Kinetic Energy Stored up in, 253
—— Riveting, 170
Machinery, Quick Speed, 530
—— Water Pressure, 83
Machines, Balancing of, 606
—— Forging, Welding, Punching, Stamping, Shearing, 196, 198
—— Shearing, 335
—— Steadiness of, 252
Magnet, 564
Mahogany, 278
Mainspring of Timekeeper, 554
Malleable Castings, 284
Manganese, 284
—— Bronze, 289
Marbles, 275
Masonry Joint, 459
Mass, 40
—— Centre of, 135
Materials, Behaviour of, 290
—— Used in Construction, 275
—— Weights of, 9
Mathematical Tables, 24
Mean Depth, Hydraulic, 85
Mechanical Advantage, 59, 88
—— Hypothetical, 102
—— of Hydraulic Jack, 174
Mechanism, 220, 570
—— Four-Link, 35
—— Kinetics of, 220
—— Quick Return, 578
Mechanisms, Acceleration of, 583
—— Velocity of, 583
Memel, 277
Mensuration, 1, 6
Metal, 288
—— Arches, 478
—— Babbit's, 289
—— Delta, 289
—— Flow of, 298, 301, 335
—— Gun, 289
—— Muntz, 289
—— Sterro, 289
—— White, 289
Mills, Driving of, 515
Milne, Prof., 401
Mitis Castings, 287
Modulus of Elasticity of Bulk, 317
—— of Rigidity, 333, 563
—— of Section, 398
—— Young's, 299, 305
Moment, Bending, 128, 384
—— Law of, 115
—— of a Force, 115
—— of Inertia, 135, 136
—— of Inertia of a Circle, 141
—— of Inertia of Area, 149
—— of Inertia of Rectangle, 141
—— of Inertia of Sections, 398
—— of Momentum, 500, 528, 587
—— Work done by, 115
Momental Ellipse, 142
Momentum, 263
Momentum, Important Example, 274
—— of Cannon, 486
—— of Shot, 486
—— Tangential, 524
Mortar, 276
Motion, Fluids in, 505
—— of Rotation, 127, 592
—— of Translation, 121, 592
—— Periodic, 546
—— Produced by Blow, 498
Motions, Link Value, 584
Moulder, 9
Moulding, 282
Muntz Metal, 88

N

Neutral Axis, 381
—— Surface, 381
Newton's Second Law, 602, 604
Nickel, 287, 289
Non-Redundant Frame, Criterion for, 148
Norway Spruce, 277
Notch, 514
—— Gauge, 515, 536
—— Rectangular, 536
—— Triangular, 514, 536
Nozzles, 513
Nuts, 311

O

Oak, English, 278
Oil Engines, Experiments with, 91
—— Engines, 43
—— Sperm, 66
—— Tester, Thurston, 81
Orifice, 512
—— Gas Flowing from, 537
—— Rectangular, 535
—— Sharp-Edged, 513, 534
—— Triangular, 535
—— Water from, 512
Oscillating Cylinder Engine, 578
Outward Radial flow Turbine, 531
Overhauling, 96, 104, 109
Overhung Crank, 355

P

Packing Hay, 185
Parabola, 162, 168
Parabolic Rib, 162
Paraboloids of Revolution, 218
Paradox, Ferguson's, 36
Parallel Motion, Watt's, 69
Parallelogram of Forces, 30
Patterns, 10, 282
Peaucellier Cell, 578
Pendulum, Ballistic, 499, 500
—— Compound, 559
—— Conical, 546

The References are to Pages.

Pendulum, Equivalent Simple, 550
—— Impulse given to, 554
—— Simple, 242
—— Simple, Time of Swing, 551
Percussion, Centre of, 499
Periodic Motion, 546, 592
—— Not S.H.M., 554
—— Time, 20, 546
—— Time of Balance, 558
Permanent Axes, 607
—— Set in Wire, 300
Perry's, Mr. James, Syphon, 517
Phosphor Bronze, 288
Phosphorus, 284, 287
Physics, 290
Pianoforte Wire, 289
Piezometer, 173
Pig-Iron, Puddling and Refining of, 284
Pile Driver, 273, 489
Pin, Resultant Force at, 112
Pine, Red, 277
Pins, 101
Pipe, Bell-Mouthed, 522
—— Flexible, 197
—— Resistance to Motion of Fluid in a, 76
—— Strength of, 182, 319
—— Suddenly Enlarged, 518
—— Wooden, 183
Pipes, Bends in, 519
—— Flow of Water in, 53
Piston-Rod, 309
Pitch of Screw, 100
Plane Motion, 576
Planimeter, 2, 7
Plastic Elongations, 301
Plasticity, 301
Plate, Flat, 330
Platform, Weight of, 402
Plotting on Squared Paper, 62, 80, 89, 91, 92, 229
Poisson's Ratio, 343
Polygon, Closed, 120
—— Gauche, 121
—— Link, 129
—— of Forces, 124
—— Unclosed, 124
Portland Cement, 277
Potential Energy, 40, 242, 505, 511
Poundage of Steam, 95
Power, Loss of, at Different Parts of Engine, 90
—— Misuse of this Expression, 89
—— of a Stream, 515
—— Transmission by Shafts, 224
Precession of Top, 596
Press, Cotton, 184
—— for Packing Hay, 185
—— for Warehouses, 185
—— Hand, 185
—— Hydraulic, 170
Pressure Energy, 505, 511
—— Gauge, 505
—— *of* a Fluid, 215
—— *of* Earth, 167, 348

The References are to Pages.

Pressure of Water, 167
—— on Immersed Surface, 215
Principal Moment of Inertia, 139
—— Stress, 354
—— Stresses, 364
Principle of Work, 120
Prints, 10, 282
Prism, Centre of Gravity of, 9
—— of Elliptic Section, 376
—— Twisting or Bending of, 369
—— Volume of, 9
Prismatic Body, Volume of, 9
Projectile, 245
Projection of Area, 27
—— of Line, 27
Propeller Shaft, 82, 100
Propulsion of Ships, 52, 525
Protractor, 23
Puddling of Pig-Iron, 284
Pull, How it is Exerted, 292
Pulley Block, 98
—— Block, Efficiency of, 98
—— Coned, 37
—— Friction of, 58
—— Rim of, 283
Pump, 505
—— Centrifugal, 33, 264, 508
—— Double-Acting Force, 506
—— Efficiency of, 43
—— Force, 506
—— Jet, 516
—— Lifting, 505
—— Mechanism, 578
Punching, 335
—— Bear, 179

Q

Quantity, Scalar, 29
—— Vector, 29
Quartz Fibres, 299
Quasi-Hinge, 166
—— Rigidity, 497
—— Solid Friction, 177
Quicklime, 276
Quick Return Mechanism, 578

R

Rack, 573
Radial Gears, 584
Radian, Definition of, 23
Radius of Curvature, 22, 388
—— of Gyration, 136, 251, 560
Rafford's Dynamometer, 86
Railway Axles, 83
—— Brake, Friction of, 75
—— Girder, 133, 402
Ram, Hydraulic, 503
Rankine, 319
Rankine's Rule for Earth, 349
Rate, Algebraical Representation of, 15
Ratio, Velocity, 59

Rectangular Notch, 536
—— Orifice, 534
Red Pine, 277
Redundant Bars, 395
Refining of Pig-Iron, 284
Regulation of Turbine, 529
Relative Velocity, 32, 34, 35
—— Viscosities, 569
Resilience, 41, 308
—— Compressive, 316
—— Shear, 316
—— Tensile, 316
Resistance, Line of, 113, 114, 161, 480
—— of a Moving Train, 44
—— to Motion of Fluid in a Pipe, 76
—— to Rolling, 85
—— Wave, 54
Resolved Part of a Vector, 29
Resultant Force, 29, 129
—— on Forces in One Plane, 122
Reynolds, Prof. O., 73, 78, 81-4, 541
Rib Arched, 162
—— Parabolic, 162
Rigidity, Flexural, 634
—— Modulus of, 333, 563
—— Quasi, 497
—— Torsional, 634
Rim of Wheel, Volume of, 9
River Weaver, 206
Riveted Joints, 336
—— Work, 297
Riveting Machine, 170
Rocks, Slaty, Stratified, 275
Rolled Girder Section, 419, 421
Rollers, Curved, 572
—— for Girder, 183
Rolling, 106
—— and Hammering of Iron, 285
—— of a Cone, 577
—— Load, 403
—— Resistance to, 85
—— True, 571
Roof, Calculation of Forces in, 151, 153
—— Principal, 133
Rope, Centrifugal Force in, 322
—— Weight of, 315
Ropes, Wire, 232
Rotating Body, Energy in, 247
—— Cylinder, 368
Rotation, Motion of, 127
Rubbing, Distance of, 68
Rules for Beams, 410
—— for Deflection of Beams, 431
—— for Flow of Water through Notch, 515, 536
—— for Periodic Time in S.H.M., 549
—— for Simple Pendulum, 551
—— for Struts, 466
—— for Strength of Pipe, 182
Rupert's Drop, 183, 280

S

Safety, Factor of, 30
Safety-valve, 116
Sandstones, 275
—— Green, Dry, 282
Scalar Quantity, 29
Scraped Surfaces, 73
Screw, The, 99
—— Cutting Lathe, 101
—— Jack, 63
—— Jack, Efficiency of, 110
—— Piles, 101
—— Pitch of, 100
—— Propeller, 101
—— Propeller Blades, 289
—— Square-Threaded, 109
—— Thread, Whitworth, Sellers, Buttress, 101
Seasoning of Timber, 101
Section, Change of Shape of, 374
Sections of Structures, 159
Seizing, 73
Sellers' Thread, 101
Sense of Vector, 29
Shaft, Hollow Round, 354
—— Indiarubber, 318
—— Overtwisted, 290
—— Section, an Equilateral Triangle, 377
Shafts of Various Sections, 357
—— Stability of, 475
—— Strength of, 351, 352
—— Subjected to Twisting and Bending, 353
—— Whirling of Loaded, 476
Shape of Stream from Orifice, 534
—— of Teeth, 572
—— of Worm Thread, 574
Sharp-edged Notch, 536
—— Orifices, 534
Shear and Twist, 332
—— Strain, 332, 341, 350
—— Stress, 332
—— Stress in Beams, 460
Shearing Force, 128, 383, 355, 631
—— Force in Beams, 414, 427
—— Machines, 335
Ship, 496
—— Displacement of a, 55
—— Heeling Angle of, 260
—— Righting Moment of, 260
Ships, Friction of, 56
—— Horse-power of, 54
—— Models of, 55
—— Propulsion of, 52, 525
—— Resistance to Motion of, 54
—— Speed of, 53
Shot, Momentum of, 486
Silicon, 284
—— Bronze, 289
Silver, Alloys of, 289
Similar Figures, Area of, 8
—— Structures, Similarly Loaded, 424
Simple Harmonic Motion, 20, 222, 546
—— Harmonic Motion, Amplitude of, 20
—— Harmonic Motion, Frequency of, 20
—— Harmonic Motion, Periodic Time of, 20

Simple Harmonic Motions, Combination of, 555
Simpson's Rule, 7
Sine of an Angle, 23
Sines, Curve of, 78
Skeleton Drawings, 221, 570
Skew Bevel Wheels, 572
Slider Crank Chain, 577
Sliding, 106
—— Contact, 571
Slip of Belt, 86, 232
Slope of a Curve, 15
Smith, Prof. R. H., 45, 582
Snatch Block, 102
Soapy Water, Use of, 237
Space Average of a Force, 45
—— Average, Time Average, 272, 274
—— Rate, Time Rate, 269
Specific Gravity, 10
Speed of Bullet, 14
—— of Commercial Ships, 53
—— of Train, 12
Sperm Oil, 66
Sphere, Surface of, 8
—— Volume of, 8
Spherical Shell, Thick, 325
Spinning, 106
—— Tops, 498
Spiral Flow of Water, 539, 540
—— Spring, Vibration of, 550
Spring Balance, 40
—— of Indiarubber, 41
—— of Steel, 41
—— Spiral, Vibration of, 550
Springs, 613
—— Buffer Stop, 613
—— C, 643
—— Carriage, 618, 646
—— Clock, 613
—— Cylindrical Spiral, 624, 633
—— Different Forms of, 618
—— Elongation of, 630
—— Flat Spiral, 624, 627, 628, 629
—— Hardening and Tempering of, 622, 649
—— Materials Used in, 620
—— Phosphor Bronze, 622
—— Resilience of, 620, 641
—— Spiral, 622
—— Tubular Spiral, 642, 643
—— Uses of, 620
—— Vibration, 615
—— which Bend, 462
Spruce, Norway, 277
Spur Gearing, 95, 102
—— Wheel, 36
Squared Paper, 1, 13, 15, 21, 44, 60, 89, 90, 229, 293, 300, 565
Square-threaded Screw, Efficiency of, 109
Statical Friction, 75
Steadiness of Machines, 252
Steady Motion in Fluids, 532
Steam Engine, 43
—— Turbine, 531
Steel, 285
—— Basic, 286

Steel, Bessemer, 285
—— Cast, 183
—— Castings, Annealing of, 286
—— Crucible, 286
—— Hardening of, 290
—— Ingots, 285
—— Martin, 289
—— Rails, 286
—— Shear, 285
—— Siemens', 286
—— Spring, 41
—— Strength of, 286
—— Tempering of, 285
—— Weight of, 9
Step, 68
Stepped Cones, 26
Stiffness of Beams, 432
Stilling of Vibrations, 565
Stone, 275
—— Artificial, 276
Stoppage (Sudden) of Water in a Pipe, 192, 501
Storage of Energy in Fluids, 321
Stores of Energy, 42, 509
Straight Line Law, 62
Strain, 293, 296, 312
—— Energy, 85, 307, 489
—— in Castings, 183
—— Nature of, 311
—— Potential, 362
—— Shear, 331, 341
Strained Spring, Energy of, 41
Stream Lines, Circular, 541
Strength of Boiler, 319
—— of Chains, 315
—— Modulus of Sections, 399
—— of Pipe, 182, 319
—— of Ropes, 315
Stress, 296
—— Principal, 354
—— Shear, 331
Structures, Determination of Forces in, 156
—— Method of Sections, 159
Struts, 151, 463
—— Bending of, 464
—— Euler's Formula, 465
—— Rules for, 467
—— Shortening of, 298
—— with Lateral Loads, 470
St. Venant, 296, 318, 348, 366, 370, 379
Suddenly Applied Load, 306
Sulphur, 284
Sun and Planet Motion, 36
Surface, Level, 22
—— of Fracture, 306
—— Scraped, 73
Surfaces, Equi-potential, 218
Suspension, Axis of, 560, 561
—— Bifilar, 564
—— Bridge, 161
Switchback Railway, 41
Syphon, 79
—— Lubricator, 83
—— Mr. James Perry's, 517

The References are to Pages.

T

Table of Constants, 654
—— I. of Sections, 397
—— of Strength Moduli, 397
Tables, Mathematical, 24
Tangent of an Angle, 23
Tangential and Normal Components, 106
Teak, 278
Teeth of Spur-wheel, 572
—— of Wheels, 423
—— of Worm-wheel, 572
Telegraph Wire, Dip of, 168
Tempering Colours, 291, 292
—— of Steel, 290
Template, 218
Testing Machine, 1
—— Machine, Bailey's, 294
—— of Water-tube Boiler, 93
Theorem of Three Moments, 454
Theory of Bending, 318
—— of Fluid Friction, 80
Thick Cylinder, 183, 321, 328, 367
—— Spherical Shell, 325
Thin Cylinder, 324
Thomson's Jet Pump, 516
—— Prof. J., Dynamometer, 235
—— Prof. J., Overtwisted Shaft, 290
—— Triangular Notch, 514, 536
—— Turbine, 526
—— Whirlpool Chamber, 522
Thorneycroft's Dynamometer, 234
Thread, Whitworth, 100
Threads, Shapes of, 101
Thurston Oil Tester, 81
Tide-predicting Machine, Kelvin, 555
Tide, Rise and Fall of, 554
Tie Rod, 151
Tilt Hammer, 500
Timber, Felling of, 278
—— Preserving of, 279
—— Seasoning of, 277
—— Strength of, 279
Time Average of a Force, 45
Time Average, Space Average, 272, 274
Time, Periodic, 546
Time Rate, Space Rate, 269
Tin, 289
Top, Precession of, 596
Torque, 77, 95, 127, 260, 351
Torsional Rigidity, 634
—— of a Shaft, 352
Tortuosity of Path, 606
Total Energy of 1 lb. of Water, 515, 533
Toughened Cast Iron, 284
—— Glass, 280
Tower, Mr. Beauchamp, Experiments of, 73, 81
Train, Moving, 44
—— Epicyclic, 35
—— Speed of, 12
Tramcar, Pull on, 44
—— Work Done on, 38
Trammels, Elliptic, 577
Translation, Motion of, 121

The References are to Pages.

Transmission Dynamometers, 92, 233
—— of Power by Chains, 233
—— of Power by Shafts, 224
Travelling Loads, 436
Tresca, M., Experiments by, 301
Triangle of Forces, 56
Triangular Orifice, 534
Trochoidal Curves, 573
Tube, Built up, 323
—— Strengthening of, 324
Tungsten, 287
Turbines, 515, 526
—— Arranging of, 530
—— Axial Flow, 531
—— Efficiency of, 43
—— Guide Blades, 529
—— Hydraulic Efficiency of, 529
—— Inward Radial Flow, 531
—— Outward Radial Flow, 531
—— Regulation of, 529
—— Steam, 531
Tweddell's Hydraulic Tools, 199
Twist, Angle of, 349-352
—— and Shear, 332
Twisting, 349
—— and Bending in Shafts, 353
—— in Wire, 349
—— Moment, 350, 352, 353
—— of Prism, 369

U

Unclosed Polygon, 124, 129
Unit of Heat Energy, 142
Useful Work, 39
U-Tube, Motion in, 553

V

Valve Motions, 584
Vanes, Radial, 521
—— of Centrifugal Pump, 508
Vector, 29, 123
—— Clinure of, 29
—— Resolved Part of, 29
—— Sense of, 29
Vectors, Difference of, 30
—— Sum of, 30
Vehicle, Pulling Force on, 108
Velocity, 12, 14
—— Angular, 25
—— at Any Instant, 12
—— Average, 12
—— Critical, 83
—— Ratio, 59, 103, 220
—— Relative, 32, 34, 35
—— of Rubbing, 74
Venant, St., 296, 318, 348, 366, 370, 379
Vertical Line, 22
Vibrating Adjustable Masses, 561
—— Weight, 41
Vibration, Amplitude of, 565
—— Damping of, 566

Vibration Indicator, 590
—— of Heavy Disc, 568
—— of Spiral Spring, 550
—— of Strip of Steel, 552
—— Stilling of, 565
Viscosities, Relative, 569
Viscosity, 41
—— Co-efficient of, 76
Volts, 92
Volumes of Solids, 88
Voussoirs, 163, 165

W

Waste of Hydraulic Power, 58
Watchmaker's Drill, 291
Water Flowing Spirally, 589
—— Lake of, 512
—— Pressure Engine, 189, 191
—— Pressure Machinery, 88
—— Pressure of, 167
Waterfall, Energy of, 59
"Waterwitch" Steamship, 486
Watt's Parallel Motion, 69
—— Sun and Planet Motion, 86
Wave Propagation, 367
Weighbridge, 117
—— Graduation of Lever of, 118
Weights of Materials, 9
Wertheim's Experiments, 302
Wheels and Axle, 101
—— and Axle, Compound, 102
—— Friction, 69
—— Teeth of, 428
—— Teeth, Shapes of, 221
—— Value of Train of, 102
—— Worm, 102
Whirling Fluids, 216
—— Loaded Shaft, 476
Whirlpools, 518
—— Chamber, Thomson's, 522
White Fir, 277
—— Metal, 75, 289
Whitworth Thread, 100

Willans' Engine, 611
Wind Pressure, 156
Windmills, 101
Wire, Area of Section, 10
—— Copper, 289
—— Drawing Through Die, 298
—— Experiment on, 293, 300
—— Gun, 825
—— Pianoforte, 289
—— Ropes, 232
—— Telegraph, 168
—— Telephone, 289
Wöhler's Experiments, 305, 309
Wooden Pipe, 183
Work, 38
—— by Expanding Fluid, 19
—— Cutting of Metals, 46
—— Done by a Moment, 115
—— Done by Turbine, 521
—— Law of, 58
—— Lost in Friction, 67
—— Principle of, 120
—— Useful, 39
Workshop, 1
Worm and Worm Wheel, 102, 221
—— Thread, 574
Wrought Iron, 284
—— Case-hardening of, 285
—— Charcoal, 284
—— Forging of, 284
—— Lowmoor, 284
—— Red-short, 284
—— Staffordshire, 284
—— Weight of, 9

Y

Yield Point, 300
Young's Modulus, 299, 305

Z

Zinc, 287, 288

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